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THE RELATION BETWEEN ADDING MACHINE AND $p-$ADIC INTEGERS

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Abstract. In this paper, we equip $Aut(X^*)$ with a natural metric and give an elementary proof that the closure of the adding machine group, a subgroup of the automorphism group, is both isometric and isomorphic to the group of $p-$adic integers. This also shows that the group of $p-$adic integers can be isometrically embedded into the metric space $Aut(X^*)$.

1. Introduction

The adding machine group is one of the most important examples of self-similar automorphism groups of the rooted tree X^* ([2], [5], [7]). In this paper, we denote this group by A. A is a cyclic group generated by

$$a = (\underbrace{1, 1, \ldots, 1}_{p-1 \text{ times}}, a)\sigma$$

where a is an automorphism of the $p-$ary rooted tree and $\sigma = (012\ldots(p-1))$ is a permutation in S_p on $X = \{0, 1, 2, \ldots, (p-1)\}$. Since A is a infinite cyclic group, it is isomorphic to $\mathbb{Z}$. On the other hand, one can consider the automorphism a as adding one to a $p-$adic integer. This is a reason of the term adding machine introduced in [3]. In [6], a $p-$adic integer is pictured on a tree. This picture shows that any ultrametric space can be drawn on a tree. Moreover, in [3], the properties of $p-$adic adding machine are given in detail.

It is well-known that the closure of the group generated by the adding machine automorphism of a regular rooted tree is topologically isomorphic to the group of $p-$adic integers. In this paper, more clearly, by using a different way, we present a proof. So, we firstly equip $Aut(X^*)$ with a natural metric and prove that the group of $p-$adic integers is both isometric and isomorphic to the closure of the adding machine group which is denoted by $\overline{A}$, a subgroup of the automorphism group of the $p-$ary rooted tree. Consequently, we identify any $p-$adic integers with an element of $\overline{A}$.

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2. Preliminaries

The following definitions and notions are given in [4], [8] and [9].
$p-$adic integers: A $p-$adic integer is a formal series

$$\sum_{i\geq 0} x_i p^i$$

for each $x_i \in \{0,1,2,\ldots,(p-1)\}$ and the set of all $p-$adic integers is denoted by $\mathbb{Z}_p$ ([8]).

Suppose that $x = \sum_{i\geq 0} x_i p^i$ and $y = \sum_{i\geq 0} y_i p^i$ be elements of $\mathbb{Z}_p$. Then, the addition $z = \sum_{i\geq 0} z_i p^i$ of x and y is defined by

$$\sum_{i=0}^{m} z_i p^i \equiv \sum_{i=0}^{m} (x_i + y_i) p^i \pmod{p^{m+1}} \tag{2.1}$$

for each $m \in \{0,1,2,\ldots\}$ where $z_i \in \{0,1,\ldots,(p-1)\}$. If $x = \sum_{i\geq 0} x_i p^i$ is an element of $\mathbb{Z}_p$, then $-x = \sigma(x) + 1$ is the inverse of x where

$$\sigma(x) = \sum_{i\geq 0} (p - 1 - x_i) p^i.$$

$\mathbb{Z}_p$ is a group with this operation and is called the group of $p-$adic integers.

Let $x = \sum_{i\geq 0} x_i p^i$ be an element of $\mathbb{Z}_p$ and let $x \neq 0$. Thus, there is a first index $v(x) \geq 0$ such that $x_v \neq 0$. This index is called the order of x and is denoted by $ord_p(x)$. If $ord_p(x) = \infty$, then $x_i = 0$ for $i = 0,1,2,\ldots$. On the other hand, the $p-$adic value of x is denoted by

$$|x|_p = \begin{cases} 0 & \text{if } x_i = 0 \text{ for } i = 0,1,2,\ldots, \\ p^{-ord_p(x)} & \text{otherwise} \end{cases}$$

and induces the metric $d_p(x,y) = |x-y|_p$ for $x, y \in \mathbb{Z}_p$ ([8]).

A $p-$adic number is a formal series

$$\sum_{i=-\infty}^{\infty} a_i p^i$$

where $a_i \in \{0,1,2,\ldots,(p-1)\}$ for each $i \in \mathbb{Z}$ and $a_{-i} = 0$ for large i. The set of all $p-$adic numbers is denoted by $\mathbb{Q}_p$. Addition in $\mathbb{Z}_p$ which is defined by equation (2.1) can be naturally extended to $\mathbb{Q}_p$. Hence, $\mathbb{Q}_p$ is a group. Moreover, $\mathbb{Q}_p$ is the metric completion of $\mathbb{Q}$ with respect to the $p-$adic metric. It is easily seen that the group of $p-$adic numbers is a topological group. Moreover, the group of $p-$adic integers is expressed as

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p | \ |x|_p \leq 1\}$$

and is an important subgroup of $\mathbb{Q}_p$.

The following definitions and notions are given in [2], [3], [5] and [7].
The automorphism group of the rooted tree: Let X be a finite set (alphabet) and let

$$X^* = \{x_1 x_2 \ldots x_n \mid x_i \in X, n \geqslant 0 \}$$

be the set of all finite words over the alphabet X, including the empty word $\emptyset$. In other terms, X^* is the free monoid generated by X ([7]). The length of a word $v = x_1 x_2 \ldots x_n \in X^*$ is the number of its letters and is denoted by $|v|$. The product of $v_1, v_2 \in X^*$ is naturally defined by concatenation $v_1 v_2$. One can think of X^* as vertex set of a rooted tree.

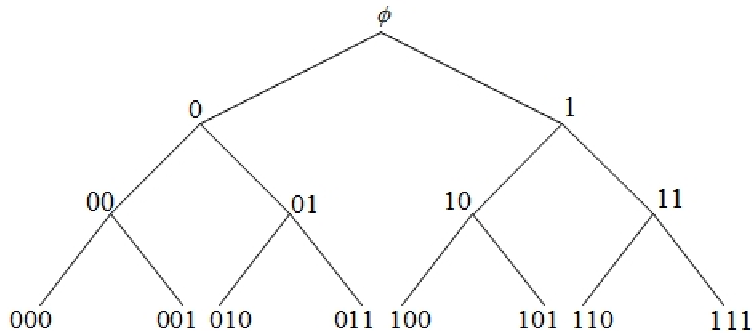

Figure 1. The first three levels of the binary rooted tree X^* for $X = \{0,1\}$.

The set $X^n = \{v \in X^* \mid |v| = n\}$ is called the nth level of X^*. The empty word $\emptyset$ is the root of the tree X^*. Two words are connected by an edge if and only if they are of the form v, vx where $v \in X^*$ and $x \in X$.

A map $f : X^* \to X^*$ is an endomorphism of the tree X^* if it preserves the root and adjacency of the vertices. An automorphism is a bijective endomorphism. The group of all automorphisms of the tree X^* is denoted by $Aut(X^*)$.

If G is a subgroup of the automorphism group $Aut(X^*)$ of the rooted tree X^*, then for $v \in X^*$, the subgroup

$$G_v = \{g \in G \mid g(v) = v\}$$

is called the vertex stabilizer where $g(v)$ is the image of v under the action of g. The nth level stabilizer is the subgroup

$$St_G(n) = \bigcap_{v \in X^n} G_v.$$

We need a useful way to express the automorphisms the rooted tree X^* and to perform computations with them. For this aim, we give a definition and a proposition from [7].

Definition 2.1 ([7])**.** Let H be a group acting (from the right) by permutations on a set X and let G be an arbitrary group. Then the (permutational) wreath product $G \wr H$ is the semi-direct product $G^X \rtimes H$, where H acts on the direct power G^X by the respective permutations of the direct factors.

If $|X| = d$, then the elements of the wreath product are given by the forms $(g_1, g_2, \ldots, g_d)h$ for $g_i \in G$ and $h \in H$. The multiplication in the wreath product is given by

$$(g_1, g_2, \ldots, g_d)\alpha(h_1, h_2, \ldots, h_d)\beta = (g_1 h_{\alpha(1)}, g_2 h_{\alpha(2)}, \ldots, g_d h_{\alpha(d)})\alpha\beta$$

where $g_i, h_i \in G, \alpha, \beta \in H$ and $\alpha(i)$ is the image of i under the action of α.

Let $g : X^* \to X^*$ be an endomorphism of the rooted tree X^*. Then, $g : vX^* \to g(v)X^*$ is a morphism of the rooted trees where $v \in X^*$. The subtrees vX^* and $g(v)X^*$ are naturally isomorphic to the whole tree X^*. Identifying vX^* and $g(v)X^*$ with X^* we get an endomorphism $g|_v : X^* \to X^*$. It is uniquely determined by the condition

$$g(vw) = g(v)g|_v(w).$$

We call the endomorphism $g|_v$ the restriction of g in v (for details see [7]).

Proposition 2.1 ([7]). *Denote by $S(X)$ the symmetric group of all permutations of X. Fix some indexing $\{x_1, x_2, \ldots, x_d\}$ of X. Then we have an isomorphism*

$$\psi : Aut(X^*) \to Aut(X^*) \wr S(X),$$

given by

$$\psi(g) = (g|_{x_1}, g|_{x_2}, \ldots, g|_{x_d})\alpha,$$

where α is the permutation equal to the action of g on $X \subset X^*$.

Thus, $g \in Aut(X^*)$ is identified with the image $\psi(g) \in Aut(X^*) \wr S(X)$ and it is written as

$$g = (g|_{x_1}, g|_{x_2}, \ldots, g|_{x_d})\alpha.$$

The adding machine group: Let a be the transformation on X^* defined by the wreath recursion

$$a = (\underbrace{1, 1, \ldots, 1}_{p-1 \text{ times}}, a)\sigma$$

where $\sigma = (012\ldots(p-1))$ is a permutation in S_p on $X = \{0, 1, 2, \ldots, (p-1)\}$.

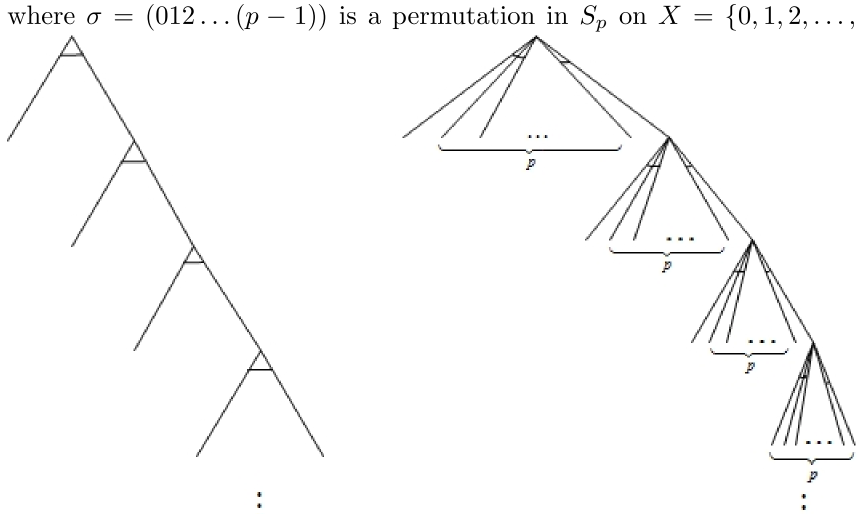

Figure 2. Portrait of the transformation a for $X = \{0, 1\}$ and $X = \{0, 1, \ldots, p-1\}$

The transformation a generates an infinite cyclic group on X^*. This group is called the adding machine group and we denote this group by A. For example, using permutational wreath product we obtain that

$$\begin{aligned} a^p &= (1, \ldots, 1, a)\sigma(1, \ldots, 1, a)\sigma \ldots (1, \ldots, 1, a)\sigma \\ &= (a, a, \ldots, a)\sigma^p \\ &= (a, a, \ldots, a) \end{aligned}$$

(for details see [2], [7]).

The Metric Space $(Aut(X^*), d)$: In the following definition, we equip the automorphism group of the $p-$ary rooted tree X^* with a natural metric where $X = \{0, 1, 2, \ldots, p-1\}$. This metric is also used in [1].

Definition 2.2. The metric function $d : Aut(X^*) \times Aut(X^*) \to \mathbb{R}$ can be defined by

$$d(g_1, g_2) = \begin{cases} \frac{1}{p^k} & \text{for } g_1^{-1}g_2 \in St_{Aut(X^*)}(k) \text{ and } g_1^{-1}g_2 \notin St_{Aut(X^*)}(k+1), \\ 0 & \text{for } g_1 = g_2 \end{cases}$$

where $g_1, g_2 \in Aut(X^*)$. In other words, if g_1 and g_2 agree on all vertices of the level k but do not agree at least one vertex of the level $(k+1)$ of the tree X^*, then the distance between g_1 and g_2 is $\frac{1}{p^k}$.

$(Aut(X^*), d)$ is a compact metric space and is a topological group. It is obvious that $\overline{A}$, the closure of A, is a subgroup of $Aut(X^*)$.

3. An Isometry between the Group of $p-$adic Integers and the Closure of Adding machine group

Now we give a formula for the distance between two elements of the adding machine group. Notice that this expression is similar to the distance between two $p-$adic integers.

Proposition 3.1. *For $a^n, a^m \in A$, the distance $d(a^n, a^m)$ can be defined by*

$$\begin{array}{rcl} d \;:\; A \times A & \to & A \\ (a^n, a^m) & \mapsto & d(a^n, a^m) = \begin{cases} 0 & \text{for } n = m, \\ \frac{1}{p^k} & \text{for } n - m = tp^k \end{cases} \end{array}$$

where $t, k \in \mathbb{Z}$, p is prime number and $(p, t) = 1$.

Proof. First we compute $St_A(1)$. Using permutational wreath product we obtain that

$$\begin{array}{rcl} a^p & = & (1,1,\ldots,a)\sigma(1,1,\ldots,a)\sigma\ldots(1,1,\ldots,a)\sigma \\ & = & (a,a,\ldots,a). \end{array}$$

This shows that $St_A(1) = \langle a^p \rangle$. Moreover, we get

$$\begin{array}{rcl} a^{p^2} & = & a^p a^p \ldots a^p \\ & = & (a,a,\ldots,a)(a,a,\ldots,a)\ldots(a,a,\ldots,a) \\ & = & (a^p, a^p, \ldots, a^p) \end{array}$$

We have $a^{p^2} \in St_A(2)$ because $a^p \in St_A(1)$. Therefore, it is obtained that $St_A(2) = \langle a^{p^2} \rangle$. By proceeding in a similar manner, we compute $St_A(k) = \langle a^{p^k} \rangle$.

So, elements of A which are in $St_A(1)$ but are not in $St_A(2)$ can be expressed as

$$St_A(1) - St_A(2) = \{a^{tp} : (p,t) = 1\}$$

and by using the induction method, it is easily seen that

$$St_A(k) - St_A(k+1) = \{a^{tp^k} : (p,t) = 1\}.$$

Let us take arbitrary $a^n, a^m \in A$. If $n = m$, then it is $a^n = a^m$ and $d(a^n, a^m) = 0$. If $n \neq m$, then there exists a unique expression $n - m = tp^k$ such that $(p,t) = 1$. Then we obtain

$$a^{-m}a^n = a^{n-m} = a^{tp^k} \in St_A(k) - St_A(k+1)$$

and thus it is $d(a^n, a^m) = \frac{1}{p^k}$. □

Proposition 3.2. *Let* $\sum_{i\geq 0}\alpha_i p^i \in \mathbb{Z}_p$. *Then, the sequence*

$$a^{\alpha_0}, a^{\alpha_0+\alpha_1 p}, a^{\alpha_0+\alpha_1 p+\alpha_2 p^2}, \ldots$$

is convergent.

Proof. For any $\varepsilon > 0$, there is a positive integer n_0 such that $\frac{1}{p^{n_0}} < \varepsilon$. If $k > l$ and $k, l \geq n_0$, then it is obtained that

$$d(a^{\alpha_0+\alpha_1 p+\ldots+\alpha_k p^k}, a^{\alpha_0+\alpha_1 p+\ldots+\alpha_l p^l}) = \frac{1}{p^l} < \varepsilon$$

from Proposition 3.1. Thus, it is a Cauchy sequence. Since $Aut(X^*)$ is a complete metric space, this sequence is convergent. □

Now we give our main proposition:

Proposition 3.3. *We define*

$$\varphi \ : \ \mathbb{Z}_p \ \rightarrow \ \overline{A}$$

such that $\varphi(\sum_{i\geq 0}\alpha_i p^i)$ *is the limit of the sequence* $a^{\alpha_0}, a^{\alpha_0+\alpha_1 p}, a^{\alpha_0+\alpha_1 p+\alpha_2 p^2}, \ldots$. *Then,* φ *is both an isometry and a group isomorphism.*

Proof. From Proposition 3.2, φ is well-defined. Now we show that φ is an isometry. In other words, we show that $d_p(\alpha, \beta) = d(\varphi(\alpha), \varphi(\beta))$ for every $\alpha, \beta \in \mathbb{Z}_p$. Let $\alpha = \sum_{i\geq 0}\alpha_i p^i$ and $\beta = \sum_{i\geq 0}\beta_i p^i$.

If $d_p(\alpha, \beta) = 0$, then we obtain $d(\varphi(\alpha), \varphi(\beta)) = 0$ since $\alpha_i = \beta_i$ for $i = 0, 1, 2, \ldots$.

If $d_p(\alpha, \beta) = \frac{1}{p^k}$, then $\alpha_i = \beta_i$ for $i < k$ and $\alpha_k \neq \beta_k$. We must show that $d(\varphi(\alpha), \varphi(\beta)) = \frac{1}{p^k}$. Because $\varphi(\alpha)$ and $\varphi(\beta)$ are the limits of the sequences

$$a^{\alpha_0}, a^{\alpha_0+\alpha_1 p}, a^{\alpha_0+\alpha_1 p+\alpha_2 p^2}, \ldots \quad \text{and} \quad a^{\beta_0}, a^{\beta_0+\beta_1 p}, a^{\beta_0+\beta_1 p+\beta_2 p^2}, \ldots$$

respectively, it is written the equality

$$\lim_{k\rightarrow\infty}(a^{\alpha_0+\alpha_1 p+\ldots+\alpha_k p^k}, a^{\beta_0+\beta_1 p+\ldots+\beta_k p^k}) = (\varphi(\alpha), \varphi(\beta)).$$

Since any metric function is continuous, we obtain that

$$d(a^{\alpha_0}, a^{\beta_0}), d(a^{\alpha_0+\alpha_1 p}, a^{\beta_0+\beta_1 p}), \ldots \rightarrow d(\varphi(\alpha), \varphi(\beta)).$$

From Proposition 3.1, we get

$$0, 0, ..., 0, \frac{1}{p^k}, \frac{1}{p^k}, \ldots, \frac{1}{p^k}, \ldots \rightarrow \frac{1}{p^k}.$$

This shows that $d(\varphi(\alpha), \varphi(\beta)) = \frac{1}{p^k}$. Namely, φ is an isometry map.

Moreover, φ is injective since φ is an isometry map.

Now we show that φ is surjective. Let b be an arbitrary element of $\overline{A}$. Thus, there exists a sequence

$$a^{n_0}, a^{n_1}, \ldots, a^{n_k}, \ldots \rightarrow b$$

whose elements are in A. Furthermore, every integer n_k can be expressed in $\mathbb{Z}_p$ as

$$\begin{aligned} n_0 &= \alpha_0^0 + \alpha_1^0 p + \alpha_2^0 p^2 + \ldots \\ n_1 &= \alpha_0^1 + \alpha_1^1 p + \alpha_2^1 p^2 + \ldots \\ &\vdots \\ n_k &= \alpha_0^k + \alpha_1^k p + \alpha_2^k p^2 + \ldots \\ &\vdots \end{aligned} \tag{3.1}$$

At least one of the numbers $0, 1, 2, ..., (p-1)$ occurs infinitely many times in the sequence $(\alpha_0^k)_k$. We choose one of them and denote it by β_0. Let $(\alpha_1^{k_l})_l$ be a subsequence of $(\alpha_1^k)_k$ such that $\alpha_0^{k_l} = \beta_0$ for $l = 0, 1, 2, \ldots$. Similarly, we denote by β_1, any one of the numbers that appears infinitely many times in the sequence $(\alpha_1^{k_l})_l$. Proceeding in this manner, we obtain a sequence

$$a^{\beta_0}, a^{\beta_0+\beta_1 p}, \ldots, a^{\beta_0+\beta_1 p+\ldots+\beta_k p^k}, \ldots.$$

From Proposition 3.2, this sequence is convergent. Now we show this sequence converges to b. Due to the construction of (3.1), there exists a subsequence (n_{k_s}) of the sequence (n_k) whose $p-$adic expression of term sth such that

$$\beta_0 + \beta_1 p + \beta_2 p^2 + \ldots + \beta_s p^s + \gamma_{s+1} p^{s+1} + \gamma_{s+2} p^{s+2} + \ldots.$$

Owing to the fact that

$$\lim_{s\to\infty} d(a^{\beta_0+\beta_1 p+\ldots+\beta_s p^s}, a^{n_{k_s}}) = 0$$

and from the triangle inequality, the sequence $(a^{\beta_0+\beta_1 p+\ldots+\beta_k p^k})$ converges to b. This shows that $\varphi(\sum_{i\geq 0} \beta_i p^i) = b$ and hence φ is surjective.

Finally, we prove that φ is a homomorphism. In other words, we prove that

$$\varphi(\alpha+\beta) = \varphi(\alpha)\varphi(\beta)$$

for every $\alpha, \beta \in \mathbb{Z}_p$. Let

$$\alpha = \alpha_0 + \alpha_1 p + \alpha_2 p^2 + \ldots,$$
$$\beta = \beta_0 + \beta_1 p + \beta_2 p^2 + \ldots$$

and

$$\alpha + \beta = \gamma_0 + \gamma_1 p + \gamma_2 p^2 + \ldots.$$

From the definition of φ, we have

$$a^{\gamma_0}, a^{\gamma_0+\gamma_1 p}, a^{\gamma_0+\gamma_1 p+\gamma_2 p^2}, \ldots \to \varphi(\alpha+\beta).$$

Moreover, it follows that

$$a^{(\alpha_0+\beta_0)}, a^{(\alpha_0+\beta_0)+(\alpha_1+\beta_1)p}, a^{(\alpha_0+\beta_0)+(\alpha_1+\beta_1)p+(\alpha_2+\beta_2)p^2}, \ldots \to \varphi(\alpha)\varphi(\beta)$$

due to the fact that $Aut(X^*)$ is a topological group,

$$a^{\alpha_0}, a^{\alpha_0+\alpha_1 p}, a^{\alpha_0+\alpha_1 p+\alpha_2 p^2}, \ldots \to \varphi(\alpha)$$

and

$$a^{\beta_0}, a^{\beta_0+\beta_1 p}, a^{\beta_0+\beta_1 p+\beta_2 p^2}, \ldots \to \varphi(\beta).$$

In $\mathbb{Z}_p$, we have

$$\begin{array}{rcl} \alpha_0 + \beta_0 & = & \gamma_0 + \overline{\gamma_0} p + 0p^2 + 0p^3 + 0p^4 + \ldots \\ (\alpha_0+\beta_0) + (\alpha_1+\beta_1)p & = & \gamma_0 + \gamma_1 p + \overline{\gamma_1} p^2 + 0p^3 + 0p^4 + 0p^5 + \ldots \\ & \vdots & \\ (\alpha_0+\beta_0) + \ldots + (\alpha_k+\beta_k)p^k & = & \gamma_0 + \gamma_1 p + \ldots + \gamma_k p^k + \overline{\gamma_k} p^{k+1} + 0p^{k+2} \\ & & + 0p^{k+3} + +0p^{k+4} + \ldots. \\ & \vdots & \end{array}$$

Let

$$x = (\alpha_0+\beta_0) + \ldots + (\alpha_k+\beta_k)p^k$$

and

$$y = \gamma_0 + \gamma_1 p + \ldots + \gamma_k p^k + \overline{\gamma_k} p^{k+1} + 0p^{k+2} + 0p^{k+3} + \ldots.$$

Then, we have

$$d(a^x, a^y) = \begin{cases} \frac{1}{p^k} & \text{if } \overline{\gamma_k} \neq 0, \\ 0 & \text{if } \overline{\gamma_k} = 0. \end{cases}$$

It follows that $\varphi(\alpha + \beta) = \varphi(\alpha)\varphi(\beta)$ since

$$d(a^{\alpha_0+\beta_0}, a^{\gamma_0}), d(a^{\alpha_0+\beta_0+(\alpha_1+\beta_1)p}, a^{\gamma_0+\gamma_1 p}), \ldots \to d(\varphi(\alpha)\varphi(\beta), \varphi(\alpha+\beta))$$

and

$$\lim_{k\to\infty} d(a^x, a^y) = 0.$$

Hence, the proof is completed. □

Consequently, the group of $p-$adic integers $\mathbb{Z}_p$ can be isometrically embedded into the metric space $Aut(X^*)$ since $\overline{A} \subseteq Aut(X^*)$.

Example 3.1. We show $\varphi(-1)$ for $p = 2$ in Figure **??**. It is well-known that

$$-1 = 1 + 1.2^1 + 1.2^2 + \ldots + 1.2^k + \ldots \in \mathbb{Z}_2.$$

Due to the definition of φ, $\varphi(-1)$ is the limit of the sequence

$$a^1, a^{1+1.2^1}, a^{1+1.2^1+1.2^2}, \ldots$$

in A for $X = \{0, 1\}$. This limit equals to $a^{-1} = (a^{-1}, 1)\sigma$ because of Proposition 3.1.

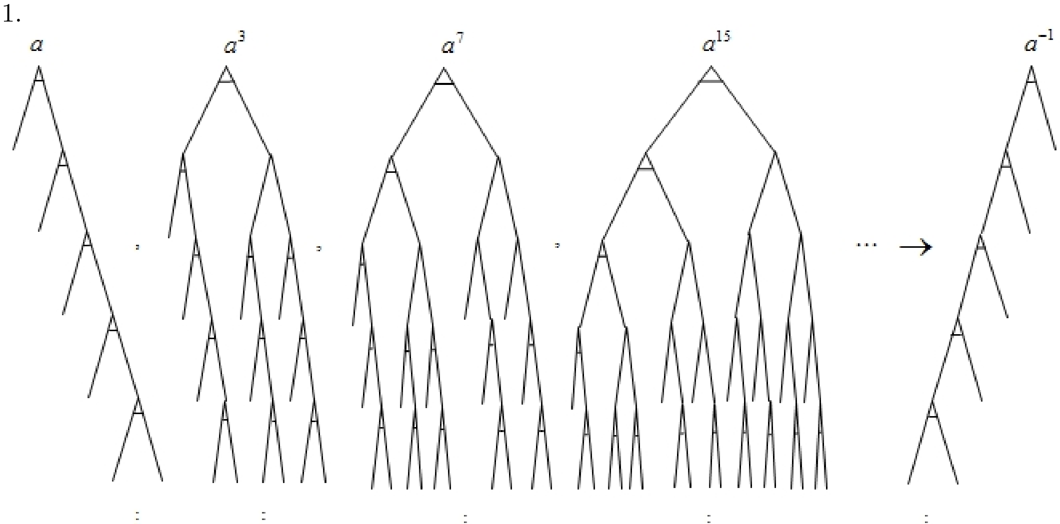

Figure 3. The image of $-1 \in Z_2$ under the map φ.

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