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A NOTE ON CROSSED MODULES OF LEIBNIZ ALGEBRAS

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Abstract. In this paper we give the relation between precrossed modules and crossed modules of Leibniz algebras. Also construct the coproduct object in the category of crossed L-modules on Leibniz algebras.

1. Introduction

Leibniz algebras were first introduced by Loday in [5]. They can be thought as a generalisation of Lie algebras. The main difference is that the bracket of Leibniz algebra is non-skew-symmetric. They have many applications in some branches of Mathematics and Physics.We refer the references given in [3] for a survey about Leibniz algebras.

Crossed modules were introduced by Whitehead in [9] as a model for connected homotopy 2-types. After then, crossed modules used in many branches of mathematics such as category theory, cohomology of algebraic structures, differential geometry and in physics. This makes the crossed modules one of the fundamental algebraic gadget. For some different usage, crossed modules were defined in different categoriess such as Lie algebras, commutative algebras etc.([7],[4]). The (pre)crossed modules are generalisations of Leibniz algebras. This is why the subject is important. At this vein, in the paper, we construct the relation between pre-crossed and crossed modules of Leibniz algebras and construct the coproducts in the category of crossed L-modules on Leibniz algebras.

2. Preliminaries

Definition 2.1. A Leibniz algebra L is a $\mathbb{k}$-vector space equipped with a bilinear map $[-,-] : L \times L \longrightarrow L$, satisfying the Leibniz identity

$$[x,[y,z]] = [[x,y],z] - [[x,z],y]$$

for all $x, y, z \in L$.

If $[x,x] = 0$, for all $x \in L$, then the Leibniz identity becomes to the Jacobi identity, so L will be a Lie algebra.

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Definition 2.2. A morphism of Leibniz algebras is a $\mathbb{k}$-linear map $f : L \longrightarrow L'$ such that

$$f[x,y] = [f(x), f(y)]$$

for all $x, y \in L$.

By this definition we have the category of Leibniz algebras which will be denoted by **Lbnz** in this work.

Definition 2.3. Let L be a Leibniz algebra and I be a subalgebra (a vector subspace of L closed under the bracket operation). If $[x,y], [y,x] \in I$, for all $x \in L$ and $y \in I$, then I is called a two-sided ideal of L and this is denoted by $I \trianglelefteq L$.

If $I \trianglelefteq L$ then, the quotient I/L inherits a Leibniz structure with the operations induced from L.

Definition 2.4. An abelian Leibniz algebra is a Leibniz algebra with the trivial bracket.

Example 2.1. Every $\mathbb{k}$-vector space has an abelian Leibniz algebra structure.

Definition 2.5. Let L and L' be Leibniz algebras. A Leibniz action from L over L' consist of $\mathbb{k}$-bilinear maps

$$\begin{array}{ll} \lambda : L \times L' \longrightarrow L', & (x,m) \longmapsto {}^{x}m \\ \rho : L' \times L \longrightarrow L', & (m,x) \longmapsto m^{x} \end{array}$$

satisfying

$$\begin{array}{lll} 1. & {}^{x}[m,n] & = [{}^{x}m, n] - [{}^{x}n, m] \\ 2. & [m, {}^{x}n] & = [m^{x}, n] - [m,n]^{x} \\ 3. & [m, n^{x}] & = [m,n]^{x} - [m^{x}, n] \\ 4. & {}^{x}({}^{y}m) & = {}^{[x,y]}m - ({}^{x}m)^{y} \\ 5. & ({}^{x}m^{y}) & = ({}^{x}m)^{y} - {}^{[x,y]}m \\ 6. & m^{[x,y]} & = (m^{x})^{y} - (m^{y})^{x} \end{array}$$

for all $x, y \in L, m, n \in L'$.

Now we will recall the definition of crossed modules on Leibniz algebras from [7].

Definition 2.6. A precrossed module on Leibniz algebras is a Leibniz algebra homomorphism $\partial : L_1 \longrightarrow L_0$ with an action of L_0 on L_1 such that

$$\partial({}^{l_0}l_1) = [l_0, \partial(l_1)], \quad \partial(l_1^{l_0}) = [\partial(l_1), l_0],$$

for all $l_0 \in L_0, l_1 \in L_1$. This is a crossed module if in addition it satisfy the Peiffer identities

$$l_1^{\partial(l_1')} = [l_1, l_1'], \quad {}^{\partial(l_1')}l_1 = [l_1', l_1],$$

for all $l_1, l_1' \in L_1$.

Definition 2.7. Let $\partial : L_1 \longrightarrow L_0$, $\delta : M_1 \longrightarrow M_0$ be (pre)crossed modules. A morphism of (pre)crossed modules is a pair (f_1, f_0) of Leibniz homomorphisms $f_1 : L_1 \longrightarrow M_1$, $f_2 : L_0 \longrightarrow M_0$ such that

$$f_0\partial = \delta f_1, \quad f_1({}^{l_0}l_1) = {}^{f_0(l_0)}f_1(l_1), \quad f_1(l_1^{l_0}) = (f_1(l_1))^{f_0(l_0)},$$

for all $l_0 \in L_0, l_1 \in L_1$.

Consequently, we have the category of precrossed modules and category of crossed modules which will denoted by **PXLbnz**, **XLbnz**, respectively.

Example 2.2. Let I be a two-sided ideal of a Leibniz algebra L. Then

$$inc. : I \longrightarrow L$$

is a crossed module with the conjugate action of L on I defined by $\ {}^{l}i = [l, i]\ $, $\ i^{l} = [i, l]\ $, for all $i \in I, l \in L$.

Example 2.3. For any Leibniz algebra L, $L \longrightarrow 0$ is a precrossed module, which is called a trivial precrossed module.

Remark 2.1. We have the full inclusion

$$\mathbf{Lbnz} \subseteq \mathbf{PXLbnz}$$

where a Leibniz algebra L is identified with the trivial precrossed module $\ L \longrightarrow 0\ $. So precrossed modules can be thought as a generalisation of Leibniz algebras. We have the functor

$$inc. : \mathbf{Lbnz} \longrightarrow \mathbf{PXLbnz}$$

Also we have the forgetful functor

$$U : \mathbf{XLbnz} \longrightarrow \mathbf{PXLbnz}$$

which forgets the Peiffer identities.

3. From precrossed modules to crossed modules

In this section we will define the Peiffer ideals of Leibniz algebras and give a functor from precrossed modules to crossed modules. The group case of this work was given in [1].

Definition 3.1. A commutator in a Leibniz algebra is defined by $[x, y]$, for $x, y \in L$. The commutator ideal of L is the two-sided ideal generated by all commutators of L.

Definition 3.2. Let $\partial : L_1 \longrightarrow L_0$ be a precrossed module. A left Peiffer commutator is defined by

$$\langle a, b\rangle_l = \ {}^{\partial(a)}b - [a, b]$$

and the right Peiffer commutator is defined by

$$\langle a, b\rangle_r = b^{\partial(a)} - [b, a]$$

for all $a, b \in L_1$. The two-sided ideal generated by the set $\{\langle a, b\rangle_l, \langle a, b\rangle_r \mid a, b \in L_1\}$ is called the Peiffer ideal of L_1 and is denoted by $P_2(\partial)$.

Theorem 3.1. *Let $\partial : L_1 \longrightarrow L_0$ be a precrossed module. Then*

$$\partial^{cr} : L_1^{cr} = L_1/P_2(\partial) \longrightarrow L_0$$

is a crossed module where ${}^{l_0}(\overline{l_1}) = \overline{{}^{l_0}l_1}, \partial^{cr}(\overline{l_1}) = \partial(l_1)$ *for all* $l_0 \in L_0, l_1 \in L_1$.

Proof. We will only look the Peiffer conditions. Since ${}^{\partial(l_1)}m_1 \equiv [l_1, m_1]$, $(m_1)^{\partial(l_1)} \equiv [m_1, l_1]$ modulo $P_2(\partial)$, we have

$${}^{\partial_2(\overline{m_1})}(\overline{l_1}) = {}^{\partial_2(m_1)}\ (\overline{l_1}) = \overline{({}^{\partial_2(m_1)}l_1)} = \overline{[m_1, l_1]},$$

and by a similar way

$$(\overline{l_1})^{\overline{\partial_2(m_1)}} = \overline{[l_1, m_1]},$$

for all $l_1, m_1 \in L_1$ as required. □

Remark 3.1. As a consequence of Theorem 14, we have the functor

$$()^{cr} : \mathbf{PXLbnz} \longrightarrow \mathbf{Lbnz}$$

defined by

$$()^{cr}(\partial : L_1 \longrightarrow L_0) = (\partial^{cr} : L_1^{cr} \longrightarrow L_0)$$

for any precrossed module $\partial : L_1 \longrightarrow L_0$.

4. Some properties of the categories of crossed L-modules

In this section we will define the full subcategory $\mathbf{XLbnz}/L$ of $\mathbf{XLbnz}$ and construct coproduct object in this subcategory.

Let L be a fixed Leibniz algebra. We define a subcategory of $\mathbf{XLbnz}$ whose objects are crossed modules with same base L. We will denote this category by $\mathbf{XLbnz}/L$. The objects of this category will be called as crossed L-modules.

Proposition 4.1. *a) Two morphisms with same source and target have equaliser in* $\mathbf{XLbnz}/L$
b) $\mathbf{XLbnz}/L$ *has pullbacks*
c) $inc : 0 \longrightarrow L$ *is the terminal object in* $\mathbf{XLbnz}/L$

Proof. Follows from a direct calculation similar to commutative algebra case given in [6]. □

Corollary 4.1. $\mathbf{XLbnz}/L$ *is finitely complete.*

Proof. It is an obvious result of proposition 16. □

Remark 4.1. Let $(N, \partial), (M, \delta)$ be crossed L-modules. M has a Leibniz action on N, via the homomorphism δ,thanks to the action of L on N. This action gives rise to the semi-direct product $N \rtimes M$ where the bracket is defined as follows;

$$[(n, m)(n_0, m_0)] = ([n, n_0] + {}^{\delta(m)}n_0 + n^{\delta(m_0)}, [m, m_0])$$

for all $m, m_0 \in M, n, n_0 \in N$. It is obvious that L has an action on $N \rtimes M$ defined by

$${}^{l}(n, m) = ({}^{l}n, {}^{l}m) \quad , \quad (n, m)^l = (n^l, m^l)$$

for all $l \in L, n \in N, m \in M$.

Define $\alpha : N \rtimes M \longrightarrow L$ by $\alpha(n, m) = \partial(n) + \delta(m)$. $\alpha : N \rtimes M \longrightarrow L$ is a precrossed L-module with the action of L on $N \rtimes M$ defined above. Indeed

$$\begin{aligned} \alpha({}^{l}(n, m)) &= \alpha({}^{l}n, {}^{l}m) \\ &= \partial({}^{l}n) + \delta({}^{l}m) \\ &= [l, \partial(n)] + [l, \delta(m)] \\ &= [l, \partial(n) + \delta(m)] \\ &= [l, \alpha(n, m)] \end{aligned}$$

for all $l \in L, (n, m) \in N \rtimes M$, as required. By a similar calculation we have $\alpha((n, m)^l) = [\alpha(n, m), l]$, for all $l \in L, (n, m) \in N \rtimes M$.

But $\alpha : N \rtimes M \longrightarrow L$ do not satisfy the Peiffer identities in general. By theorem 14

$$\alpha^{cr} : (N \rtimes M)/P_2(\alpha) \longrightarrow L$$

is a crossed module.

Corollary 4.2. $\alpha^{cr} : (N \rtimes M)/P_2(\alpha) \longrightarrow L$ *is the coproduct of the crossed L-modules* $\partial : N \longrightarrow L$, $\delta : M \longrightarrow L$.

Proof. Direct checking. Details can be found in [6], for commutative algebra case. □

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