Konuralp Journal of Mathematics
Volume 1 No. 2 pp. 17–29 (2013) ©KJM

SOME INEQUALITIES FOR DIFFERENTIABLE PREQUASIINVEX FUNCTIONS WITH APPLICATIONS

MUHAMMAD AMER LATIF

Abstract. In this paper, we present several inequalities of Hermite-Hadamard type for differentiable prequasiinvex functions.Our results generalize those results proved in [2] and hence generalize those given in [7], [11] and [23]. Applications of the obtained results are given as well.

1. Introduction

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard inequality, due to its rich geometrical significance and applications, which is stated as (see [25]):

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I on real numbers and $a, b \in I$ with $a < b$. Then

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Both the inequalities hold in reversed direction if f is concave.

For several results which generalize, improve and extend the inequalities (1.1), we refer the interested reader to [7, 8, 9], [11]-[14], [23, 24], [27]-[32].

In [7], Dragomir and Agarwal obtained the following inequalities for differentiable functions which estimate the difference between the middle and the rightmost terms in (1.1):

Theorem 1.1. [7] *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$, and $f' \in L(a, b)$. If $\left|f'\right|$ is convex function on $[a, b]$, then the*

Date: January 1, 2013 and, in revised form, February 2, 2013.
2000 *Mathematics Subject Classification.* 26D15, 26D20, 26D07.
Key words and phrases. Hermite-Hadamard's inequality, invex set, preinvex function, prequasiinvex, Hölder's integral inequality.
The author is supported by ...

following inequality holds:

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}\left[\left|f^{'}(a)\right|+\left|f^{'}(b)\right|\right]. \tag{1.2}$$

Theorem 1.2. [7] *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$,and $f^{'} \in L(a,b)$. If $\left|f^{'}\right|^{\frac{p}{p-1}}$ is convex function on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left[\left|f^{'}(a)\right|^{\frac{p}{p-1}}+\left|f^{'}(b)\right|^{\frac{p}{p-1}}\right], \tag{1.3}$$

where $p > 1$ and $\frac{1}{p}+\frac{1}{q}=1$.

In [23], Pearce and J. Pečarić gave an improvement and simplification of the constant in Theorem 1.2 and consolidated this results with Theorem 1.1. The following is the main result from [23]:

Theorem 1.3. [23] *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$, and $f^{'} \in L(a,b)$. If $\left|f^{'}\right|^{q}$ is convex function on $[a,b]$, for some $q \geq 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left[\frac{\left|f^{'}(a)\right|^{q}+\left|f^{'}(b)\right|^{q}}{2}\right]^{\frac{1}{q}}. \tag{1.4}$$

If $\left|f^{'}\right|^{q}$ is concave on $[a,b]$, for some $q \geq 1$. Then

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\left|f^{'}\left(\frac{a+b}{2}\right)\right|. \tag{1.5}$$

Now, we recall that the notion of quasi-convex functions generalizes the notion of convex functions. More exactly, a function $f : [a,b] \to \mathbb{R}$ is said quasi-convex on $[a,b]$ if

$$f(tx+(1-t)y) \leq \max\{f(x), f(y)\}$$

for all $x; y \in [a; b]$ and $t \in [0,1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex
functions which are not convex, (see [11]).

Recently, Ion [11] introduced two inequalities of the right hand side of Hadamard's type for quasi-convex functions, as follows:

Theorem 1.4. [11] *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$. If $\left|f^{'}\right|$ is quasi-convex function on $[a,b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{4}\sup\left\{\left|f^{'}(a)\right|,\left|f^{'}(b)\right|\right\} \tag{1.6}$$

Theorem 1.5. [11] *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$. If $\left|f'\right|^p$ is quasi-convex function on $[a, b]$, for some $p > 1$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}\int_a^b g(x)dx - \frac{1}{b-a}\int_a^b f(x)g(x)\,dx\right| \leq \frac{b-a}{2(p+1)^{\frac{1}{p}}}\left(\sup\left\{\left|f'(a)\right|^{\frac{p}{p-1}}, \left|f'(b)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}, \tag{1.7}$$

where $\frac{1}{p} + \frac{1}{q} = 1$.

In [2], Alomari, Darus and Kirmaci established Hermite-Hadamard-type inequalities for quasi-convex functions which give refiments of those given above in Theorem 1.4 and Theorem 1.5.

Theorem 1.6. [2] *Let $f : I \subseteq [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$. If the mapping $\left|f'\right|$ is quasi-convex function on $[a, b]$, then the following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| \leq \frac{b-a}{8}\left[\sup\left\{\left|f'(a)\right|, \left|f'\left(\frac{a+b}{2}\right)\right|\right\} + \sup\left\{\left|f'(b)\right|, \left|f'\left(\frac{a+b}{2}\right)\right|\right\}\right]. \tag{1.8}$$

Theorem 1.7. [2] *Let $f : I \subseteq [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$. If $\left|f'\right|^p$ is convex function on $[a, b]$, for $p > 1$, then the following inequality holds:*

$$\begin{aligned}\left|\frac{f(a)+f(b)}{2} - \frac{1}{b-a}\int_a^b f(x)dx\right| &\leq \frac{b-a}{4(p+1)^{\frac{1}{p}}}\left[\left(\sup\left\{\left|f'(a)\right|^{\frac{p}{p-1}}, \left|f'\left(\frac{a+b}{2}\right)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right. \\ &\quad \left. + \left(\sup\left\{\left|f'(b)\right|^{\frac{p}{p-1}}, \left|f'\left(\frac{a+b}{2}\right)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right].\end{aligned} \tag{1.9}$$

Theorem 1.8. [2] *Let $f : I \subseteq [0, \infty) \to \mathbb{R}$ be a differentiable mapping on I°, where a, $b \in I$ with $a < b$. If $\left|f'\right|^q$ is convex function on $[a, b]$, for $p > 1$, then the* *following inequality holds:*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)dx\right| \le \frac{b-a}{8}\left[\left(\sup\left\{\left|f^{'}(a)\right|^q,\left|f^{'}\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}} + \left(\sup\left\{\left|f^{'}(b)\right|^q,\left|f^{'}\left(\frac{a+b}{2}\right)\right|^q\right\}\right)^{\frac{1}{q}}\right]. \tag{1.10}$$

In recent years, lot of efforts have been made by many mathematicians to generalize the classical convexity. These studies include among others the work of Hanson in [10], Ben-Israel and Mond [5], Pini [22], M.A.Noor [19, 20], Yang and Li [34] and Weir [33]. Mond [5], Weir [32] and Noor [18, 19], have studied the basic properties of the preinvex functions and their role in optimization, variational inequalities and equilibrium problems. Hanson in [10], introduced invex functions as a significant generalization of convex functions. Ben-Israel and Mond [4], gave the concept of preinvex function which is special case of invexity. Pini [22], introduced the concept of prequasiinvex functions as a generalization of invex functions.

Let us recall some known results concerning preinvexity and prequasiinvexity.

Let K be a closed set in $\mathbb{R}^n$ and let $f : K \to \mathbb{R}$ and $\eta : K \times K \to \mathbb{R}$ be continuous functions. Let $x \in K$, then the set K is said to be invex at x with respect to $\eta(\cdot,\cdot)$, if

$$x + t\eta(y, x) \in K, \forall x, y \in K, t \in [0, 1].$$

K is said to be an invex set with respect to η if K is invex at each $x \in K$. The invex set K is also called a η-connected set.

Definition 1.1. [33] The function f on the invex set K is said to be preinvex with respect to η, if

$$f(u + t\eta(v, u)) \le (1-t) f(u) + tf(v), \forall u, v \in K, t \in [0, 1].$$

The function f is said to be preconcave if and only if $-f$ is preinvex.

It is to be noted that every convex function is preinvex with respect to the map $\eta(x, y) = x - y$ but the converse is not true see for instance [32].

Definition 1.2. [21] The function f on the invex set K is said to be prequasiinvex with respect to η, if

$$f(u + t\eta(v, u)) \le \max\{f(u), f(v)\}, \forall u, v \in K, t \in [0, 1].$$

Also Every quasi-convex function is a prequasiinvex with respect to the map $\eta(v, u)$ but the converse does not hold, see for example [35].

In the recent paper, Noor [17] has obtained the following Hermite-Hadamard inequalities for the preinvex functions:

Theorem 1.9. [17]*Let* $f : [a, a + \eta(b, a)] \to (0, \infty)$ *be a preinvex function on the interval of the real numbers* K° *(the interior of* K*) and* a, $b \in K^\circ$ *with* $a < a + \eta(b, a)$*. Then the following inequality holds:*

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \le \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx \le \frac{f(a)+f(b)}{2}.$$

Barani, Ghazanfari and Dragomir in [4], presented the following estimates of the right-side of a Hermite- Hadamard type inequality in which some preinvex functions are involved.

Theorem 1.10. [4] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $\left|f'\right|$ is preinvex on K then, for every $a, b \in K$ with $\eta(b,a) \neq 0$, then the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right| \leq \frac{|\eta(b,a)|}{8}\left(\left|f'(a)\right|+\left|f'(b)\right|\right). \tag{1.11}$$

Theorem 1.11. [4] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. Assume $p \in \mathbb{R}$ with $p > 1$. If $\left|f'\right|^{\frac{p}{p-1}}$ is preinvex on K then, for every $a, b \in K$ with $\eta(b,a) \neq 0$, then the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right| \leq \frac{|\eta(b,a)|}{2(1+p)^{\frac{1}{p}}}\left[\frac{\left|f'(a)\right|^{\frac{p}{p-1}}+\left|f'(b)\right|^{\frac{p}{p-1}}}{2}\right]^{\frac{p-1}{p}}. \tag{1.12}$$

In [3], Barani, Ghazanfari and Dragomir gave similar results for quasi-preinvex functions as follows:

Theorem 1.12. [3] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. If $\left|f'\right|$ is prequasiinvex on K then, for every $a, b \in K$ with $\eta(b,a) \neq 0$, then the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right| \leq \frac{|\eta(b,a)|}{8}\sup\left\{\left|f'(a)\right|,\left|f'(b)\right|\right\}. \tag{1.13}$$

Theorem 1.13. [3] *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$. Suppose that $f : K \to \mathbb{R}$ is a differentiable function. Assume $p \in \mathbb{R}$ with $p > 1$. If $\left|f'\right|^{\frac{p}{p-1}}$ is preinvex on K then, for every $a, b \in K$ with $\eta(b,a) \neq 0$, then the following inequality holds:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)dx\right| \leq \frac{|\eta(b,a)|}{2(1+p)^{\frac{1}{p}}}\left(\sup\left\{\left|f'(a)\right|^{\frac{p}{p-1}},\left|f'(b)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}. \tag{1.14}$$

For several new results on inequalities for preinvex functions we refer the interested reader to [4, 21, 26] and the references therein.

In the present paper we give new inequalities of Hermite-Hadamard for functions whose derivatives in absolute value are preinvex and prequasiinvex. Our results generalize those results presented in a very recent paper of Alomari, Darus and Kirmaci [2].

2. Main Results

The following Lemma is essential in establishing our main results in this section:

Lemma 2.1. *Let $K \subseteq \mathbb{R}$ be an open invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a,\ b \in K$ with $a < a + \eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f^{'} \in L([a, a+\eta(b,a)])$. Then the following equality holds:*

$$\frac{f(a) + f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx = \frac{\eta(b,a)}{4}$$
$$\times \left[\int_0^1 (-t) f^{'}\left(a + \left(\frac{1-t}{2}\right)\eta(b,a)\right) dt + \int_0^1 t f^{'}\left(a + \left(\frac{1+t}{2}\right)\eta(b,a)\right) dt \right],$$

Proof. It suffices to note that

$$\begin{aligned} I_1 &= \int_0^1 (-t) f^{'}\left(a + \left(\frac{1-t}{2}\right)\eta(b,a)\right) dt \\ &= \left.\frac{2(-t) f\left(a + \left(\frac{1-t}{2}\right)\eta(b,a)\right)}{-\eta(b,a)}\right|_0^1 - \frac{2}{\eta(b,a)} \int_0^1 f\left(a + \left(\frac{1-t}{2}\right)\eta(b,a)\right) dt \\ &= \frac{2f(a)}{\eta(b,a)} - \frac{2}{\eta(b,a)} \int_0^1 f\left(a + \left(\frac{1-t}{2}\right)\eta(b,a)\right) dt. \end{aligned}$$

Setting $x = a + \left(\frac{1-t}{2}\right)\eta(b,a)$ and $dx = -\frac{\eta(b,a)}{2}dt$, which gives

$$I_1 = \frac{2f(a)}{\eta(b,a)} - \frac{4}{(\eta(b,a))^2} \int_a^{a+\frac{1}{2}\eta(b,a)} f(x)\,dx.$$

Similarly, we also have

$$I_2 = \frac{2f(a+\eta(b,a))}{\eta(b,a)} - \frac{4}{(\eta(b,a))^2} \int_{a+\frac{1}{2}\eta(b,a)}^{a+\eta(b,a)} f(x)\,dx.$$

Thus

$$\frac{\eta(b,a)}{4}[I_1 + I_2] = \frac{f(a) + f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\,dx.$$

which is the required result. □

Remark 2.1. If we take $\eta(b,a) = b - a$, then Lemma 2.1 reduces to Lemma 2.1 from [2].

Now using Lemma 2.1, we shall propose some new upper bound for the right-hand side of Hadamard's inequality for prequasiinvex mappings, which is better than the inequality had done in [3]. our results generalize those results proved in [2] as well.

Theorem 2.1. *Let $K \subseteq [0,\infty)$ be an open invex subset with respect to $\eta : K\times K \to \mathbb{R}$ and $a,\, b \in K$ with $a < a+\eta(b,a)$. Suppose $f : K \to \mathbb{R}$ is a differentiable mapping on K such that $f' \in L([a, a+\eta(b,a)])$. If $\left|f'\right|$ is prequasiinvex on K, then we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{\eta(b,a)}{8}\left[\sup\left\{\left|f'(a)\right|, \left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right\} + \sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|, \left|f'(a+\eta(b,a))\right|\right\}\right]. \tag{2.1}$$

Proof. From Lemma 2.1 and by using the prequasiinvex of $\left|f'\right|$ on K, for any $t\in[0,1]$ we have

$$\begin{aligned}
&\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \\
&\leq \frac{\eta(b,a)}{4}\left[\int_0^1 t\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|dt + \int_0^1 t\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|dt\right] \\
&\leq \frac{\eta(b,a)}{4}\left[\sup\left\{\left|f'(a)\right|, \left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right\}\int_0^1 t\,dt \right.\\
&\left. + \sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|, \left|f'(a+\eta(b,a))\right|\right\}\int_0^1 t\,dt\right] \\
&= \frac{\eta(b,a)}{8}\left[\sup\left\{\left|f'(a)\right|, \left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right\} + \sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|, \left|f'(a+\eta(b,a))\right|\right\}\right].
\end{aligned}$$

This completes the proof of the theorem. ∎

Corollary 2.1. *Let f be as in Theorem 2.1, if in addition*

(1) $\left|f'\right|$ *is increasing, then we have*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{\eta(b,a)}{8}\left[\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right| + \left|f'(a+\eta(b,a))\right|\right] \tag{2.2}$$

(2) $\left|f'\right|$ *is decreasing, then we have*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\,dx\right| \leq \frac{\eta(b,a)}{8}\left[\left|f'(a)\right| + \left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right] \tag{2.3}$$

Proof. The proof follows directly from Theorem 2.1. □

Remark 2.2. We note that the inequalities (2.2) and (2.3) are two new refinements of the trapezoid inequality for prequasiinvex functions, and thus for preinvex functions.

Remark 2.3. If we take $\eta(b,a)=b-a$ in Theorem 2.1, then the inequality reduces to the inequality (1.8). If we take $\eta(b,a)=b-a$ in corollary 2.1, then (2.2) and (2.3) reduce to the related corollary of Theorem 1.6 from [2].

Another similar result may be extended in the following theorem.

Theorem 2.2. *Let $K\subseteq[0,\infty)$ be an open invex subset with respect to $\eta:K\times K\to\mathbb{R}$ and $a,\,b\in K$ with $a<a+\eta(b,a)$. Suppose $f:K\to\mathbb{R}$ is a differentiable mapping on K such that $f'\in L([a,a+\eta(b,a)])$. If $\left|f'\right|^p$ is prequasiinvex on K from some $p>1$, then we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right| \tag{2.4}$$
$$\le\frac{\eta(b,a)}{4(p+1)^{\frac{1}{p}}}\left[\left(\sup\left\{\left|f'(a)\right|^{\frac{p}{p-1}},\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right.$$
$$\left.+\sup\left\{\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{\frac{p}{p-1}},\left|f'(a+\eta(b,a))\right|^{\frac{p}{p-1}}\right\}^{\frac{p-1}{p}}\right].$$

Proof. From Lemma 2.1 and using the well known Hölder's inequality, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\,dx\right| \tag{2.5}$$
$$\le\frac{\eta(b,a)}{4}\left[\int_0^1 t\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|dt+\int_0^1 t\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|dt\right]$$
$$\le\frac{\eta(b,a)}{4}\left[\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_0^1 t^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q dt\right)^{\frac{1}{q}}\right].$$

By the prequasiinvexity of $\left|f'\right|^p$ on K from some $p>1$, we have for every $a,b\in K$ with $\eta(b,a)>0$ and $t\in[0,1]$ that

$$\left|f'\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^q\le\sup\left\{\left|f'(a)\right|^q,\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q\right\}$$

and

$$\left|f'\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^q\le\sup\left\{\left|f'(a+\eta(b,a))\right|^q,\left|f'\left(a+\frac{1}{2}\eta(b,a)\right)\right|^q\right\},$$

where $\frac{1}{p}+\frac{1}{q}=1$. Using the above inequalities in (2.5), we get the required result. This completes the proof of the theorem as well. □

Corollary 2.2. *Let f be as in Theorem 2.2, if in addition*

(1) $\left|f^{'}\right|^{\frac{p}{p-1}}$ *is increasing, then we have*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(x)\,dx\right| \leq \frac{\eta(b,a)}{4(p+1)^{\frac{1}{p}}}\left[\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|+\left|f^{'}(a+\eta(b,a))\right|\right] \tag{2.6}$$

(2) $\left|f^{'}\right|^{\frac{p}{p-1}}$ *is decreasing, then we have*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(x)\,dx\right| \leq \frac{\eta(b,a)}{4(p+1)^{\frac{1}{p}}}\left[\left|f^{'}(a)\right|+\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|\right] \tag{2.7}$$

Proof. It is a direct consequence of Theorem 2.2. □

Remark 2.4. If we take $\eta(b,a)=b-a$ in Theorem 2.2, then the inequality reduces to the inequality (1.9). If we take $\eta(b,a)=b-a$ in corollary 2.2, then (2.6) and (2.7) reduce to the related corollary of Theorem 1.7 from [2].

An improvement of the constants in Theorem 2.2 and a consolidation of this result with Theorem 2.1 are given in the following theorem.

Theorem 2.3. *Let $K\subseteq\mathbb{R}$ be an open invex subset with respect to $\eta: K\times K\to\mathbb{R}$ and $a,\ b\in K$ with $a<a+\eta(b,a)$ Suppose $f: K\to\mathbb{R}$ is a differentiable mapping on K such that $f^{'}\in L([a,a+\eta(b,a)])$. If $\left|f^{'}\right|^{q}$ for $q\geq 1$, is prequasiinvex on K, then for every $a,b\in K$ with $\eta(b,a)>0$ we have the following inequality:*

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(x)\,dx\right| \leq \frac{\eta(b,a)}{8}\left[\left(\sup\left\{\left|f^{'}(a)\right|^{q},\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{q}\right\}\right)^{\frac{1}{q}} + \sup\left\{\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{q},\left|f^{'}(a+\eta(b,a))\right|^{q}\right\}^{\frac{1}{q}}\right]. \tag{2.8}$$

Proof. From Lemma 2.1, using the power-mean integral inequality and using the prequasiinvexity of $\left|f^{'}\right|^{q}$ on K for $q \geq 1$, we have

$$\left|\frac{f(a)+f(a+\eta(b,a))}{2}-\frac{1}{\eta(b,a)}\int_{a}^{a+\eta(b,a)}f(x)\,dx\right| \tag{2.9}$$
$$\leq \frac{\eta(b,a)}{4}\left[\int_{0}^{1}t\left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|dt+\int_{0}^{1}t\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|dt\right]$$
$$\leq \frac{\eta(b,a)}{4}\left[\left(\int_{0}^{1}tdt\right)^{1-\frac{1}{q}}\left(\int_{0}^{1}t\left|f^{'}\left(a+\left(\frac{1-t}{2}\right)\eta(b,a)\right)\right|^{q}dt\right)^{\frac{1}{q}}\right.$$
$$\left.+\left(\int_{0}^{1}tdt\right)^{1-\frac{1}{q}}\left(\int_{0}^{1}t\left|f^{'}\left(a+\left(\frac{1+t}{2}\right)\eta(b,a)\right)\right|^{q}dt\right)^{\frac{1}{q}}\right]$$
$$\leq \frac{\eta(b,a)}{8}\left[\left(\sup\left\{\left|f^{'}(a)\right|^{q},\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{q}\right\}\right)^{\frac{1}{q}}\right.$$
$$\left.++\sup\left\{\left|f^{'}\left(a+\frac{1}{2}\eta(b,a)\right)\right|^{q},\left|f^{'}(a+\eta(b,a))\right|^{q}\right\}^{\frac{1}{q}}\right].$$

which completes the proof □

Corollary 2.3. *Let f be as in Theorem 2.3, if in addition*

(1) $\left|f^{'}\right|^{\frac{1}{q}}$ *is increasing, then we have the inequality (2.2).*
(2) $\left|f^{'}\right|^{\frac{1}{q}}$ *is decreasing, then we have the inequality (2.3).*

Remark 2.5. If we take $\eta(b,a)=b-a$ in Theorem 2.3, then the inequality reduces to the inequality (1.10). If we take $\eta(b,a)=b-a$ in corollary 2.3, then we get the results of the related corollary of Theorem 1.8 from [2].

Remark 2.6. For $q=1$, (2.8) reduces to Theorem 2.1. For $q=\frac{p}{p-1}$ $(p>1)$ we have an improvement of the constants in Theorem 2.2, since $4^{p}>p+1$ if $p>1$ and accordingly

$$\frac{1}{8}<\frac{1}{(p+1)^{\frac{1}{p}}}.$$

3. Applications to Special Means

In what follows we give certain generalizations of some notions for a positive valued function of a positive variable.

Definition 3.1. [6]A function $M:\mathbb{R}_{+}^{2}\to\mathbb{R}_{+}$, is called a Mean function if it has the following properties:

(1) Homogeneity: $M(ax,ay)=aM(x,y)$, for all $a>0$,
(2) Symmetry : $M(x,y)=M(y,x)$,
(3) Reflexivity : $M(x,x)=x$,
(4) Monotonicity: If $x\leq x^{'}$ and $y\leq y^{'}$, then $M(x,y)\leq M(x^{'},y^{'})$,
(5) Internality: $\min\{x,y\}\leq M(x,y)\leq\max\{x,y\}$.

We consider some means for arbitrary positive real numbers α, β (see for instance [6]).

(1) The arithmetic mean:

$$A := A(\alpha, \beta) = \frac{\alpha + \beta}{2}$$

(2) The geometric mean:

$$G := G(\alpha, \beta) = \sqrt{\alpha\beta}$$

(3) The harmonic mean:

$$H := H(\alpha, \beta) = \frac{2}{\frac{1}{\alpha} + \frac{1}{\beta}}$$

(4) The power mean:

$$P_r := P_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2}\right)^{\frac{1}{r}}, \ r \geq 1$$

(5) The identric mean:

$$I := I(\alpha, \beta) = \begin{cases} \frac{1}{e}\left(\frac{\beta^\beta}{\alpha^\alpha}\right)^{\frac{1}{\beta-\alpha}}, & \alpha \neq \beta \\ \alpha, & \alpha = \beta \end{cases}$$

(6) The logarithmic mean:

$$L := L(\alpha, \beta) = \frac{\alpha - \beta}{\ln|\alpha| - \ln|\beta|}, \quad |\alpha| \neq |\beta|$$

(7) The generalized log-mean:

$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)}\right]^{\frac{1}{p}}, \ \alpha \neq \beta, \ p \in \mathbb{R}\backslash\{-1, 0\}.$$

It is well known that L_p is monotonic nondecreasing over $p \in \mathbb{R}$, with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$.

Now, let a and b be positive real numbers such that $a < b$. Consider the function $M := M(a, b) : [a, a + \eta(b, a)] \times [a, a + \eta(b, a)] \to \mathbb{R}^+$, which is one of the above mentioned means, therefore one can obtain variant inequalities for these means as follows:

Setting $\eta(b, a) = M(b, a)$ in (2.1), (2.4) and (2.8), one can obtain the following interesting inequalities involving means:

$$\left|\frac{f(a) + f(a + M(b,a))}{2} - \frac{1}{M(b,a)}\int_a^{a+M(b,a)} f(x)\,dx\right| \leq \frac{M(b,a)}{8}\left[\sup\left\{\left|f'(a)\right|, \left|f'\left(a + \frac{1}{2}M(b,a)\right)\right|\right\} + \sup\left\{\left|f'\left(a + \frac{1}{2}M(b,a)\right)\right|, \left|f'(a + M(b,a))\right|\right\}\right]. \tag{3.1}$$

$$
\begin{aligned}
(3.2)\quad & \left|\frac{f(a)+f(a+M(b,a))}{2}-\frac{1}{M(b,a)}\int_{a}^{a+M(b,a)}f(x)\,dx\right| \\
& \leq \frac{M(b,a)}{4(p+1)^{\frac{1}{p}}}\left[\left(\sup\left\{\left|f^{'}(a)\right|^{\frac{p}{p-1}},\left|f^{'}\left(a+\frac{1}{2}M(b,a)\right)\right|^{\frac{p}{p-1}}\right\}\right)^{\frac{p-1}{p}}\right. \\
& \qquad\left.+\sup\left\{\left|f^{'}\left(a+\frac{1}{2}M(b,a)\right)\right|^{\frac{p}{p-1}},\left|f^{'}(a+M(b,a))\right|^{\frac{p}{p-1}}\right\}^{\frac{p-1}{p}}\right],
\end{aligned}
$$

for $p > 1$, and

$$
\begin{aligned}
(3.3)\quad & \left|\frac{f(a)+f(a+M(b,a))}{2}-\frac{1}{M(b,a)}\int_{a}^{a+M(b,a)}f(x)\,dx\right| \\
& \leq \frac{M(b,a)}{8}\left[\left(\sup\left\{\left|f^{'}(a)\right|^{q},\left|f^{'}\left(a+\frac{1}{2}M(b,a)\right)\right|^{q}\right\}\right)^{\frac{1}{q}}\right. \\
& \qquad\left.+\sup\left\{\left|f^{'}\left(a+\frac{1}{2}M(b,a)\right)\right|^{q},\left|f^{'}(a+M(b,a))\right|^{q}\right\}^{\frac{1}{q}}\right],
\end{aligned}
$$

for $q \geq 1$. Letting $M = A$, G, H, P_r, I, L, L_p in (3.1), (3.2) and (3.3), we can get the required inequalities, and the details are left to the interested reader.

4. Acknowledgment

The author is very thankful to the anonymous referee for his/her very constructive comments which have improved the final version of the paper.

References

[1] T. Antczak, Mean value in invexity analysis, Nonl. Anal., 60 (2005), 1473-1484.

[2] M. Alomari , M. Darus, U.S. Kirmaci, Refinements of Hadamard-type inequalities for quasi-convex functions with applications to trapezoidal formula and to special means, Computers and Mathematics with Applications 59 (2010) 225 232.

[3] A. Barani, A.G. Ghazanfari, S.S. Dragomir, Hermite-Hadamard inequality through prequsi-invex functions, RGMIA Research Report Collection, 14(2011), Article 48, 7 pp.

[4] A. Barani, A.G. Ghazanfari, S.S. Dragomir, Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex, RGMIA Research Report Collection, 14(2011), Article 64, 11 pp.

[5] A. Ben-Israel and B. Mond, What is invexity?, J. Austral. Math. Soc., Ser. B, 28(1986), No. 1, 1-9.

[6] P.S. Bullen, Handbook of Means and Their Inequalities, Kluwer Academic Publishers, Dordrecht, 2003.

[7] S. S. Dragomir, and R. P. Agarwal, Two inequalities for differentiable mappings and applications to special means of real numbers and trapezoidal formula, Appl. Math. Lett., 11(5)(1998), 91–95.

[8] S. S. Dragomir, Two mappings in connection to Hadamard's inequalities, J. Math. Anal. Appl., 167(1992), 42–56.

[9] D. -Y. Hwang, Some inequalities for differentiable convex mapping with application to weighted trapezoidal formula and higher moments of random variables, Appl. Math. Comp., 217(23)(2011), 9598-9605.

[10] M. A. Hanson, On sufficiency of the Kuhn-Tucker conditions, J. Math. Anal. Appl. 80 (1981) 545-550.

[11] D. A. Ion, Some estimates on the Hermite-Hadamard inequality through quasi-convex functions, Annals of University of Craiova, Math. Comp. Sci. Ser. 34 (2007) 82 87.
[12] U. S. Kırmacı, Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comp., 147(1)(2004), 137-146.
[13] U. S. Kırmacı and M. E. Özdemir, On some inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula, Appl. Math. Comp., 153(2)(2004), 361-368.
[14] K. C. Lee and K. L. Tseng, On a weighted generalization of Hadamard's inequality for G-convex functions, Tamsui-Oxford J. Math. Sci., 16(1)(2000), 91–104.
[15] A. Lupas, A generalization of Hadamard's inequality for convex functions, Univ. Beograd. Publ. Elek. Fak. Ser. Mat. Fiz., 544-576(1976), 115–121.
[16] S. R. Mohan and S. K. Neogy, On invex sets and preinvex functions, J. Math. Anal. Appl. 189 (1995), 901–908.
[17] M. Aslam Noor, Hermite-Hadamard integral inequalities for log-preinvex functions, Preprint.
[18] M. A. Noor, Variational-like inequalities, Optimization, 30 (1994), 323–330.
[19] M. A. Noor, Invex equilibrium problems, J. Math. Anal. Appl., 302 (2005), 463–475.
[20] M. A. Noor, Some new classes of nonconvex functions, Nonl. Funct. Anal. Appl.,11(2006),165-171
[21] M. A. Noor, On Hadamard integral inequalities involving two log-preinvex functions, J. Inequal. Pure Appl. Math., 8(2007), No. 3, 1-14.
[22] R. Pini, Invexity and generalized convexity, Optimization 22 (1991) 513-525.
[23] C. E. M. Pearce and J. Pečarić, Inequalities for differentiable mappings with application to special means and quadrature formulae, Appl. Math. Lett., 13(2)(2000), 51–55.
[24] F. Qi, Z. -L.Wei and Q. Yang, Generalizations and refinements of Hermite-Hadamard's inequality, Rocky Mountain J. Math., 35(2005), 235–251.
[25] J. Pečarić, F. Proschan and Y. L. Tong, Convex functions, partial ordering and statistical applications, Academic Press, New York, 1991.
[26] M. Z. Sarikaya, H. Bozkurt and N. Alp, On Hermite-Hadamard type integral inequalities for preinvex and log-preinvex functions, arXiv:1203.4759v1.
[27] M. Z. Sarıkaya and N. Aktan, On the generalization some integral inequalities and their applications Mathematical and Computer Modelling, 54(9-10)(2011), 2175-2182.
[28] M. Z. Sarikaya, M. Avci and H. Kavurmaci, On some inequalities of Hermite-Hadamard type for convex functions, ICMS International Conference on Mathematical Science, AIP Conference Proceedings 1309, 852(2010).
[29] M. Z. Sarikaya, O new Hermite-Hadamard Fejér type integral inequalities, Stud. Univ. Babeş-Bolyai Math. 57(2012), No. 3, 377-386.
[30] A. Saglam, M. Z. Sarikaya and H. Yıldırım and, Some new inequalities of Hermite-Hadamard's type, Kyungpook Mathematical Journal, 50(2010), 399-410.
[31] C. -L. Wang and X. -H. Wang, On an extension of Hadamard inequality for convex functions, Chin. Ann. Math., 3(1982), 567–570.
[32] S. -H. Wu , On the weighted generalization of the Hermite-Hadamard inequality and its applications, The Rocky Mountain J. of Math., 39(2009), no. 5, 1741–1749.
[33] T. Weir, and B. Mond, Preinvex functions in multiple bjective optimization, Journal of Mathematical Analysis and Applications, 136 (1998) 29-38.
[34] X. M. Yang and D. Li, On properties of preinvex functions, J. Math. Anal. Appl. 256 (2001), 229-241.
[35] X.M. Yang, X.Q. Yang and K.L. Teo, Characterizations and applications of prequasiinvex functions, properties of preinvex functions, J. Optim. Theo. Appl. 110 (2001) 645-668.
[36] X. M. Yang, X. Q. Yang, K.L. Teo, Generalized invexity and generalized invariant monotonocity, Journal of Optimization Theory and Applications 117 (2003) 607-625.

College of Science, Department of Mathematics,, University of Hail, Hail 2440, Saudi Arabia
E-mail address: m_amer_latif@hotmail.com