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On the Richard and Raoul numbers

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Padovan numbers, Perrin numbers, Generating functions, Binet-like formula, Binomial sum

Abstract — In this study, we define and examine the Richard and Raoul sequences and we deal with, in detail, two special cases, namely, Richard and Raoul sequences. We indicate that there are close relations between Richard and Raoul numbers and Padovan and Perrin numbers. Moreover, we present the Binet-like formulas, generating functions, summation formulas, and some identities for these sequences.

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1. Introduction

There has been a considerable deal of interest in the existing literature on the study of integer sequences such as Fibonacci, Lucas, Pell, and Jacobsthal and their applications in various scientific disciplines. One of the most studied sequences is the Padovan sequence $\{P_n\}_{n\geq 0}$, which is defined by the recurrence relation

$$P_{n+3} = P_{n+1} + P_n \tag{1.1}$$

with $P_0 = 1$, $P_1 = 1$, and $P_2 = 1$. The first few Padovan numbers are 1, 1, 1, 2, 2, 3, 4, 5, 7, 9, 12, 16, 21, 28. This sequence corresponds to the sequence A000931 in the online encyclopedia of integer sequences (OEIS) in [1].

This and similar sequences have been presented in various math articles, including numbers theory, analysis, calculus, applied mathematics, algebra, and statistics, as well as architectural, physics, and various scientific articles [2]. In [3], the Padovan p-numbers are defined and various properties are discussed. In [4], some families of Toeplitz-Hessenberg determinants the entries of which are the Padovan numbers are investigated. In [5], the Fermat numbers are determined in the Padovan and Perrin sequences. Matrices formula and sequences for the Padovan and Perrin sequences are given in [6, 7]. In [8], the Padovan and Pell-Padovan quaternions are introduced and their some properties are investigated. The split (s,t)-Padovan and (s,t)-Perrin quaternions are studied in [9]. In [10], some geometric interpretations of the plastic ratio

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related to the Padovan numbers are given. A historical analysis of the Padovan numbers is studied in [11]. Now let us give the definition of Perrin numbers which have the same recurrence as the Padovan numbers, but they have different initial conditions than the initial conditions of the Padovan numbers. Edouard Lucas (1876) studied Perrin numbers for the first time. However, the sequence was later named after Raoul Perrin, who worked on this sequence in 1899. The Perrin sequence $\{R_n\}_{n\geq 0}$ is defined by the recurrence

$$R_{n+3} = R_{n+1} + R_n \tag{1.2}$$

with $R_0 = 3$, $R_1 = 0$, and $R_2 = 2$. The first few Perrin numbers are 3, 0, 2, 3, 2, 5, 5, 7, 10, 12, 17, 22, 29, 39. This sequence corresponds to the sequence A001608 in the online encyclopedia of integer sequences (OEIS) in [12].

The recurrences (1.1) and (1.2) involve the characteristic equation

$$x^3 - x - 1 = 0 \tag{1.3}$$

If its roots are denoted by α, β, and γ then, the following equalities can be derived

$$\alpha + \beta + \gamma = 0$$

$$\alpha\beta + \alpha\gamma + \beta\gamma = -1$$

$$\alpha\beta\gamma = 1$$

Moreover, the Binet-like formula for the Padovan sequence is

$$P_n = a\alpha^n + b\beta^n + c\gamma^n \tag{1.4}$$

Here,

$$a = \frac{(\beta-1)(\gamma-1)}{(\alpha-\beta)(\alpha-\gamma)}, \quad b = \frac{(\alpha-1)(\gamma-1)}{(\beta-\alpha)(\beta-\gamma)}, \quad \text{and} \quad c = \frac{(\alpha-1)(\beta-1)}{(\gamma-\alpha)(\gamma-\beta)} \tag{1.5}$$

and the Binet-like formula for the Perrin sequence is

$$R_n = \alpha^n + \beta^n + \gamma^n \tag{1.6}$$

The negative indices of Padovan and Perrin numbers are obtained with the following recurrences, respectively:

$$P_n = P_{n+3} - P_{n+1} \text{ and } R_n = R_{n+3} - R_{n+1}$$

For $n \geq 0$, the Padovan and Perrin sequences the following identities are valid:

$$P_n = P_{n-1} + P_{n-5} \tag{1.7}$$

$$R_n = P_{n+1} + P_{n-10} \tag{1.8}$$

$$P_n^2 - P_{n+1}P_{n-1} = P_{-n-7} \tag{1.9}$$

$$\sum_{k=0}^{n} P_k = P_{n+5} - 2 \tag{1.10}$$

$$\sum_{k=0}^{n} P_{2k} = P_{2n+3} - 1 \tag{1.11}$$

$$\sum_{k=0}^{n} P_{2k+1} = P_{2n+4} - 1 \tag{1.12}$$

$$\sum_{k=0}^{n} P_k^2 = P_{n+2}^2 - P_{n-1}^2 - P_{n-3}^2 \tag{1.13}$$

$$\sum_{n=0}^{\infty} P_n x^n = \frac{x + x^2}{1 - x^2 - x^3} \tag{1.14}$$

$$\sum_{n=0}^{\infty} R_n x^n = \frac{3 - x^2}{1 - x^2 - x^3} \tag{1.15}$$

$$\sum_{n=0}^{m} \binom{m}{n} P_n = P_{3m} \tag{1.16}$$

$$\sum_{k=0}^{m} \binom{m}{k} P_{n-k} = P_{n+2m} \tag{1.17}$$

The above identities and properties can be seen in [13–15].

2. The Richard, Raoul, and A023434 Numbers

In this part, we give definitions of the Richard (for honor Richard Padovan) and Raoul (for honor Raoul Perrin) sequences and investigate some identities of these sequences such as the Binet-like formula, generating functions, certain binomial sums and various identities. Moreover, we define a new sequence associated with Richard and Raoul sequences, and also examine some identities of this sequence. Studies similar to the numbers in this work, namely generalized Edouard, Ernst, Oresme, Pisano and John numbers, were studied by Soykan [14, 16–21]. Catarino and Borges [22] studied the Leonardo numbers.

Definition 2.1. The Richard sequence $\{\mathscr{P}_n\}_{n\geq 0}$ is defined by the following recurrence

$$\mathscr{P}_{n+3} = \mathscr{P}_{n+1} + \mathscr{P}_n + 1, \quad \mathscr{P}_0 = \mathscr{P}_1 = \mathscr{P}_2 = 1 \tag{2.1}$$

The first few terms of the Richard numbers are 1, 1, 1, **3**, **3**, **5**, **7**, 9, **13**, **17**, **23**, **31**, **41**, 55, **73**, **97**, 129, 171, **227**, 301, 399, 529, **701**, 929, **1231**, 1631, **2161**, 2863, **3793**, 5025, 6657, **8819**, 11683, 15477, 20503, 27161 where bold values are prime numbers and every Richard numbers are odd, which is easily verified. Furthermore, the unit digits of Richard numbers are periodic with period 24.

Lemma 2.2. For each $n \geq 0$, the Richard number $\mathscr{P}_n$ is an odd number.

Proof.

We prove this by using the induction method on n. For $n = 0,1,2$, the assertion is true. Assume that the assertion is valid for $2 < k \leq n$. Then, we may verify it for $n+1$. Since by recurrence (2.1) we have $\mathscr{P}_{n+4} = \mathscr{P}_{n+2} + \mathscr{P}_{n+1} + 1$ and as the sum of two odd numbers which are $\mathscr{P}_{n+2}$ and $\mathscr{P}_{n+1}$ by induction hypothesis is even and, in order, the sum of an even number with the number 1 is an odd number, The proof is finished.

The recurrence (2.1) can also be written as follows

$$\mathscr{P}_{n+4} = \mathscr{P}_{n+3} + \mathscr{P}_{n+2} - \mathscr{P}_n \tag{2.2}$$

In fact, by the equalities $\mathscr{P}_{n+4} = \mathscr{P}_{n+2} + \mathscr{P}_{n+1} + 1$ and $\mathscr{P}_{n+3} = \mathscr{P}_{n+1} + \mathscr{P}_n + 1$, we reach equality (2.2)

The recurrence (2.2) involves the characteristic equation

$$r^4 - r^3 - r^2 + 1 = 0 \tag{2.3}$$

The roots of the characteristic Equation (2.3) are 1, α, β, and γ where the other roots except 1 are the same as the roots of the characteristic Equation (1.3).

If we take the initial conditions of the above recurrence (2.2) as $\mathscr{P}_0 = \mathscr{P}_1 = \mathscr{P}_2 = 1$, and $\mathscr{P}_3 = 3$, we can easily reach the following Binet-like formula.

Theorem 2.3. The Binet-like formula for the Richard sequence is

$$\mathscr{P}_n = 2a\alpha^n + 2b\beta^n + 2c\gamma^n - 1 \tag{2.4}$$

where the values a, b, c are given in Equation (1.5).

Proof.

Assume that $\mathscr{P}_n = u\alpha^n + v\beta^n + w\gamma^n + k$. So, we have

$$\mathscr{P}_0 = u + v + w + k = 1$$

$$\mathscr{P}_1 = u\alpha + v\beta + w\gamma + k = 1$$

$$\mathscr{P}_2 = u\alpha^2 + v\beta^2 + w\gamma^2 + k = 1$$

$$\mathscr{P}_3 = u\alpha^3 + v\beta^3 + w\gamma^3 + k = 3$$

By performing the solution with the Gaussian elimination method, we find $u = 2a$, $v = 2b$, $w = 2c$, and $k = -1$.

Theorem 2.4. For $n \geq 0$, the following identity is valid:

$$\mathscr{P}_n = 2P_n - 1 \tag{2.5}$$

Proof.
Using te identities (2.4) and (1.4), we get

$$\begin{aligned}\mathscr{P}_n &= 2a\alpha^n + 2b\beta^n + 2c\gamma^n - 1\\ &= 2\left(a\alpha^n + b\beta^n + c\gamma^n\right) - 1\\ &= 2P_n - 1\end{aligned}$$

Definition 2.5. The Raoul sequence $\{\mathscr{R}_n\}_{n\geq 0}$ is defined by the following recurrence

$$\mathscr{R}_{n+3} = \mathscr{R}_{n+1} + \mathscr{R}_n + 1, \quad \mathscr{R}_0 = 3, \quad \mathscr{R}_1 = 0, \quad \text{and} \quad \mathscr{R}_2 = 2 \tag{2.6}$$

The first few terms of the Raoul numbers are 3, 0, 2, 4, 3, 7, 8, 11, 16, 20, 28, 37, 49, 66, 87, 116, 154, 204, 271. Similarly, the recurrence (2.6) can also be written as follows

$$\mathscr{R}_{n+4} = \mathscr{R}_{n+3} + \mathscr{R}_{n+2} - \mathscr{R}_n \tag{2.7}$$

If we take the initial conditions of the above recurrence (2.7) as $\mathscr{R}_0 = 3$, $\mathscr{R}_1 = 0$, $\mathscr{R}_2 = 2$, and $\mathscr{R}_3 = 4$, we can easily reach the following Binet-like formula.

Theorem 2.6. The Binet-like formula for the Raoul sequence is

$$\mathscr{R}_n = (a+1)\alpha^n + (b+1)\beta^n + (c+1)\gamma^n - 1 \tag{2.8}$$

where the values a, b, c are given in Equation (1.5).

Proof.
Assume that $\mathscr{R}_n = x\alpha^n + y\beta^n + z\gamma^n + l$. Therefore, we have

$$\mathscr{R}_0 = x + y + z + l = 3$$

$$\mathscr{R}_1 = x\alpha + y\beta + z\gamma + l = 0$$

$$\mathscr{R}_2 = x\alpha^2 + y\beta^2 + z\gamma^2 + l = 2$$

$$\mathscr{R}_3 = x\alpha^3 + y\beta^3 + z\gamma^3 + l = 4$$

By performing the solution with the Gaussian elimination method, we find $x = a+1$, $y = b+1$, $z = c+1$, and $l = -1$. Now, we give a new integers sequence (called the sequence A023434) which we will relate to some sequences such as Padovan, Perrin, Richard and Raoul sequences.

Definition 2.7. The sequence A023434 $\{\mathscr{T}_n\}_{n\geq 0}$ is defined by the following recurrence relation

$$\mathscr{T}_{n+4} = \mathscr{T}_{n+3} + \mathscr{T}_{n+2} - \mathscr{T}_n, \quad \mathscr{T}_0 = 0, \quad \mathscr{T}_1 = 0, \quad \mathscr{T}_2 = 0, \quad \text{and} \quad \mathscr{T}_3 = 1 \tag{2.9}$$

The first few numbers of the sequence A023434 are 0, 0, 0, 1, 1, 2, 3, 4, 6, 8, 11, 15, 20, 27, 36, 48, 64, 85. This sequence corresponds to the sequence A023434 in the online encyclopedia of integer sequences (OEIS) in [23].

Theorem 2.8. The Binet-like formula for the sequence A023434 is

$$\mathscr{T}_n = a\alpha^n + b\beta^n + c\gamma^n - 1 \tag{2.10}$$

where the values a, b, c are given in Equation (1.5).

Proof.
It is proved similarly to the proofs of Theorem 2.3 and Theorem 2.6.
The negative indices of Richard, Raoul and A023434 numbers are obtained with the following recurrences, respectively:

$$\mathscr{P}_n = \mathscr{P}_{n+3} - \mathscr{P}_{n+1} - 1$$
$$\mathscr{R}_n = \mathscr{R}_{n+3} - \mathscr{R}_{n+1} - 1$$
$$\mathscr{T}_n = \mathscr{T}_{n+3} + \mathscr{T}_{n+2} - \mathscr{T}_{n+4}$$

The relations of the above sequences with each other are given below.

Proposition 2.9. For $n \geq 0$, the following identities are valid:

i. $\mathscr{R}_n = P_n + R_n - 1$

ii. $\mathscr{P}_n = 2\mathscr{R}_n - 2R_n + 1$

iii. $\mathscr{T}_n = P_n - 1$

iv. $\mathscr{R}_n = R_n + \mathscr{T}_n$

v. $\mathscr{P}_n = 2\mathscr{T}_n + 1$

Proof.
By using the identities (1.4), (1.6), (2.4), (2.8), (2.9), and (2.10), the above identities are proved.

Proposition 2.10. For $n \geq 0$, the following identities are valid:

i. $\mathscr{P}_n = \mathscr{P}_{n-1} + \mathscr{P}_{n-5} + 1$

ii. $\mathscr{R}_n = \dfrac{\mathscr{P}_{n+3} + \mathscr{P}_{n-10}}{2}$

iii. $\mathscr{P}_n^2 - \mathscr{P}_{n+1}\mathscr{P}_{n-1} = 2\mathscr{P}_{-n-7} - 2\mathscr{P}_n + \mathscr{P}_{n+1} + \mathscr{P}_{n-1} + 2$

Proof.
By using the identities (1.7), (1.8), (1.9), and (2.5) the above identities are proved.

Proposition 2.11. For $n \geq 0$, the following sum formulas of terms of the Richard sequence are valid:

i. $\sum_{k=0}^{n} \mathscr{P}_k = \mathscr{P}_{n+5} - n - 3$

ii. $\sum_{k=0}^{n} \mathscr{P}_{2k} = \mathscr{P}_{2n+3} - n - 1$

iii. $\sum_{k=0}^{n} \mathscr{P}_{2k+1} = \mathscr{P}_{2n+4} - n - 1$

iv. $\sum_{k=0}^{n} \mathscr{P}_k^2 = \mathscr{P}_{n+2}^2 - \mathscr{P}_{n-1}^2 - \mathscr{P}_{n-3}^2 + 2\mathscr{P}_{n-2} - \mathscr{P}_{n+5} + n + 6$

Proof.
By using the identities (1.10), (1.11), (1.12), (1.13), and (2.5) the above identities are proved.

Theorem 2.12. The generating functions of the Richard, Raoul and A023434 sequences are as follows:

i. $\sum_{n=0}^{\infty} \mathscr{P}_n x^n = \frac{-x^3+x^2+2x-1}{(1-x)(1-x^2-x^3)}$

ii. $\sum_{n=0}^{\infty} \mathscr{R}_n x^n = \frac{x^3-2x+2}{(1-x)(1-x^2-x^3)}$

iii. $\sum_{n=0}^{\infty} \mathscr{T}_n x^n = \frac{2x^3+x^2}{(1-x)(1-x^2-x^3)}$

Proof.
By using the identities (1.14), (1.15), and (2.5) the above identities are proved.
The reader can look for similar results in the above for the Raoul sequence.

Theorem 2.13. The exponential generating functions of the Richard, Raoul, and A023434 sequences are as follows:

i. $\sum_{n=0}^{\infty} \mathscr{P}_n \frac{t^n}{n!} = 2ae^{\alpha t} + 2be^{\beta t} + 2ce^{\gamma t} - e^t$

ii. $\sum_{n=0}^{\infty} \mathscr{R}_n \frac{t^n}{n!} = (a+1)e^{\alpha t} + (b+1)e^{\beta t} + (c+1)e^{\gamma t} - e^t$

iii. $\sum_{n=0}^{\infty} \mathscr{T}_n \frac{t^n}{n!} = ae^{\alpha t} + be^{\beta t} + ce^{\gamma t} - e^t$

Proof.
By using the identities (2.4), (2.8), and (2.10) the above identities are proved.

Proposition 2.14. For $n \geq 0$, the following identities are valid:

i. $\sum_{n=0}^{m} \binom{m}{n} \mathscr{P}_n = 2\mathscr{P}_{3m} + 1 - 2^m$

ii. $\sum_{k=0}^{m} \binom{m}{k} \mathscr{P}_{n-k} = 2\mathscr{P}_{n+2m} + 1 - 2^m$

Proof.
By using the identities (1.16), (1.17), and (2.5) the above identities are proved.

The reader can look for similar results in the above for the Raoul and sequence A023434.

3. Conclusion

In this present work, the sequences of the Richard and Raoul numbers are introduced. In addition, the sequence A023434 associated with these sequences is identified. The Binet-like formulas, generating functions, and a few identities, among other characteristics affecting these sequences, are presented. There are also created a number of expressions including sums and products with the terms of these sequences. In the future, the various new properties of these sequences can be examined.

Author Contributions

All the authors contributed equally to this work. They all read and approved the last version of the paper.

Conflicts of Interest

The authors declare no conflict of interest.

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