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ON THE HADAMARD'S TYPE INEQUALITIES FOR L-LIPSCHITZIAN MAPPING

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Abstract. In this paper, we establish some new inequalities of Hadamard's type for L-Lipschitzian mapping in two variables.

1. Introduction

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex mapping defined on the interval I of real numbers and $a, b \in I$, with $a < b$. the following double inequality is well known in the literature as the Hermite-Hadamard inequality:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)\,dx \leq \frac{f(a)+f(b)}{2}.$$

Let us now consider a bidemensional interval $\Delta =: [a,b] \times [c,d]$ in $\mathbb{R}^2$ with $a < b$ and $c < d$. A mapping $f : \Delta \to \mathbb{R}$ is said to be convex on Δ if the following inequality:

$$f(tx + (1-t)z, ty + (1-t)w) \leq tf(x,y) + (1-t)f(z,w)$$

holds, for all $(x,y), (z,w) \in \Delta$ and $t \in [0,1]$ .A function $f : \Delta \to \mathbb{R}$ is said to be on the co-ordinates on Δ if the partial mappings $f_y : [a,b] \to \mathbb{R}$, $f_y(u) = f(u,y)$ and $f_x : [c,d] \to \mathbb{R}$, $f_x(v) = f(x,v)$ are convex where defined for all $x \in [a,b]$ and $y \in [c,d]$ (see [3]).

A formal definition for co-ordinated convex function may be stated as follows:

Definition 1.1. A function $f : \Delta \to \mathbb{R}$ will be called co-ordinated canvex on Δ, for all $t, s \in [0,1]$ and $(x,y), (u,w) \in \Delta$,if the following inequality holds:

$$\begin{aligned} & f(tx + (1-t)y, su + (1-s)w) \\ (1.1) \quad \leq \quad & tsf(x,u) + s(1-t)f(y,u) + t(1-s)f(x,w) + (1-t)(1-s)f(y,w). \end{aligned}$$

Clearly, every convex function is co-ordinated convex. Furthermore, there exist co-ordinated convex function which is not convex, (see, [3]). For several recent results concerning Hermite-Hadamard's inequality for some convex function on the

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co-ordinates on a rectangle from the plane $\mathbb{R}^2$, we refer the reader to ([1]-[3], [5], [6], [8], [9] and [11]).

In [3], Dragomir establish the following similar inequality of Hadamard's type for co-ordinated convex mapping on a rectangle from the plane $\mathbb{R}^2$.

Theorem 1.1. *Suppose that $f : \Delta \to \mathbb{R}$ is co-ordinated convex on Δ. Then one has the inequalities:*

$$\begin{aligned}
& f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\
\leq\ & \frac{1}{2}\left[\frac{1}{b-a}\int_a^b f\left(x, \frac{c+d}{2}\right)dx + \frac{1}{d-c}\int_c^d f\left(\frac{a+b}{2}, y\right)dy\right] \\
\leq\ & \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\
\leq\ & \frac{1}{4}\left[\frac{1}{b-a}\int_a^b f(x,c)\,dx + \frac{1}{b-a}\int_a^b f(x,d)\,dx \right. \\
& \left. + \frac{1}{d-c}\int_c^d f(a,y)\,dy + \frac{1}{d-c}\int_c^d f(b,y)\,dy\right] \\
\leq\ & \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned}$$

The above inequalities are sharp.

Definition 1.2. Consider a function $f : V \to \mathbb{R}$ defined on a subset V of $\mathbb{R}^n$, $n \in \mathbb{N}$. Let $L = (L_1, L_2, ..., L_n)$ where $L_i \geq 0,\ i = 1, 2, ..., n$. We say that f is L-Lipschitzian function if

$$|f(x) - f(y)| \leq \sum_{i=1}^{n} L\,|x_i - y_i|$$

for all $x, y \in V$.

For several recent results concerning Hadamard's type inequality for some L-Lipschitzian function, we refer the reader to ([4], [7], [10]).

The main purpose of this paper is to establish some Hadamard's type ineqaulities for L-Lipschitzian mapping in two variables.

2. Hadamard's Type Inequalities

Firstly, we will start the proof of the Theorem 1.1 by using the definition of the co-ordinated convex functions as follows:

Theorem 2.1. *Suppose that $f : \Delta \to \mathbb{R}$ is co-ordinated convex on Δ. Then one has the inequalities:*

$$\begin{aligned}
f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) &\leq \frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx \\
&\leq \frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}.
\end{aligned} \tag{2.1}$$

Proof. According to (1.1) with $x = t_1a + (1 - t_1)b,\ y = (1 - t_1)a + t_1b,\ u = s_1c + (1 - s_1)d,\ w = (1 - s_1)c + s_1d$ and $t = s = \frac{1}{2}$, we find that

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ \leq\ & \frac{1}{4}\left[f(t_1a + (1-t_1)b, s_1c + (1-s_1)d) + f((1-t_1)a + t_1b, s_1c + (1-s_1)d)\right. \\ & \left. + f(t_1a + (1-t_1)b, (1-s_1)c + s_1d) + f((1-t_1)a + t_1b, (1-s_1)c + s_1d)\right]. \end{aligned}$$

Thus, by integrating with respect to t_1, s_1 on $[0,1] \times [0,1]$, we obtain

$$\begin{aligned} & f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \\ \leq\ & \frac{1}{4}\left[\int_0^1 \int_0^1 [f(t_1a + (1-t_1)b, s_1c + (1-s_1)d) + f((1-t_1)a + t_1b, s_1c + (1-s_1)d)]\, ds_1dt_1\right. \\ & \left. + \int_0^1 \int_0^1 [f(t_1a + (1-t_1)b, (1-s_1)c + s_1d) + f((1-t_1)a + t_1b, (1-s_1)c + s_1d)]\, ds_1dt_1\right]. \end{aligned}$$

Using the change of the variable, we get

$$f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) \leq \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dydx, \tag{2.2}$$

which the first inequality is proved. The proof of the second inequality follows by using (1.1) with $x = a,\ y = b,\ u = c$ and $w = d$, and integrating with respect to t, s over $[0,1] \times [0,1]$,

$$\begin{aligned} & \int_0^1 \int_0^1 f(ta + (1-t)b, sc + (1-s)d)\, dsdt \\ \leq\ & \int_0^1 \int_0^1 [tsf(a,c) + s(1-t)f(b,c) + t(1-s)f(a,d) + (1-t)(1-s)f(b,d)]\, dsdt \\ =\ & \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}. \end{aligned}$$

Here, using the change of the variable $x = ta + (1-t)b$ and $y = sc + (1-s)d$ for $s, t \in [0,1]$, we have

$$\frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dydx \leq \frac{f(a,c) + f(a,d) + f(b,c) + f(b,d)}{4}. \tag{2.3}$$

We get the inequality (2.1) from (2.2) and (2.3). The proof is complete. □

Theorem 2.2. *Let $f : \Delta \subset \mathbb{R}^2 \to \mathbb{R}$ satisfy L-Lipschitzian conditions. That is, for (t_1, s_1) and (t_2, s_2) belong to $\Delta := [a,b] \times [c,d]$, then we have*

$$|f(t_1,s_1) - f(t_2,s_2)| \leq L_1 |t_1 - t_2| + L_2 |s_1 - s_2|$$

where L_1 and L_2 are positive constants. Then, we have the following inequalities:

$$\left| f\left(\frac{a+b}{2}, \frac{c+d}{2}\right) - \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d f(x,y)\, dydx \right| \leq \frac{1}{16}\left(M_1 |b-a| + M_2 |d-c|\right) \tag{2.4}$$

$$\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\right|$$
$$\leq \frac{1}{12}\left(M_1|b-a|+M_2|d-c|\right) \tag{2.5}$$

where $M_1=[L_1+L_3+L_5+L_7]$ *and* $M_2=[L_2+L_4+L_6+L_8]$.

Proof. Let $t,s\in[0,1]$. Since $ts+s(1-t)+t(1-s)+(1-t)(1-s)=1$, then we have

$$\begin{aligned}
&|tsf(a,c)+s(1-t)f(b,c)+t(1-s)f(a,d)+(1-t)(1-s)f(b,d)\\
&-f(ta+(1-t)b,sc+(1-s)d)|\\
=\;&|ts[f(a,c)-f(ta+(1-t)b,sc+(1-s)d)]\\
&+s(1-t)[f(b,c)-f(ta+(1-t)b,sc+(1-s)d)]\\
&+t(1-s)[f(a,d)-f(ta+(1-t)b,sc+(1-s)d)]\\
&+(1-t)(1-s)[f(b,d)-f(ta+(1-t)b,sc+(1-s)d)]|\\
\leq\;& ts[(1-t)L_1|b-a|+(1-s)L_2|d-c|]+s(1-t)[tL_3|b-a|+(1-s)L_4|d-c|]\\
&+t(1-s)[(1-t)L_5|b-a|+sL_6|d-c|]+(1-t)(1-s)[tL_7|b-a|+sL_8|d-c|]\\
=\;&(ts(1-t)[L_1+L_3]+t(1-s)(1-t)[L_5+L_7])|b-a|\\
&+(ts(1-s)[L_2+L_6]+s(1-s)(1-t)[L_4+L_8])|d-c|.
\end{aligned} \tag{2.6}$$

If we choose $t=s=\frac{1}{2}$ in (2.6), we get

$$\begin{aligned}
&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\\
\leq\;&\frac{1}{8}\left([L_1+L_3+L_5+L_7]|b-a|+[L_2+L_6+L_4+L_8]|d-c|\right).
\end{aligned} \tag{2.7}$$

Thus, if we put $ta+(1-t)b$ instead of a, $(1-t)a+tb$ instead of b, $sc+(1-s)d$ instead of c and $(1-s)c+sd$ instead of d in (2.7), respectively, then it follows that

$$\begin{aligned}
&\Bigg|\frac{f(ta+(1-t)b,sc+(1-s)d)+f(ta+(1-t)b,(1-s)c+sd)}{4}\\
&+\frac{f((1-t)a+tb,sc+(1-s)d)+f((1-t)a+tb,(1-s)c+sd)}{4}\\
&-f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\Bigg|\\
\leq\ &\frac{1}{8}\left([L_1+L_3+L_5+L_7]\,|1-2t|\,|b-a|+[L_2+L_6+L_4+L_8]\,|1-2s|\,|d-c|\right)
\end{aligned} \tag{2.8}$$

for all $t,s\in[0,1]$. If we integrate the inequality (2.8) with respect to s,t on $[0,1]\times[0,1]$

$$\begin{aligned}
&\Bigg|\frac{1}{4}\int_0^1\int_0^1\left[f(ta+(1-t)b,sc+(1-s)d)+f(ta+(1-t)b,(1-s)c+sd)\right]dsdt\\
&+\frac{1}{4}\int_0^1\int_0^1\left[f((1-t)a+tb,sc+(1-s)d)+f((1-t)a+tb,(1-s)c+sd)\right]dsdt\\
&-f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\Bigg|\\
\leq\ &\frac{1}{8}\Bigg\{[L_1+L_3+L_5+L_7]\,|b-a|\int_0^1\int_0^1|1-2t|\,dsdt\\
&+[L_2+L_6+L_4+L_8]\,|d-c|\int_0^1\int_0^1|1-2s|\,dsdt\Bigg\}.
\end{aligned}$$

Thus, using the change of the variable $x=ta+(1-t)b$, $y=(1-t)a+tb$, $u=sc+(1-s)d$ and $w=(1-s)c+sd$ for $t,s\in[0,1]$, and

$$\int_0^1\int_0^1|1-2t|\,dsdt=\int_0^1\int_0^1|1-2s|\,dsdt=\frac{1}{2}$$

we obtain the inequality (2.4).

Note that, by the inequality (2.6), we write

$$\begin{aligned}
&|tsf(a,c)+s(1-t)f(b,c)+t(1-s)f(a,d)+(1-t)(1-s)f(b,d)\\
&-f(ta+(1-t)b,sc+(1-s)d)|\\
\leq\ &(ts(1-t)[L_1+L_3]+t(1-s)(1-t)[L_5+L_7])\,|b-a|\\
&+(ts(1-s)[L_2+L_6]+s(1-s)(1-t)[L_4+L_8])\,|d-c|\,.
\end{aligned} \tag{2.9}$$

for all $t,s\in[0,1]$. If we integrate the inequality (2.9) with respect to s,t on $[0,1]\times[0,1]$, we have

$$\begin{aligned}&\left|\frac{f(a,c)+f(a,d)+f(b,c)+f(b,d)}{4}-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\right|\\ \leq\ &\frac{1}{12}\left([L_1+L_3+L_5+L_7]\,|b-a|+[L_2+L_6+L_4+L_8]\,|d-c|\right)\end{aligned}$$

and so we have the inequality (2.5), where we use the fact that

$$\int_0^1\int_0^1 st(1-t)dsdt=\int_0^1\int_0^1 s(1-s)(1-t)dsdt=\frac{1}{12}.$$

This completes the proof. □

3. The Mapping H

For a L-Lipschitzian function $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$, we can define a mapping $H:[0,1]\times[0,1]\to\mathbb{R}$ by

$$H(t,s):=\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f\left(tx+(1-t)\frac{a+b}{2},sy+(1-s)\frac{c+d}{2}\right)dydx.$$

Now, we give some properties of this mapping as follows:

Theorem 3.1. *Suppose that $f:\Delta\subset\mathbb{R}^2\to\mathbb{R}$ be L-Lipschitzian on $\Delta:=[a,b]\times[c,d]$. Then:*

(i) The mapping H is L-Lipschitzian on $[0,1]\times[0,1]$.
(ii) We have the following inequalities

$$\left|H(t,s)-f\left(\frac{a+b}{2},\frac{c+d}{2}\right)\right|\leq\frac{L_1t}{4}(b-a)+\frac{L_2s}{4}(d-c) \tag{3.1}$$

$$\left|H(t,s)-\frac{1}{(b-a)(d-c)}\int_a^b\int_c^d f(x,y)\,dydx\right|\leq\frac{L_1(1-t)}{4}(b-a)+\frac{L_2(1-s)}{4}(d-c). \tag{3.2}$$

Proof. *(i)* Let $t_1, t_2, s_1, s_2 \in [0,1]$. Then, we have

$$
\begin{aligned}
& |H(t_2,s_2) - H(t_1,s_1)| \\
= & \frac{1}{(b-a)(d-c)} \left| \int_a^b \int_c^d f\left(t_2 x + (1-t_2)\frac{a+b}{2}, s_2 y + (1-s_2)\frac{c+d}{2}\right) dydx \right. \\
& \left. - \int_a^b \int_c^d f\left(t_1 x + (1-t_1)\frac{a+b}{2}, s_1 y + (1-s_1)\frac{c+d}{2}\right) dydx \right| \\
\leq & \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left| f\left(t_2 x + (1-t_2)\frac{a+b}{2}, s_2 y + (1-s_2)\frac{c+d}{2}\right) \right. \\
& \left. - f\left(t_1 x + (1-t_1)\frac{a+b}{2}, s_1 y + (1-s_1)\frac{c+d}{2}\right) dydx \right| \\
= & \frac{1}{(b-a)(d-c)} \int_a^b \int_c^d \left[L_1 |t_2 - t_1| \left| x - \frac{a+b}{2} \right| + L_2 |s_2 - s_1| \left| y - \frac{c+d}{2} \right| \right] dydx \\
= & \frac{L_1(b-a)}{4} |t_2 - t_1| + \frac{L_2(d-c)}{4} |s_2 - s_1|,
\end{aligned}
$$

i.e., for all $t_1, t_2, s_1, s_2 \in [0,1]$,

$$
|H(t_2,s_2) - H(t_1,s_1)| \leq \frac{L_1(b-a)}{4} |t_2 - t_1| + \frac{L_2(d-c)}{4} |s_2 - s_1|, \tag{3.3}
$$

which yields that the mapping H is L-Lipschitzian on $[0,1] \times [0,1]$.

(ii) The inequalities (3.1) and (3.2) follow from (3.3) by choosing $t_1 = 0,\ t_2 = t,\ s_1 = 0,\ s_2 = s$ and $t_1 = 1,\ t_2 = t,\ s_1 = 1,\ s_2 = s$, respectively. □

Another result which is connected in a sense with the inequality (2.5) is also given in the following:

Theorem 3.2. *Under the assumptions Theorem 3.1, then we get the following inequality*

$$
\begin{aligned}
& \left| \frac{f\left(at + (1-t)\frac{a+b}{2}, cs + (1-s)\frac{c+d}{2}\right) + f\left(at + (1-t)\frac{a+b}{2}, ds + (1-s)\frac{c+d}{2}\right)}{4} \right. \\
& + \frac{f\left(bt + (1-t)\frac{a+b}{2}, cs + (1-s)\frac{c+d}{2}\right) + f\left(bt + (1-t)\frac{a+b}{2}, ds + (1-s)\frac{c+d}{2}\right)}{4} \\
& \left. - \frac{1}{(n_2-n_1)(m_2-m_1)} \int_{n_1}^{n_2} \int_{m_1}^{m_2} f(u,w) dwdu \right| \\
\leq & \frac{1}{12} \left(M_1 |n_2 - n_1| t + M_2 |m_2 - m_1| s \right)
\end{aligned} \tag{3.4}
$$

where $M_1 = [L_1 + L_3 + L_5 + L_7]$ *and* $M_2 = [L_2 + L_4 + L_6 + L_8]$.

Proof. If we denote $n_1 = at + (1-t)\frac{a+b}{2}$, $n_2 = bt + (1-t)\frac{a+b}{2}$, $m_1 = cs + (1-s)\frac{c+d}{2}$ and $m_2 = ds + (1-s)\frac{c+d}{2}$, then, we have

$$H(t,s) = \frac{1}{(n_2-n_1)(m_2-m_1)} \int_{n_1}^{n_2} \int_{m_1}^{m_2} f(u,w)dwdu.$$

Now, using the inequality (2.5) applied for n_1, n_2, m_1 and m_2, we have

$$\left| \frac{f(n_1,m_1) + f(n_1,m_2) + f(n_2,m_1) + f(n_2,m_2)}{4} - \frac{1}{(n_2-n_1)(m_2-m_1)} \int_{n_1}^{n_2} \int_{m_1}^{m_2} f(u,w)dwdu \right| \le \frac{1}{12} \left(M_1 |n_2 - n_1| + M_2 |m_2 - m_1| \right)$$

from which we have the inequality (3.4). This completes the proof. □

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