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GENERALIZATION OF DIFFERENT TYPE INTEGRAL INEQUALITIES VIA FRACTIONAL INTEGRALS FOR FUNCTIONS WHOSE SECOND DERIVATIVES ABSOLUTE VALUES ARE QUASI-CONVEX

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Abstract. In this paper, the author establish some new estimates on Hermite-Hadamard type and Simpson type inequalities via Riemann Liouville fractional integral for functions whose second derivatives in absolute values at certain power are quasi-convex.

1. Introduction

The following definition for convex functions is well known in the mathematical literature:

Definition 1.1. A function $f:[a,b]\subset\mathbb{R}\to\mathbb{R}$ is said to be convex on $[a,b]$ if

$$f(tx+(1-t)y)\leq tf(x)+(1-t)f(y)$$

for all $x,y\in[a,b]$ and $t\in[0,1]$.

Many inequalities have been established for convex functions but the most famous is the Hermite-Hadamard inequality, due to its rich geometrical significance and applications, which is stated as follow:

Let $f:I\subseteq\mathbb{R}\to\mathbb{R}$ be a convex function defined on the interval I of real numbers and $a,b\in I$ with $a<b$. Then

$$f\left(\frac{a+b}{2}\right)\leq\frac{1}{b-a}\int_a^b f(x)dx\leq\frac{f(a)+f(b)}{2}. \tag{1.1}$$

The following inequality is one of the best-known results in the literature as Simpson's inequality:

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Theorem 1.1. *Let $f:[a,b]\rightarrow\mathbb{R}$ be a four times continuously differentiable mapping on (a,b) and $\left\|f^{(4)}\right\|_{\infty}=\sup\limits_{x\in(a,b)}\left|f^{(4)}(x)\right|<\infty$. Then the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2}+2f\left(\frac{a+b}{2}\right)\right]-\frac{1}{b-a}\int_{a}^{b}f(x)dx\right|\leq\frac{1}{2880}\left\|f^{(4)}\right\|_{\infty}(b-a)^{4}.$$

In recent years many authors were established an error estimations for both the Hermite-Hadamard inequality and the Simpson's inequality, for refinements, counterparts, generalizations and new inequalities for them see [2, 3, 4, 6, 7, 8, 10, 11].

We recall that the notion of quasi-convex function generalizes the notion of convex function. More exactly, a function $f:[a,b]\subset\mathbb{R}\rightarrow\mathbb{R}$ is said to be quasi-convex on $[a,b]$ if

$$f(tx+(1-t)y)\leq max\{f(x),f(y)\}$$

for all $x,y\in[a,b]$ and $t\in[0,1]$. Clearly, any convex function is a quasi-convex function. Furthermore, there exist quasi-convex functions which are not convex ([5]).

We give some necessary definitions and mathematical preliminaries of fractional calculus theory which are used throughout this paper.

Definition 1.2. Let $f\in L[a,b]$. The Riemann-Liouville integrals $J_{a^{+}}^{\alpha}f$ and $J_{b^{-}}^{\alpha}f$ of oder $\alpha>0$ with $a\geq0$ are defined by

$$J_{a^{+}}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_{a}^{x}(x-t)^{\alpha-1}f(t)dt,\ x>a$$

and

$$J_{b^{-}}^{\alpha}f(x)=\frac{1}{\Gamma(\alpha)}\int_{x}^{b}(t-x)^{\alpha-1}f(t)dt,\ x<b$$

respectively, where $\Gamma(\alpha)$ is the Gamma function defined by $\Gamma(\alpha)=\int_{0}^{\infty}e^{-t}t^{\alpha-1}dt$ and $J_{a^{+}}^{0}f(x)=J_{b^{-}}^{0}f(x)=f(x)$.

In the case of $\alpha=1$, the fractional integral reduces to the classical integral. For some recent results connected with fractional integral ineqalities, see [9, 10, 11, 12, 13].

In [4], Barani et al. obtained the following theorems related to the right-hand side of (1.1) for functions whose second derivatives in absolute values at certain power are quasiconvex.

Theorem 1.2. *Let $f:I\rightarrow\mathbb{R}$ be a twice differentiable function on I° such that $f''\in L[a,b]$, where $a,b\in I^{\circ}$ with $a<b$. If $|f''|^{q}$ is quasi-convex on $[a,b]$ for $q\geq1$,*

then the following inequality holds

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{(b-a)}\int_a^b f(x)dx\right| \tag{1.2}$$
$$\leq \frac{(b-a)^2}{24}\left\{\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q, \left|f''(a)\right|^q\right\}\right)^{\frac{1}{q}} + \left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q, \left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}}\right\}.$$

Theorem 1.3. *Let $f: I \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L[a,b]$, where $a,b \in I^\circ$ with $a<b$. If $|f''|^{p/p-1}$ is quasi-convex on $[a,b]$, for $p>1$, then the following inequality holds*

$$\left|\frac{f(a)+f(b)}{2}-\frac{1}{(b-a)}\int_a^b f(x)dx\right| \tag{1.3}$$
$$\leq \frac{(b-a)^2}{16}\left(\frac{\sqrt{\pi}}{2}\right)^{\frac{1}{p}}\left(\frac{\Gamma(1+p)}{\Gamma\left(\frac{3}{2}+p\right)}\right)^{\frac{1}{p}}\left\{\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q, \left|f''(a)\right|^q\right\}\right)^{\frac{1}{q}} + \left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q, \left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}}\right\},$$

where $1/p+1/q=1$.

In [2], Alomari et al. established the following result connected with Simpson's type inequalities for twice differentiable functions:

Theorem 1.4. *Let $f': I \subset [0,\infty) \to \mathbb{R}$ be an absolutely continuous function on I° and $a,b \in I^\circ$ with $a<b$, such that $f'' \in L[a,b]$. If $|f''|^q$ is quasi-convex on $[a,b]$, $q \geq 1$, then the following inequality holds*

$$\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{1}{(b-a)}\int_a^b f(x)dx\right| \tag{1.4}$$
$$\leq \frac{(b-a)^2}{162}\left\{\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q, \left|f''(a)\right|^q\right\}\right)^{\frac{1}{q}} + \left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q, \left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}}\right\}.$$

In [10], Sarıkaya et al. established some results connected with the left-hand side of (1.1) as follows:

Theorem 1.5. *Let $f: I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L[a,b]$, where $a,b \in I^\circ$ with $a<b$. If $|f''|^q$ is quasi-convex on $[a,b]$ for $q \geq 1$, then the following inequality holds:*

$$\left|f\left(\frac{a+b}{2}\right)-\frac{1}{(b-a)}\int_a^b f(x)dx\right| \leq \frac{(b-a)^2}{24}\left(\max\left\{\left|f''(a)\right|^q, \left|f''(a)\right|^q\right\}\right)^{\frac{1}{q}}. \tag{1.5}$$

Theorem 1.6. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L[a,b]$, where $a, b \in I^{\circ}$ with $a < b$. If $|f''|^{q}$ is quasi-convex on $[a,b]$ for $q > 1$, then the following inequality holds:*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)} \int_{a}^{b} f(x)dx \right| \leq \frac{(b-a)^{2}}{8\left(2p+1\right)^{\frac{1}{p}}} \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right)\right|^{q}, \left|f''(a)\right|^{q} \right\} \right)^{\frac{1}{q}}, \tag{1.6}$$

where $1/p + 1/q = 1$.

We will establish some new results using the following Lemma:

Lemma 1.1. *Let $f : I \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L[a,b]$, where $a, b \in I$ with $a < b$. Then for all $x \in [a,b]$, $\lambda \in [0,1]$ and $\alpha > 0$ we have:*

$$\begin{aligned} & (1-\lambda)\left[\frac{(x-a)^{\alpha} + (b-x)^{\alpha}}{b-a}\right] f(x) + \lambda \left[\frac{(x-a)^{\alpha} f(a) + (b-x)^{\alpha} f(b)}{b-a}\right] \\ & + \left(\frac{1}{\alpha+1} - \lambda\right)\left[\frac{(b-x)^{\alpha+1} - (x-a)^{\alpha+1}}{b-a}\right] f'(x) - \frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^{\alpha} f(a) + J_{x^+}^{\alpha} f(b)\right] \\ = & \frac{(x-a)^{\alpha+2}}{(\alpha+1)(b-a)} \int_{0}^{1} t\left((\alpha+1)\lambda - t^{\alpha}\right) f''\left(tx + (1-t)a\right) dt \\ & + \frac{(b-x)^{\alpha+2}}{(\alpha+1)(b-a)} \int_{0}^{1} t\left((\alpha+1)\lambda - t^{\alpha}\right) f''\left(tx + (1-t)b\right) dt. \end{aligned} \tag{1.7}$$

A simple proof of equality can be given by performing an twice integration by parts in the integrals from the right side and changing the variable (see [6]).

The main aim of this article is to establish a generalization of Hermite Hadamard-type and Simpson-type inequalities via fractional integrals for functions whose absolute values of second derivatives are quasi-convex. By using the integral equality (1.7), the author establish some new inequalities of the Simpson-like and the Hermite-Hadamard-like type for these functions.

2. Main Results

Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable function on I°, the interior of I, throughout this section we will take

$$\begin{aligned} & S_{f}\left(x, \lambda, \alpha; a, b\right) \\ = & (1-\lambda)\left[\frac{(x-a)^{\alpha} + (b-x)^{\alpha}}{b-a}\right] f(x) + \lambda \left[\frac{(x-a)^{\alpha} f(a) + (b-x)^{\alpha} f(b)}{b-a}\right] \\ & + \left(\frac{1}{\alpha+1} - \lambda\right)\left[\frac{(b-x)^{\alpha+1} - (x-a)^{\alpha+1}}{b-a}\right] f'(x) - \frac{\Gamma(\alpha+1)}{b-a}\left[J_{x^-}^{\alpha} f(a) + J_{x^+}^{\alpha} f(b)\right] \end{aligned}$$

where $a, b \in I$ with $a < b$, $x \in [a,b]$, $\lambda \in [0,1]$, $\alpha > 0$ and Γ is Euler Gamma function.

Theorem 2.1. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be a twice differentiable function on I° such that $f'' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$. If $|f''|^q$ is quasi-convex on $[a,b]$ for some fixed $q \geq 1$, then for $x \in [a,b]$, $\lambda \in [0,1]$ and $\alpha > 0$, the following inequality for fractional integrals holds*

$$
\begin{aligned}
&|S_f(x,\lambda,\alpha;a,b)| \\
\leq\ & C_1(\alpha,\lambda)\left\{\frac{(x-a)^{\alpha+2}}{(\alpha+1)(b-a)}\left(\max\left\{|f''(x)|^q, |f''(a)|^q\right\}\right)^{\frac{1}{q}}\right. \\
&\left.+\frac{(b-x)^{\alpha+2}}{(\alpha+1)(b-a)}\left(\max\left\{|f''(x)|^q, |f''(b)|^q\right\}\right)^{\frac{1}{q}}\right\},
\end{aligned}
\tag{2.1}
$$

where

$$
C_1(\alpha,\lambda) = \left\{\begin{array}{ll} \frac{\alpha[(\alpha+1)\lambda]^{\frac{\alpha+2}{\alpha}}+1}{\alpha+2} - \frac{(\alpha+1)\lambda}{2}, & 0 \leq \lambda \leq \frac{1}{\alpha+1} \\ \frac{(\alpha+1)(\alpha+2)\lambda-2}{2(\alpha+2)}, & \frac{1}{\alpha+1} < \lambda \leq 1 \end{array}\right. .
$$

Proof. From Lemma 1.1, property of the modulus and using the power-mean inequality we have

$$
\begin{aligned}
|S_f(x,\lambda,\alpha;a,b)| &\leq \frac{(x-a)^{\alpha+2}}{(\alpha+1)(b-a)}\int_0^1 |t|\,|(\alpha+1)\lambda - t^\alpha|\,|f''(tx+(1-t)a)|\,dt \\
&+\frac{(b-x)^{\alpha+2}}{(\alpha+1)(b-a)}\int_0^1 |t|\,|(\alpha+1)\lambda - t^\alpha|\,|f''(tx+(1-t)b)|\,dt \\
\leq\ & \frac{(x-a)^{\alpha+2}}{(\alpha+1)(b-a)}\left(\int_0^1 t\,|(\alpha+1)\lambda - t^\alpha|\,dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 t\,|(\alpha+1)\lambda - t^\alpha|\,|f''(tx+(1-t)a)|^q\,dt\right)^{\frac{1}{q}} \\
&+\frac{(b-x)^{\alpha+2}}{(\alpha+1)(b-a)}\left(\int_0^1 t\,|t(\alpha+1)\lambda - t^\alpha|\,dt\right)^{1-\frac{1}{q}} \\
&\times\left(\int_0^1 t\,|(\alpha+1)\lambda - t^\alpha|\,|f''(tx+(1-t)b)|^q\,dt\right)^{\frac{1}{q}}.
\end{aligned}
\tag{2.2}
$$

Since $|f''|^q$ is quasi-convex on $[a,b]$ we get

$$
\begin{aligned}
\int_0^1 t\,|(\alpha+1)\lambda - t^\alpha|\,|f''(tx+(1-t)a)|^q\,dt &\leq \int_0^1 t\,|(\alpha+1)\lambda - t^\alpha|\max\left\{|f''(x)|^q, |f''(a)|^q\right\}dt \\
&= \max\left\{|f''(x)|^q, |f''(a)|^q\right\}
\end{aligned}
\tag{2.3}
$$

$$\int_0^1 t\left|(\alpha+1)\lambda-t^{\alpha}\right|\left|f''(tx+(1-t)b)\right|^q dt \le \int_0^1 t\left|(\alpha+1)\lambda-t^{\alpha}\right|\max\left\{\left|f''(x)\right|^q,\left|f''(b)\right|^q\right\}dt$$

$$(2.4)\qquad = \max\left\{\left|f''(x)\right|^q,\left|f''(a)\right|^q\right\},$$

where we use the fact that

$$(2.5)\qquad C_1(\alpha,\lambda)=\int_0^1 t\left|(\alpha+1)\lambda-t^{\alpha}\right|dt$$

$$=\begin{cases}\begin{array}{c}(\alpha+1)\lambda\int_0^{[(\alpha+1)\lambda]^{\frac{1}{\alpha}}} tdt-\int_0^{[(\alpha+1)\lambda]^{\frac{1}{\alpha}}} t^{\alpha+1}dt\\ -(\alpha+1)\lambda\int_{[(\alpha+1)\lambda]^{\frac{1}{\alpha}}}^1 tdt+\int_{[(\alpha+1)\lambda]^{\frac{1}{\alpha}}}^1 t^{\alpha+1}dt\end{array}, & 0\le\lambda\le\frac{1}{\alpha+1}\\ (\alpha+1)\lambda\int_0^1 tdt-\int_0^1 t^{\alpha+1}dt, & \frac{1}{\alpha+1}<\lambda\le 1\end{cases}$$

$$=\begin{cases}\frac{\alpha[(\alpha+1)\lambda]^{\frac{\alpha+2}{\alpha}}+1}{\alpha+2}-\frac{(\alpha+1)\lambda}{2}, & 0\le\lambda\le\frac{1}{\alpha+1}\\ \frac{(\alpha+1)(\alpha+2)\lambda-2}{2(\alpha+2)}, & \frac{1}{\alpha+1}<\lambda\le 1\end{cases},$$

Hence, If we use (2.3), (2.4) and (2.5) in (2.2), we obtain the desired result. This completes the proof. □

Corollary 2.1. *In Theorem 2.1, if we take $q=1$, then we have*

$$\begin{aligned}&\left|S_f(x,\lambda,\alpha;a,b)\right|\\ \le\ &\left\{\frac{(x-a)^{\alpha+2}}{(\alpha+1)(b-a)}\left(\max\left\{\left|f''(x)\right|,\left|f''(a)\right|\right\}\right)\right.\\ &\left.+\frac{(b-x)^{\alpha+2}}{(\alpha+1)(b-a)}\left(\max\left\{\left|f''(x)\right|,\left|f''(b)\right|\right\}\right)\right\}.\end{aligned}$$

Corollary 2.2. *In Theorem 2.1, if we take $x=\frac{a+b}{2}$, then we have*

$$\begin{aligned}&\left|\frac{2^{\alpha-1}}{(b-a)^{\alpha-1}}S_f\left(\frac{a+b}{2},\lambda,\alpha;a,b\right)\right|\\ =\ &\left|(1-\lambda)f\left(\frac{a+b}{2}\right)+\lambda\left(\frac{f(a)+f(b)}{2}\right)-\frac{\Gamma(\alpha+1)2^{\alpha-1}}{(b-a)^{\alpha}}\left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-}f(a)+J^{\alpha}_{\left(\frac{a+b}{2}\right)^+}f(b)\right]\right|\\ \le\ &\frac{(b-a)^2}{8(\alpha+1)}C_1(\alpha,\lambda)\left\{\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,\left|f''(a)\right|^q\right\}\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,\left|f''(b)\right|^q\right\}\right)^{\frac{1}{q}}\right\}.\end{aligned}$$

Corollary 2.3. *In Theorem 2.1, if we take* $x = \frac{a+b}{2}$ *and* $\lambda = \frac{1}{3}$*, then we have*

$$\begin{aligned} & \left| \frac{2^{\alpha-1}}{(b-a)^{\alpha-1}} S_f\left(\frac{a+b}{2}, \frac{1}{3}, \alpha; a, b\right) \right| \\ = & \left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{\Gamma(\alpha+1)\, 2^{\alpha-1}}{(b-a)^{\alpha}} \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b)\right] \right| \\ \leq & \frac{(b-a)^2}{8(\alpha+1)} C_1\left(\alpha, \frac{1}{3}\right) \left\{ \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right) \right|^q, |f''(a)|^q \right\} \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right) \right|^q, |f''(b)|^q \right\} \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Remark 2.1. In Corollary 2.3, if we choose $\alpha = 1$, we have the following Simpson type inequality

$$\begin{aligned} & \left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{(b-a)} \int_a^b f(x)dx \right| \\ \leq & \frac{(b-a)^2}{162} \left\{ \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right) \right|^q, |f''(a)|^q \right\} \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right) \right|^q, |f''(b)|^q \right\} \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

which is the same of the inequality (1.4).

Corollary 2.4. *In Theorem 2.1, if we take* $x = \frac{a+b}{2}$ *and* $\lambda = 0$*, then we have*

$$\begin{aligned} & \left| \frac{2^{\alpha-1}}{(b-a)^{\alpha-1}} S_f\left(\frac{a+b}{2}, 0, \alpha; a, b\right) \right| \\ = & \left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)\, 2^{\alpha-1}}{(b-a)^{\alpha}} \left[J^{\alpha}_{\left(\frac{a+b}{2}\right)^-} f(a) + J^{\alpha}_{\left(\frac{a+b}{2}\right)^+} f(b)\right] \right| \\ \leq & \frac{(b-a)^2}{8(\alpha+1)(\alpha+2)} \left\{ \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right) \right|^q, |f''(a)|^q \right\} \right)^{\frac{1}{q}} \right. \\ & \left. + \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right) \right|^q, |f''(b)|^q \right\} \right)^{\frac{1}{q}} \right\}. \end{aligned}$$

Remark 2.2. In Corollary 2.4, if we choose $\alpha = 1$, we have the following midpoint inequality

$$\begin{aligned}
&\left| f\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)} \int_a^b f(x)dx \right| \\
\leq\ & \frac{(b-a)^2}{48} \left\{ \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right) \right|^q, |f''(a)|^q \right\} \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right) \right|^q, |f''(b)|^q \right\} \right)^{\frac{1}{q}} \right\},
\end{aligned}$$

which is better than the inequality (1.5).

Corollary 2.5. *In Theorem 2.1, if we take $x = \frac{a+b}{2}$ and $\lambda = 1$, then we have*

$$\begin{aligned}
&\left| \frac{2^{\alpha-1}}{(b-a)^{\alpha-1}} S_f\left(\frac{a+b}{2}, 1, \alpha; a, b\right) \right| \\
=\ & \left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)\, 2^{\alpha-1}}{(b-a)^\alpha} \left[J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b) \right] \right| \\
\leq\ & \frac{\alpha(\alpha+3)(b-a)^2}{16(\alpha+1)(\alpha+2)} \left\{ \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right) \right|^q, |f''(a)|^q \right\} \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right) \right|^q, |f''(b)|^q \right\} \right)^{\frac{1}{q}} \right\}.
\end{aligned}$$

Remark 2.3. In Corollary 2.5, if we choose $\alpha = 1$, we have the following midpoint inequality

$$\begin{aligned}
&\left| \frac{f(a)+f(b)}{2} - \frac{1}{(b-a)} \int_a^b f(x)dx \right| \\
\leq\ & \frac{(b-a)^2}{24} \left\{ \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right) \right|^q, |f''(a)|^q \right\} \right)^{\frac{1}{q}} \right. \\
&\left. + \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right) \right|^q, |f''(b)|^q \right\} \right)^{\frac{1}{q}} \right\},
\end{aligned}$$

which is the same of the inequality (1.2).

Theorem 2.2. *Let $f : I \subset \mathbb{R} \to \mathbb{R}$ be twice differentiable function on I° such that $f'' \in L[a,b]$, where $a, b \in I^\circ$ with $a < b$. If $|f''|^q$ is quasi-convex on $[a,b]$ for some fixed $q > 1$, then for $x \in [a,b]$, $\lambda \in [0,1]$ and $\alpha > 0$, the following inequality for* *fractional integrals holds*

$$\begin{aligned}
(2.6)\qquad & \left|S_f(x,\lambda,\alpha;a,b)\right| \\
\leq\ & C_2^{\frac{1}{p}}(\alpha,\lambda,p)\left\{\frac{(x-a)^{\alpha+2}}{(\alpha+1)(b-a)}\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,|f''(a)|^q\right\}\right)^{\frac{1}{q}}\right. \\
& \left.+\frac{(b-x)^{\alpha+2}}{(\alpha+1)(b-a)}\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,|f''(b)|^q\right\}\right)^{\frac{1}{q}}\right\},
\end{aligned}$$

where $p=\frac{q}{q-1}$,

$$\begin{aligned}
& C_2(\alpha,\lambda,p) \\
=\ & \begin{cases} \frac{1}{p(\alpha+1)+1}, & \lambda=0 \\ \left[\begin{array}{c}\frac{[(\alpha+1)\lambda]^{\frac{1+(\alpha+1)p}{\alpha}}}{\alpha}\beta\left(\frac{1+p}{\alpha},1+p\right) \\ +\frac{[1-(\alpha+1)\lambda]^{p+1}}{\alpha(p+1)}\cdot{}_2F_1\left(1-\frac{1+p}{\alpha},1;p+2;1-(\alpha+1)\lambda\right)\end{array}\right], & 0<\lambda\leq\frac{1}{\alpha+1} \\ \frac{[(\alpha+1)\lambda]^{\frac{1+(\alpha+1)p}{\alpha}}}{\alpha}\beta\left(\frac{1}{(\alpha+1)\lambda};\frac{1+p}{\alpha},1+p\right), & \frac{1}{\alpha+1}<\lambda\leq 1 \end{cases},
\end{aligned}$$

${}_2F_1$ *is Hypergeometric function defined by*

$${}_2F_1(a,b;c;z)=\frac{1}{\beta(b,c-b)}\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}\,dt,\ c>b>0,\ |z|<1 \text{ (see [1])},$$

β *is Euler Beta function defined by*

$$\beta(x,y)=\frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}=\int_0^1 t^{x-1}(1-t)^{y-1}\,dt,\quad x,y>0,$$

and

$$\beta(a,x,y)=\int_0^a t^{x-1}(1-t)^{y-1}\,dt,\quad 0<a<1,\ x,y>0,$$

is incomplete Beta function.

Proof. From Lemma 1.1, property of the modulus and using the Hölder inequality we have

$$\begin{aligned}
& \left|S_f(x,\lambda,\alpha;a,b)\right|\leq\frac{(x-a)^{\alpha+2}}{(\alpha+1)(b-a)}\int_0^1|t|\left|(\alpha+1)\lambda-t^\alpha\right|\left|f''(tx+(1-t)a)\right|dt \\
& +\frac{(b-x)^{\alpha+2}}{(\alpha+1)(b-a)}\int_0^1|t|\left|(\alpha+1)\lambda-t^\alpha\right|\left|f''(tx+(1-t)b)\right|dt \\
\leq\ & \frac{(x-a)^{\alpha+2}}{(\alpha+1)(b-a)}\left(\int_0^1 t^p\left|(\alpha+1)\lambda-t^\alpha\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''(tx+(1-t)a)\right|^q dt\right)^{\frac{1}{q}} \\
(2.7)\ & +\frac{(b-x)^{\alpha+2}}{(\alpha+1)(b-a)}\left(\int_0^1 t^p\left|(\alpha+1)\lambda-t^\alpha\right|^p dt\right)^{\frac{1}{p}}\left(\int_0^1\left|f''(tx+(1-t)b)\right|^q dt\right)^{\frac{1}{q}}
\end{aligned}$$

Since $|f''|^q$ is quasi-convex on $[a,b]$ we get

$$\int_0^1 |f''(tx+(1-t)a)|^q\,dt \le \max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q, |f''(a)|^q\right\} \tag{2.8}$$

$$\int_0^1 |f''(tx+(1-t)b)|^q\,dt \le \max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q, |f''(b)|^q\right\} \tag{2.9}$$

and

$$\int_0^1 t^p\,|(\alpha+1)\lambda - t^\alpha|^p\,dt \tag{2.10}$$

$$= \begin{cases} \int_0^1 t^{(\alpha+1)p}dt & \lambda=0 \\ \int_0^{[(\alpha+1)\lambda]^{\frac{1}{\alpha}}} t^p\left[(\alpha+1)\lambda - t^\alpha\right]^p dt + \int_{[(\alpha+1)\lambda]^{\frac{1}{\alpha}}}^1 t^p\left[t^\alpha-(\alpha+1)\lambda\right]^p dt, & 0<\lambda\le\frac{1}{\alpha+1} \\ \int_0^1 t^p\left[(\alpha+1)\lambda - t^\alpha\right]^p dt, & \frac{1}{\alpha+1}<\lambda\le 1 \end{cases}$$

$$= \begin{cases} \frac{1}{p(\alpha+1)+1}, & \lambda=0 \\ \left[\begin{array}{c} \frac{[(\alpha+1)\lambda]^{\frac{1+(\alpha+1)p}{\alpha}}}{\alpha}\beta\left(\frac{1+p}{\alpha},1+p\right) \\ +\frac{[1-(\alpha+1)\lambda]^{p+1}}{\alpha(p+1)}\cdot{}_2F_1\left(1-\frac{1+p}{\alpha},1;p+2;1-(\alpha+1)\lambda\right) \end{array}\right], & 0<\lambda\le\frac{1}{\alpha+1} \\ \frac{[(\alpha+1)\lambda]^{\frac{(\alpha+1)p+1}{\alpha}}}{\alpha}\beta\left(\frac{1}{(\alpha+1)\lambda};\frac{1+p}{\alpha},1+p\right), & \frac{1}{\alpha+1}<\lambda\le 1 \end{cases}$$

Hence, If we use (2.8), (2.9) and (2.10) in (2.7), we obtain the desired result. This completes the proof. □

Corollary 2.6. *In Theorem 2.2, if we take $x=\frac{a+b}{2}$, then we have*

$$\begin{aligned} &|S_f(x,\lambda,\alpha;a,b)| \\ \le\ & C_2^{\frac{1}{p}}(\alpha,\lambda,p)\frac{(b-a)^{\alpha+1}}{(\alpha+1)2^{\alpha+2}}\left\{\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,|f''(a)|^q\right\}\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,|f''(b)|^q\right\}\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

Corollary 2.7. *In Theorem 2.2, if we take $x=\frac{a+b}{2}$ and $\lambda=\frac{1}{3}$, then we have*

$$\begin{aligned} &\left|\frac{1}{6}\left[f(a)+4f\left(\frac{a+b}{2}\right)+f(b)\right]-\frac{\Gamma(\alpha+1)2^{\alpha-1}}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-}f(a)+J^\alpha_{\left(\frac{a+b}{2}\right)^+}f(b)\right]\right| \\ \le\ & C_2^{\frac{1}{p}}\left(\alpha,\frac{1}{3},p\right)\frac{(b-a)^2}{8(\alpha+1)}\left\{\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,|f''(a)|^q\right\}\right)^{\frac{1}{q}}\right. \\ &\left.+\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,|f''(b)|^q\right\}\right)^{\frac{1}{q}}\right\}. \end{aligned}$$

Corollary 2.8. *In Theorem 2.2, if we take* $x = \frac{a+b}{2}$, $\lambda = \frac{1}{3}$ *and* $\alpha = 1$ *then we have*

$$\left| \frac{1}{6}\left[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)\right] - \frac{1}{(b-a)}\int_a^b f(x)dx \right|$$
$$\leq \frac{(b-a)^2}{16} C_2^{\frac{1}{p}}\left(1, \frac{1}{3}, p\right) \left\{ \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right)\right|^q, |f''(a)|^q \right\} \right)^{\frac{1}{q}} + \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right)\right|^q, |f''(b)|^q \right\} \right)^{\frac{1}{q}} \right\},$$

where

$$C_2\left(1, \frac{1}{3}, p\right) = \left(\frac{2}{3}\right)^{1+2p} \beta(1+p, 1+p) + \left(\frac{1}{3}\right)^{1+p} . {}_2F_1\left(-p, 1; p+2; \frac{1}{3}\right).$$

Corollary 2.9. *In Theorem 2.2, if we take* $x = \frac{a+b}{2}$ *and* $\lambda = 0$*, then we have*

$$\left| f\left(\frac{a+b}{2}\right) - \frac{\Gamma(\alpha+1)2^{\alpha-1}}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b)\right]\right|$$
$$\leq \frac{(b-a)^2}{16}\left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}} \left\{ \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right)\right|^q, |f''(a)|^q \right\} \right)^{\frac{1}{q}} + \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right)\right|^q, |f''(b)|^q \right\} \right)^{\frac{1}{q}} \right\}.$$

Remark 2.4. In Corollary 2.9, if we choose $\alpha = 1$, we have the following midpoint inequality which is better than the inequality (1.6)

$$\left| f\left(\frac{a+b}{2}\right) - \frac{1}{(b-a)}\int_a^b f(x)dx \right|$$
$$\leq \frac{(b-a)^2}{16}\left(\frac{1}{2p+1}\right)^{\frac{1}{p}} \left\{ \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right)\right|^q, |f''(a)|^q \right\} \right)^{\frac{1}{q}} + \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right)\right|^q, |f''(b)|^q \right\} \right)^{\frac{1}{q}} \right\}.$$

Corollary 2.10. *In Theorem 2.2, if we take* $x = \frac{a+b}{2}$ *and* $\lambda = 1$*, then we have*

$$\left| \frac{f(a)+f(b)}{2} - \frac{\Gamma(\alpha+1)2^{\alpha-1}}{(b-a)^\alpha}\left[J^\alpha_{\left(\frac{a+b}{2}\right)^-} f(a) + J^\alpha_{\left(\frac{a+b}{2}\right)^+} f(b)\right]\right|$$
$$\leq \frac{(b-a)^2}{16} C_2^{\frac{1}{p}}(\alpha, 1, p) \left\{ \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right)\right|^q, |f''(a)|^q \right\} \right)^{\frac{1}{q}} + \left(\max\left\{ \left| f''\left(\frac{a+b}{2}\right)\right|^q, |f''(b)|^q \right\} \right)^{\frac{1}{q}} \right\}.$$

where

$$C_2(\alpha,1,p)=\frac{(1+\alpha)^{\frac{(1+\alpha)(1+p)-\alpha}{\alpha}}}{\alpha}\beta\left(\frac{1}{1+\alpha};\frac{1+p}{\alpha},1+p\right).$$

Remark 2.5. In Corollary 2.10, if we choose $\alpha=1$, we have the following trapezoid inequality which is the same of the inequality (1.3)

$$\begin{aligned}&\left|\frac{f(a)+f(b)}{2}-\frac{1}{(b-a)}\int_a^b f(x)dx\right|\\ \leq\ &\frac{(b-a)^2}{4}\left(\beta(1+p,1+p)\right)^{\frac{1}{p}}\left\{\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,|f''(a)|^q\right\}\right)^{\frac{1}{q}}\right.\\ &\left.+\left(\max\left\{\left|f''\left(\frac{a+b}{2}\right)\right|^q,|f''(b)|^q\right\}\right)^{\frac{1}{q}}\right\},\end{aligned}$$

where

$$\beta(1+p,1+p)=2\beta\left(\frac{1}{2};1+p,1+p\right)=2^{-2p-1}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma(p+1)}{\Gamma\left(\frac{3}{2}+p\right)},\ \Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}.$$

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