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On Para-Sasakian Manifold with Respect to the Schouten-van Kampen Connection

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(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

Abstract

In the present paper, we have studied the curvature properties of the Schouten-van Kampen connection on the n-dimensional Para-Sasakian manifold and obtained some new results. Also, we studied projective curvature tensor, concircular curvature tensor, and Nijenhuis tensor for the Para-Sasakian manifold with respect to the Schouten-van Kampen connection.

Keywords: *Para-Sasakian manifold, Schouten-van Kampen connection, concircular curvature tensor, projective curvature tensor, Nijenhuis tensor.*
AMS Subject Classification (2020): *Primary: 53C25 ; Secondary: 53D15; 53B05; ; 53B15.*

1. Introduction

The Schouten–van Kampen connection is one of the most natural connections bearing a pair of complementary distributions on a differentiable manifold endowed with an affine connection [2][5][19]. It was considered by Solov'ev [21][22][23][24] to investigate hyperdistributions in Riemannian manifolds. After that, Olszak[11] studied an almost contact metric structure with respect to the Schouten–van Kampen connection and characterized some classes of almost contact metric manifolds, admitting such connection by obtaining certain curvature properties of the Schouten–van Kampen connection on these manifolds. Likewise, Yıldız[28] studied projectively flat and conharmonically flat f -Kenmotsu 3-manifolds with respect to the Schouten–van Kampen connection. In 2006, Bejancu[3] studied about Schouten-van Kampen connection on Foliated manifolds. Further Schouten-van Kampen connections on different almost contact(para) structures were studied by several geometers[4][9][10][14][15].

Sato[17] defined the notion of (ϕ, ξ, η) structure of a differentiable manifold satisfying $\phi^2 X = X - \eta(X)\xi$ and $\eta(\xi) = 1$, where ϕ is (1,1)- tensor field, ξ is a vector field and η is 1-form on the manifold, and he called the manifold with this structure an almost paracontact manifold. Further, he and K. Matsumoto[18] defined and studied the special cases of an almost paracontact structure which were considered as P-Sasakian and SP-Sasakian manifolds and they obtain several interesting results on them. Further, the P-Sasakian manifold has been investigated by several authors such as Sasaki et al [16], Shukla et al[20], K. Mondal and U.C. De [7], Yildiz et al[29], Matsumoto, Ianus, and Mihai[8], Ozgur[**?**], Adati and Miyazawa[1] and many others.

A transformation of an n-dimensional differential manifold M, which transforms every geodesic circle of M into a geodesic circle, is called a concircular transformation[6][25]. A concircular transformation is always a conformal transformation[6]. Here geodesic circle means a curve in M whose first curvature is constant and whose second curvature is identically zero. Thus, the geometry of concircular transformations is a generalization of inversive geometry in the sense that the change of metric is more general than that induced by a circle preserving diffeomorphism. An interesting invariant of a concircular transformation is the concircular curvature tensor W with respect to the Levi-Civita connection. It was defined by[25][26]

$$W(X,Y)Z = R(X,Y)Z - \frac{r}{n(n-1)}\{g(Y,Z)X - g(X,Z)Y\}, \tag{1.1}$$

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Where $X, Y, Z \in \chi(M)$, $\chi(M)$ is the tangent bundle for manifold M; R and r are the curvature tensor and the scalar curvature with respect to the Levi-Civita connection ∇. Riemannian manifolds with vanishing concircular curvature tensors are of constant curvature. Thus, the concircular curvature tensor is a measure of the failure of a Riemannian manifold to be of constant curvature.

Projective curvature tensor $P(X, Y)Z$ for an n-dimensional P-Sasakian manifold with respect to the Levi-Civita connection ∇ is given by,

$$P(X,Y)Z = R(X,Y)Z - \frac{1}{(n-1)}\{S(Y,Z)X - S(X,Z)Y\}, \tag{1.2}$$

Inspired by the above studies, we will study the curvature properties of the P-Sasakian manifold concerning the Schouten-van Kampen connection. The paper is organized as follows:

In section 2, we give a brief account of the Para-Sasakian manifold and Schouten-van Kampen connection. In section 3, we find the expressions of curvature tensor, Ricci tensor, Ricci operator, and scalar curvature of the Para-Sasakain manifold for the Schouten-van Kampen connection. Section 4 is devoted to the results for the Projective curvature and concircular curvature tensor of the P-Sasakian manifold for the Schouten-van Kampen connection. Section 5 deals with the Nijenhuis tensor for Para-Sasakian manifold with respect to the Schouten-van Kampen connection. In section 6, we come up with an example of a 3-dimensional Para-Sasakian manifold admitting the Schouten-van Kampen connection which verifies the theorem 3.1 and theorem 4.2.

2. Preliminaries

An n-dimensional differentiable manifold M is said to be an almost paracontact manifold if it admits an almost paracontact structure (ϕ, ξ, η) consisting of a (1,1)- tensor field ϕ, a vector field ξ, and a 1-form η satisfying[18]

$$\phi^2 X = X - \eta(X)\xi, \eta(\xi) = 1, \phi\xi = 0, \eta(\phi X) = 0, \tag{2.1}$$

Let g be a compatible Riemannian metric with (ϕ, ξ, η), that is,

$$g(X,Y) = g(\phi X, \phi Y) + \eta(X)\eta(Y), \tag{2.2}$$

Or equivalently,

$$g(X, \phi Y) = g(\phi X, Y), g(X, \xi) = \eta(X), \tag{2.3}$$

The fundamental 2-form F of the manifold is defined by

$$F(X,Y) = g(X, \phi Y),$$

for all $X, Y \in TM$. Then M becomes an almost paracontact Riemannian manifold along with an almost paracontact Riemannian structure (ϕ, ξ, η, g).

An almost paracontact Riemannian manifold is called a P-Sasakian manifold if it satisfies

$$(\nabla_X \phi)Y = -g(X,Y)\xi - \eta(Y)X + 2\eta(X)\eta(Y)\xi, \tag{2.4}$$

for all $X, Y \in TM$, where ∇ is Levi-Civita connection of the Riemannain metric. Also,

$$(\nabla_X \eta)Y = g(X, \phi Y) = (\nabla_Y \eta)X, \tag{2.5}$$

$$\nabla_X \xi = \phi X, \tag{2.6}$$

In an n-dimensional P-Sasakian manifold M, the curvature tensor R, the Ricci tensor S and the Ricci operator Q satisfy[1][12][13],

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X, \tag{2.7}$$

$$R(\xi, X)Y = \eta(Y)X - g(X,Y)\xi, \tag{2.8}$$

$$R(\xi, X)\xi = X - \eta(X)\xi, \tag{2.9}$$

$$S(X, \xi) = -(n-1)\eta(X), \tag{2.10}$$

$$Q\xi = -(n-1)\xi, \tag{2.11}$$

$$\eta(R(X,Y)Z) = g(X,Z)\eta(Y) - g(Y,Z)\eta(X), \tag{2.12}$$

$$\eta(R(X,Y)\xi) = 0, \tag{2.13}$$

$$\eta(R(\xi,X)Y) = \eta(X)\eta(Y) - g(X,Y). \tag{2.14}$$

An almost paracontact Riemannian manifold M is said to be an η-Einstein manifold if the Ricci tensor S satisfies[7],

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{2.15}$$

Here a and b are smooth functions. Specifically, if $b = 0$, then M is an Einstein manifold.

Now we consider two naturally defined distributions in the tangent bundle TM of M as, $H =$kerη, $V =$spanξ, such that $TM = H \oplus V$, $H \cap V = \{0\}$ and $H \perp V$. This decomposition defines the Schouten Van Kampen connection $\tilde{\nabla}$ over an almost (para) contact metric structure. The Schouten-van Kampen connection $\tilde{\nabla}$ on an almost (para) contact metric manifold concerning the Levi-Civita connection ∇ is defined by[21]

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(Y)\nabla_X \xi + (\nabla_X \eta)(Y)\xi. \tag{2.16}$$

Many properties of some geometric objects connected with the distributions H, V can be characterized with the help of the Schouten-van Kampen connection [21][22][23]. For instance , g, ξ and η are parallel with respect to $\tilde{\nabla}$, that is, $\tilde{\nabla}\xi = 0$, $\tilde{\nabla}g = 0$, $\tilde{\nabla}\eta = 0$. Also, the torsion $\tilde{T}$ of $\tilde{\nabla}$ is defined by,

$$\tilde{T}(X,Y) = \eta(X)\nabla_Y \xi - \eta(Y)\nabla_X \xi + 2d\eta(X,Y)\xi.$$

3. Curvature Properties of P-Sasakian Manifold concerning $\tilde{\nabla}$

Let M be a para-Sasakian manifold, then by using equations (2.5), (2.6) and (2.16), we can define M with respect to $\tilde{\nabla}$ such that,

$$\tilde{\nabla}_X Y = \nabla_X Y - \eta(Y)\phi X + g(X,\phi Y)\xi \tag{3.1}$$

Now, put $Y = \xi$ in (3.1) and use (2.6), we get

$$\tilde{\nabla}_X \xi = 0 \tag{3.2}$$

Let R and $\tilde{R}$ be the curvature tensors of ∇ and $\tilde{\nabla}$ respectively, then

$$R(X,Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z \tag{3.3}$$

$$\tilde{R}(X,Y)Z = \tilde{\nabla}_X \tilde{\nabla}_Y Z - \tilde{\nabla}_Y \tilde{\nabla}_X Z - \tilde{\nabla}_{[X,Y]} Z \tag{3.4}$$

Then, by using the equations (2.12), (2.13), (2.14), (3.1), (3.3) and (3.4), the curvature tensor of the P-Sasakian manifold with respect to $\tilde{\nabla}$ is given by,

$$\tilde{R}(X,Y)Z = R(X,Y)Z + \eta(Z)(\eta(Y)X - \eta(X)Y) + (g(Y,Z)\eta(X) - g(X,Z)\eta(Y))\xi - g(X,\phi Z)\phi Y + g(Y,\phi Z)\phi X \tag{3.5}$$

Taking the inner product of (3.5) with W we have,

$$\begin{aligned}\tilde{R}(X,Y,Z,W) = R(X,Y,Z,W) + \eta(Z)(\eta(Y)g(X,W) - \eta(X)g(Y,W)) + (g(Y,Z)\eta(X) \\ - g(X,Z)\eta(Y))\eta(W) - g(X,\phi Z)g(\phi Y,W) + g(Y,\phi Z g(\phi X,W))\end{aligned} \tag{3.6}$$

where $\tilde{R}(X,Y,Z,W) = g(\tilde{R}(X,Y)Z,W)$. Let $\{e_i\}$ be the orthonormal basis of tangent space at each point of the manifold then put $X = W = e_i$ and contract the equation (3.6) by X and W we get,

$$\tilde{S}(Y,Z) = S(Y,Z) + (n-1)\eta(Y)\eta(Z) + g(Y,\phi Z)trace\phi \tag{3.7}$$

The Ricci operator is given by,

$$S(X,Y) = g(QX,Y)$$

Hence, the Ricci operator $\tilde{Q}$ for P-Sasakian manifold with respect to $\tilde{\nabla}$ is given by,

$$\tilde{S}(X,Y) = g(\tilde{Q}X,Y) \tag{3.8}$$

Now, from equation (3.7) and (3.8), we get

$$\tilde{Q}X = QX + (n-1)\eta(X)\xi + (\phi X)trace\phi \tag{3.9}$$

Now, from equations (2.2) and (2.10), and using the fact that $Q(\phi X) = \phi(QX)$we have,

$$\tilde{S}(\phi Y, \phi Z) = S(Y, Z) + (n-1)\eta(Y)\eta(Z) \tag{3.10}$$

also, the scalar curvature $\tilde{r}$ is given by,

$$\tilde{r} = r + n - 1 + (trace\phi)^2 \tag{3.11}$$

Hence, we can state the following theorem.

Theorem 3.1. *For a P-Sasakian manifold M, with respect to $\tilde{\nabla}$,*

(i) The curvature tensor $\tilde{R}$ is given by equation (3.5)

(ii) The Ricci tensor $\tilde{S}$ is given by equation (3.7) and it is symmetric,

(iii) The Ricci operator $\tilde{Q}$ is given by equation (3.9),

(iv) The constant curvature $\tilde{r}$ is given by the equation (3.11).

Theorem 3.2. *For a P-Sasakian manifold M, with respect to $\tilde{\nabla}$,*

(i) $\tilde{R}(X,Y)\xi = 0, \tilde{R}(\xi,Y)Z = 0, \tilde{R}(\xi,Y)\xi = 0$

(ii) $\tilde{R}(X,Y)Z + \tilde{R}(Y,X)Z = 0$

(iii) $\tilde{R}(X,Y)Z + \tilde{R}(Y,Z)X + \tilde{R}(Z,X)Y = 0$

(iv) $\tilde{S}(Y,\xi) = 0$

(v) $\tilde{Q}\xi = 0$

where X,Y,Z $\in \chi(M)$

Proof. From equations (3.5), (2.7),(2.8) and (2.9), we have (i), (ii) and (iii).
From equations (3.7) and (2.10), we have (iv).
From equations (3.9) and (2.11), we have (v).

4. Projective and Concircular Curvature Tensors for P-Sasakian Manifold concerning $\tilde{\nabla}$

Definition 4.1. The projective curvature tensor $\tilde{P}(X,Y)Z$ with respect to $\tilde{\nabla}$ is given by,

$$\tilde{P}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{1}{n-1}\{\tilde{S}(Y,Z)X - \tilde{S}(X,Z)Y\} \tag{4.1}$$

From the equations (1.2), (3.5), (3.7), and (4.1) we have,

$$\begin{aligned} \tilde{P}(X,Y)Z = P(X,Y)Z + g(Y,Z)\eta(X)\xi - g(X,Z)\eta(Y)\xi + g(X,\phi Z)\left[\frac{(trace\phi)Y}{n-1} - \phi Y\right] \\ - g(Y,\phi Z)\left[\frac{(trace\phi)X}{n-1} - \phi X\right] \end{aligned} \tag{4.2}$$

An n-dimensional P-Sasakian manifold is ξ-projectively flat with respect to $\tilde{\nabla}$ if

$$\tilde{P}(X,Y)\xi = 0$$

for all $X, Y, Z \in \chi(M)$.

Theorem 4.1. *An n-dimensional P-Sasakian manifold is ξ-projectively flat with respect to $\tilde{\nabla}$ if and only if it is ξ-projectively flat with respect to ∇.*

Proof. Put $Z = \xi$ in equation (4.2), then we obtain

$$\tilde{P}(X,Y)\xi = P(X,Y)\xi$$

Hence, the theorem.

Definition 4.2. The concircular curvature tensor $\tilde{W}(X,Y)Z$ with respect to $\tilde{\nabla}$ is given by,

$$\tilde{W}(X,Y)Z = \tilde{R}(X,Y)Z - \frac{\tilde{r}}{n(n-1)}[g(Y,Z)X - g(X,Z)Y] \tag{4.3}$$

An n -dimensional P-Sasakian manifold is ξ-concircularly flat with respect to $\tilde{\nabla}$ if,

$$\tilde{W}(X,Y)\xi = 0$$

for all $X, Y, Z \in \chi(M)$.

Theorem 4.2. *An n-dimensional P-Sasakian manifold is ξ-concircularly flat with respect to $\tilde{\nabla}$ if and only if the scalar curvature $\tilde{r}$ with respect to $\tilde{\nabla}$ vanishes.*

Proof. Put $Z = \xi$ in equations (4.3) and using equation (1.1) and theorem 3.2, we obtain

$$\tilde{W}(X,Y)\xi = -\frac{\tilde{r}}{n(n-1)}[\eta(Y)X - \eta(X)Y]$$

As $R(X,Y)\xi = \eta(Y)X - \eta(X)Y \neq 0$, so $\tilde{W}$ will vanish if $\tilde{r} = 0$ Hence, the theorem.

Definition 4.3. An n -dimensional P-Sasakian manifold is ϕ-concircularly flat with respect to $\tilde{\nabla}$ if,

$$\phi^2(\tilde{W}(\phi X, \phi Y)\phi Z = 0$$

Also, $\phi^2(\tilde{W}(\phi X, \phi Y)\phi Z = 0$ holds if and only if

$$g(\tilde{W}(\phi X, \phi Y)\phi Z, \phi W) = 0, \tag{4.4}$$

for all $X, Y, Z \in \chi(M)$.

Theorem 4.3. *An n-dimensional P-Sasakian manifold is ϕ-concircularly flat with respect to $\tilde{\nabla}$ if and only if it is η-Einstein manifold with respect to ∇.*

Proof. From equation (4.4) it follows that,

$$g(\tilde{W}(\phi X, \phi Y)\phi Z, \phi U) = g(\tilde{R}(\phi X, \phi Y)\phi Z, \phi U) - \frac{\tilde{r}}{n(n-1)}[g(\phi Y, \phi Z)g(\phi X, \phi U) - g(\phi X, \phi Z)g(\phi Y, \phi U)] \tag{4.5}$$

Suppose,

$$g(\tilde{W}(\phi X, \phi Y)\phi Z, \phi U) = 0.$$

Then from (4.5), we get

$$g(\tilde{R}(\phi X, \phi Y)\phi Z, \phi U) - \frac{\tilde{r}}{n(n-1)}[g(\phi Y, \phi Z)g(\phi X, \phi U) - g(\phi X, \phi Z)g(\phi Y, \phi U)] = 0 \tag{4.6}$$

Let $\{e_1, e_2, ..., e_n\}$ be a local orthonormal basis of the vector fields in M and use the fact that $\{\phi e_1, \phi e_2, ..., \phi e_n\}$ is also a local orthonormal basis, contracting (4.6) and summing up with respect to i, we have

$$\sum_{i=1}^{n} g(\tilde{R}(\phi e_i, \phi Y)\phi Z, \phi e_i) = \frac{\tilde{r}}{n(n-1)} \sum_{i=1}^{n} \{g(\phi Y, \phi Z)g(\phi e_i, \phi e_i) - g(\phi e_i, \phi Z)g(\phi Y, \phi e_i)\} \tag{4.7}$$

From the equation (4.7) it follows that

$$\tilde{S}(\phi Y, \phi Z) = \frac{\tilde{r}}{n}\{g(Y, Z) - \eta(Y)\eta(Z)\} \tag{4.8}$$

Put the value from equation (3.10) in (4.8) and using equations (3.7) we get

$$S(Y, Z) = \left(\frac{r + n - 1 + (trace\phi)^2}{n}\right)g(Y, Z) - \left(\frac{r + 2n - 2 + (trace\phi)^2}{n}\right)\eta(Y)\eta(Z) \tag{4.9}$$

From (2.15) we can say that the manifold is η-Einstein manifold with respect to ∇.
Conversely, let S be of the form equation (4.9), then obviously

$$g(\tilde{W}(\phi X, \phi Y)\phi Z, \phi U) = 0.$$

Hence, the theorem is proved.

5. Nijenhuis Tensor for P-Sasakian Manifold concerning $\tilde{\nabla}$

The Nijenhuis tensor with respect to ∇ is a vector-valued bilinear function defined as [25],

$$N(X, Y) = (\nabla_{\phi X}\phi)Y - (\nabla_{\phi Y}\phi)X - \phi((\nabla_X\phi)Y) + \phi((\nabla_Y\phi)X) \tag{5.1}$$

So, the Nijenhuis tensor with respect to $\tilde{\nabla}$ is defined as

$$\tilde{N}(X, Y) = (\tilde{\nabla}_{\phi X}\phi)Y - (\tilde{\nabla}_{\phi Y}\phi)X - \phi((\tilde{\nabla}_X\phi)Y) + \phi((\tilde{\nabla}_Y\phi)X) \tag{5.2}$$

Now, as we know that

$$(\tilde{\nabla}_X\phi)Y = \tilde{\nabla}_X(\phi Y) - \phi(\tilde{\nabla}_X Y)$$

Thus using (3.1)

$$(\tilde{\nabla}_X\phi)Y = (\nabla_X\phi)Y + g(X, Y)\xi - \eta(Y)(\eta(X)\xi - \phi X) \tag{5.3}$$

Now, replacing X by ϕX in equation (5.3)

$$(\nabla_{\phi X}\phi)Y = (\tilde{\nabla}_{\phi X}\phi)Y - g(\phi X, Y)\xi + \eta(Y)(X - \eta(X)\xi) \tag{5.4}$$

Interchanging X and Y in equation (5.4)

$$(\nabla_{\phi Y}\phi)X = (\tilde{\nabla}_{\phi Y}\phi)X - g(\phi Y, X)\xi - \eta(X)(Y - \eta(Y)\xi) \tag{5.5}$$

Operating ϕ on both side of equation (5.3)

$$\phi((\nabla_X\phi)Y) = \phi((\tilde{\nabla}_X\phi)Y) - g(\phi X, Y)\xi - g(X, \phi Y)\xi - \eta(Y)X + \eta(Y)\eta(X)\xi \tag{5.6}$$

Interchanging X and Y in equation (5.6)

$$\phi((\nabla_Y\phi)X) = \phi((\tilde{\nabla}_Y\phi)X) - g(\phi Y, X)\xi - g(Y, \phi X)\xi - \eta(X)Y + \eta(Y)\eta(X)\xi \tag{5.7}$$

Put the value of equations (5.4),(5.5),(5.6) and (5.7) in equations (5.1) and (5.2) we get

$$N(X, Y) = \tilde{N}(X, Y)$$

Hence, we can state the following theorem.

Theorem 5.1. *The Nijenhuis tensor of a P-Sasakian manifold with respect to $\tilde{\nabla}$ coincides with the Nijenhuis tensor of a P-Sasakian manifold with respect to ∇.*

6. Example

In this section, we reconstruct an example of 3-dimensional P-Sasakian manifolds with respect to $\tilde{\nabla}$.

Example 6.1. Let us consider a 3-dimensional manifold $M = \{(u, v, w) \in \mathbb{R}^3\}$ where (u, v, w) are the standard coordinates in $\mathbb{R}^3$. We choose the vector fields,

$$e_1 = e^u \frac{\partial}{\partial v}, e_2 = e^u \left(\frac{\partial}{\partial v} - \frac{\partial}{\partial w} \right), e_3 = -\frac{\partial}{\partial u}$$

Which are linearly independent at each point of M.

Let g be the Riemannian metric defined by,

$$g(e_i, e_j) = \begin{cases} 1 & \text{if} \quad i = j \\ 0 & \text{if} \quad i \neq j \end{cases} ; i, j = 1, 2, 3.$$

Let η be the 1-form defined by,

$$\eta(W) = g(W, e_3)$$

for any $W \in \chi(M)$. Let ϕ be the (1,1)-tensor field defined by

$$\phi(e_1) = e_1, \phi(e_2) = e_2, \phi(e_3) = 0$$

Using the linearity of ϕ and g, we have

$$\phi^2 W = W - \eta(W) e_3$$

and

$$\eta(e_3) = 1$$

Also,

$$g(\phi U, \phi W) = g(U, W) - \eta(U)\eta(W)$$

For any $U, W \in \chi(M)$. Thus $e_3 = \xi$, the structure (ϕ, ξ, η, g) defines an almost paracontact metric structure on M. For Levi-Civita connection ∇, we have

$$[e_1, e_2] = 0, [e_1, e_3] = e_1, [e_2, e_3] = e_2$$

The Levi-Civita connection ∇ of the metric g is given by Koszul's formula which is given by,

$$2g(\nabla_U V, W) = Ug(V, W) + Vg(W, U) - Wg(U, V) - g(U, [V, W]) - g(V, [U, W]) + g(W, [U, V])$$

Taking $e_3 = \xi$ and using Koszul's formula, we get

$$\begin{array}{ccc} \nabla_{e_1} e_1 = -e_3 & \nabla_{e_1} e_2 = 0 & \nabla_{e_1} e_3 = e_1, \\ \nabla_{e_2} e_1 = 0 & \nabla_{e_2} e_2 = -e_3 & \nabla_{e_2} e_3 = e_2, \\ \nabla_{e_3} e_1 = 0 & \nabla_{e_3} e_2 = 0 & \nabla_{e_3} e_3 = 0. \end{array} \tag{6.1}$$

From the above values, it is clear that (ϕ, ξ, η, g) is a Para-Sasakian structure on M, hence M is a 3-dimensional Para-Sasakian manifold.

Using the results from equation (6.1), we can obtain the components of the Riemannian curvature tensors with respect to the Levi-Civita connection ∇ as follows:

$$\begin{array}{ccc} R(e_1, e_2)e_2 = -e_1, & R(e_1, e_3)e_3 = -e_1, & R(e_2, e_1)e_1 = -e_2, \\ R(e_2, e_3)e_3 = -e_2, & R(e_3, e_1)e_1 = -e_3, & R(e_3, e_2)e_2 = -e_3, \\ R(e_1, e_2)e_3 = 0, & R(e_3, e_2)e_3 = e_2, & R(e_3, e_1)e_2 = 0. \end{array}$$

So, the Ricci tensor with respect to the Levi-Civita connection ∇ will be,

$$\begin{array}{ccc} S(e_1, e_1) = -2, & S(e_2, e_2) = -2, & S(e_3, e_3) = -2, \\ S(e_1, e_2) = 0, & S(e_2, e_3) = 0, & S(e_3, e_1) = 0. \end{array}$$

So, the scalar curvature r with respect to the Levi-Civita connection ∇, of the manifold will be,

$$r = -6$$

By using the values of (6.1) in (3.1), we obtain

$$\tilde{\nabla}_{e_i} e_j = 0$$

for $1 \leq i, j \leq 3$. Thus, the manifold M reduces to the Ricci-flat manifold with respect to the Schouten-van Kampen connection $\tilde{\nabla}$ as the Ricci curvature tensor for M vanishes with respect to the Schouten-van Kampen connection.

Hence, for Schouten-van Kampen connection, we have

$$\tilde{r} = 0$$

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