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# ON AN OPEN PROBLEM BY B. SROYSANG

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Abstract. In this short note, we answer an open problem posed by B. Sroysang [1]. More precisely, we find all solutions of the Diophantine equation $8^x+17^y=z^2$ where $x, y$ and $z$ are non-negative integers.

## 1. Introduction

In a recent paper [1], B. Sroysang showed that (1, 0, 3) is a unique solution $(x, y, z)$ to the Diophantine equation $8^x+19^y=z^2$ where $x, y$ and $z$ are non-negative integers. His findings contradicts the result suggested by Peker and Cenberci in [2]: the Diophantine equation $8^x+19^y=z^2$ has no non-negative integer solution. Also, in the end of his paper, Sroysang [1] posed the question "What is the set of all solutions $(x, y, z)$ for the Diophantine equation $8^x+17^y=z^2$ where $x, y$ and $z$ are non-negative integers?". In this short note, we answer this question of Sroysang.

## 2. Main Results

We begin this section by stating Catalan's conjecture and proving a helpul Lemma.

**Proposition 2.1.** [2] *The solution to the Diophantine equation $a^x-b^y=1$ where $a, b, x$ and $y$ are integers with min$\{a, b, x, y\}>1$ is unique and is given by $(a, b, x, y)=(3, 2, 2, 3)$.*

**Lemma 2.1.** *Let $x$ and $z$ be non-negative integers. Then, the solutions $(x, z)$ to the Diophantine equation $8^x+17=z^2$ are $(1, 5), (2, 9)$ and $(3, 23)$.*

*Proof.* The case $x=0$ and $z=0$ are obvious. So we only consider the case when $x, z>0$. We note that $1 \equiv 8^x+17 \equiv z^2 \pmod 4$. So, $z$ is either of the form $4k+1$ or $4k+3$, $k=0$ or a natural number. Hence, we have the following cases.

**Case 1.** $z=4k+1$. If $z=4k+1$ then we have $8^x+17=(4k+1)^2=16k^2+8k+1$. So, $8^x+16=16k^2+8k$ and this implies that $8^{x-1}+2=2k^2+k$.

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Thus, $(2^{x-1})^3 + 1 = 2k^2 + k - 1$. Expressing both sides as product of their prime factors, we have $(2^{x-1} + 1)((2^{x-1})^2 - 2^{x-1} + 1) = (2k - 1)(k + 1)$. Therefore, we have two possibilities.

$$\begin{cases} 2^{x-1} + 1 &= k + 1, \\ (2^{x-1})^2 - 2^{x-1} + 1 &= 2k - 1 \end{cases}$$

or

$$\begin{cases} 2^{x-1} + 1 &= 2k - 1, \\ (2^{x-1})^2 - 2^{x-1} + 1 &= k + 1. \end{cases}$$

For the first set of equalities, we have $2^{x-1} + 1 = k + 1$ implies that $k = 2^{x-1}$. So, $(2^{x-1})^2 - 2^{x-1} + 1 = 2k - 1 = 2(2^{x-1}) - 1$. Hence, $(2^{x-1})^2 - 3(2^{x-1}) + 2 = 0$, which is a quadratic equation and is factorable. In particular, $(2^{x-1} - 1)(2^{x-1} - 2) = 0$. Here we'll obtain, $2^{x-1} = 1$ and $2^{x-1} = 2$. This gives us the values $x = 1$ and $x = 2$, respectively. This follows that $k = 1$ and $k = 2$. For $k = 1$, we have $(x, z) = (1, 5)$ and, for $k = 2$, we have $(x, z) = (2, 9)$. On the otherhand, it could be verified easily that the second set of equalities will give us the solution $(x, z) = (2, 9)$.

**Case 2.** $z = 4k + 3$. If $z = 4k + 3$ then $8^x + 17 = (4k + 3)^2 = 16k^2 + 24k + 9$. Hence, $8^x + 8 = 16k^2 + 24k$ and this implies that $8^{x-1} + 1 = 2k^2 + 3k$. Therefore, $(2^{x-1})^3 + 1 = (2^{x-1} + 1)((2^{x-1})^2 - 2^{x-1} + 1) = k(2k + 3)$. So, we have the following equalities

$$\begin{cases} 2^{x-1} + 1 &= k, \\ (2^{x-1})^2 - 2^{x-1} + 1 &= 2k + 3. \end{cases}$$

Eliminating $k$ we have, $(2^{x-1})^2 - 2^{x-1} + 1 = 2(2^{x-1} + 1) + 3 = 2(2^{x-1}) + 5$. Here we obtain the quadratic equation $(2^{x-1})^2 - 3(2^{x-1}) - 4 = 0$ which is equivalent to $(2^{x-1} + 1)(2^{x-1} - 4) = 0$. Solving for zeros, we have $2^{x-1} = -1$, which is impossible and $2^{x-1} = 4$, which is true for $x = 3$. This gives us the value $k = 5$. Thus, we have the solution $(x, y) = (3, 23)$. This completes the proof of the theorem. □

**Theorem 2.1.** *The only solutions to the Diophantine equation* $8^x + 17^y = z^2$ *in non-negative integers are given by* $(x, y, z) \in \{(1, 0, 3), (1, 1, 5), (2, 1, 9), (3, 1, 23)\}$.

*Proof.* The case when $z = 0$ is obvious so we only consider the following cases.

**Case 1.** $x = 0$. Suppose $8^x + 17^y = z^2$ is possible in non-negative integers for $x = 0$ then $z^2 - 1 = (z + 1)(z - 1) = 17^y$. So, $2 = (z + 1) - (z - 1) = 17^\beta - 17^\alpha$, where $\alpha + \beta = y$ and $\alpha < \beta$. It follows that $7^\alpha(7^{\beta-\alpha} - 1) = 2$. Hence, $7^\alpha = 1$ which is true for $\alpha = 0$. Thus, $7^\beta = 3$, a contradiction. Therefore, $17^y + 1 = z^2$ is not possible in non-negative integers.

**Case 2.** $y = 0$. If $y = 0$ we have $z^2 - 1 = (z + 1)(z - 1) = 2^{3x}$. Then, $2 = (z+1) - (z-1) = 2^\beta - 2^\alpha$, where $\alpha + \beta = 3x$ and $\alpha < \beta$. So, $2^{\alpha-1}(2^{\beta-\alpha} - 1) = 1$. Here we obtain $\alpha = 1$ and $2^{\beta-1} = 2$, which is true for $\beta = 2$. Thus, $x = 1$ and $z = 3$. Therefore, we have the solution $(x, y, z) = (1, 0, 3)$ to the Diophantine equation $8^x + 17^y = z^2$.

**Case 3.** $x, y, z > 0$. First note that for $y = 1$, Lemma 2.2 implies the following solutions $(x, y, z) = (1, 1, 5), (2, 1, 9), (3, 1, 23)$. Now suppose that $8^x + 17^y = z^2$ is possible in non-negative integers $x, y, z$ for $y > 1$. We consider two sub-cases.

**Subcase 3.1** $y$ is even. If we let $y$ be even, *i.e.* $y = 2n$ for some natural number $n$, then $z^2 - (17^n)^2 = 2^{3x}$. So, $(z + 17^n) - (z - 17^n) = 2^\beta - 2^\alpha$, again, $\alpha + \beta = 3x$ and $\alpha < \beta$. Hence, $2^{\alpha-1}(2^{\beta-\alpha} - 1) = 17^n$. This gives us a value $\alpha = 1$. It follows that, $2^{\beta-1} - 17^n = 1$, a contradiction to Catalan's conjecture. Thus, $8^x + 17^y = z^2$, for $y$ even is impossible in positive integers.

**Subcase 3.2** $y$ is odd. If $y = 2n + 1$ then we have $8^x + 17^{2k+1} = z^2$. Since $1 \equiv 8^x + 17^{2k+1} \equiv z^2 \pmod 4$ then, either $z = 4k + 1$ or $z = 4k + 3$, where $k = 0$ or a natural number. Hence, we have the following

$$\begin{cases} 8^{x-1} + 1 = 2k^2 + k - \left(\dfrac{17^{2n+1} - 9}{8}\right), & \text{for } z = 4k + 1 \quad (1), \\ \\ 8^{x-1} + 1 = 2k^2 + 3k - \left(\dfrac{17^{2n+1} - 17}{8}\right), & \text{for } z = 4k + 3 \quad (2). \end{cases}$$

Take note that $k$ is an integer. Hence, the RHS of (1) must be factorable. That is,

$$(1)^2 - (4)(2)\left(-\frac{17^{2n+1} - 9}{8}\right) = m^2,$$

for some non-negative integer $m$. Then, $17^{2n+1} - 8 = m^2$. Adding both sides by -9 we obtain $17(17^{2n} - 1) = (z + 3)(z - 3) = z^2 - 9$. This gives us a value $z = 20$. Thus, $17^{2n} = 24$ which is a contradiction. On the otherhand, the RHS of (2) must also be an integer, more precisely

$$(3)^2 - (4)(2)\left(-\frac{17^{2n+1} - 17}{8}\right) = m^2,$$

where $m$ is a non-negative integer. Hence, $17^{2n+1} - 8 = m^2$, which is again a contradiction. Thus, $8^x + 17^y = z^2$, for $y$ odd is not solvable in positive integers. This proves the theorem. □

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