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Non-classical periodic boundary value problems with impulsive conditions

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Abstract — This study investigates some spectral properties of a new type of periodic Sturm-Liouville problem. The problem under consideration differs from the classical ones in that the differential equation is given on two disjoint segments that have a common end, and two additional interaction conditions are imposed on this common end (such interaction conditions are called various names, including transmission conditions, jump conditions, interface conditions, impulsive conditions, etc.). At first, we proved that all eigenvalues are real and there is a corresponding real-valued eigenfunction for each eigenvalue. Then we showed that two eigenfunctions corresponding to different eigenvalues are orthogonal. We also defined some left and right-hand solutions, in terms of which we constructed a new transfer characteristic function. Finally, we have defined asymptotic formulas for the transfer characteristic functions and also for the eigenvalues. The results obtained are a generalization of similar results of the classical Sturm-Liouville theory.

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1. Introduction

Since the middle of the 19th century, an extensive theory of Sturm-Liouville problems (SLP), as well as the spectral theory of linear differential operators in Hilbert spaces, has been developed in connection with applications in physics and engineering. Many mathematical physics problems, such as heat and mass transfer problems, the vibrations of a drum membrane or violin strings, and the motion of a particle in the matter, are modeled by SLPs [1]. For example, consider the wave equation

$$sw_{yy} = div\,(q\nabla w) - tw$$

with $w = w(x, y)$ where x changes in the domain of interest belonging to the Euclidean space $\mathbb{R}^2$ or $\mathbb{R}^3$, $s(x) > 0$, $q(x) \geq 0$, and $t(x) \geq 0$ are given functions of the spatial variables. Using the separation of independent variables method, we will look for a separate solution of the form $w = \upsilon(x)\theta(y)$. Such

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a separated solution satisfies the wave equation if and only if

$$\frac{\theta''}{\theta} = \frac{div\,(q\nabla \upsilon) - t\upsilon}{(s-q)\upsilon}$$

holds for all (x, y). Both sides of this equality must be constant, here λ. Consequently, we have a simplest Sturm-Liouville equation

$$-\theta'' + \lambda\theta = 0$$

We know that, in typical real-world applications, the separation parameter λ is a negative number. So the separated solutions $w(x, y) = \theta(y)\upsilon(x)$ are periodic in time variable y. In particular, Sturm-Liouville equations with real periodic coefficients and/or with periodic boundary conditions have been paid much attention due to the need to solve many periodic phenomena that arise in natural sciences. A description of the most qualitative properties of periodic differential equations of second order, in part summarized, is given in books [1, 2]. Further results relating to the spectral theory of differential equations are given in [3–5]. Various methods have also been developed to solve various types of Sturm-Liouville problems (see, for example, [2–8] and references cited therein)

Recently, there has been an increasing interest in Sturm-Liouville boundary value problems defined on two or more disjoint segments with common ends, the so-called many-interval SLPs (see, for example, [9–21] and references cited therein). To deal with such multi-interval boundary value problems, naturally, additional conditions (the so-called transmission conditions, jump conditions, interface conditions, and impulsive conditions) are imposed at these common endpoints.

In this study, we will consider the two-interval Sturm-Liouville equation

$$-u''(x) + (q(x) - \lambda)\,u(x) = 0,\ x \in [-1, 0) \cup (0, 1] \tag{1.1}$$

together with periodic boundary conditions, given by

$$u(-1) = u(1) \tag{1.2}$$

$$u'(-1) = u'(1) \tag{1.3}$$

and with additional conditions (the so-called transmission conditions or impulsive conditions) at the point of interaction $x = 0$, given by

$$u(0+) = u'(0-) \tag{1.4}$$

$$u'(0+) = \alpha u'(0-) - u(0-) \tag{1.5}$$

where α is a real number and $q(x)$ is a real-valued function which is continuous on each of intervals $[-1, 0)$ and $(0, 1]$ and has a finite left and right-hand limit values $q(0\pm) = \lim\limits_{x\to 0\pm} q(x)$.

2. Preliminaries

This section presents some of the basic notions employed in the following sections.

Theorem 2.1. If λ is an eigenvalue of the Sturm-Liouville boundary-value transmission problem (1.1)-(1.5), then it must be a real number.

Proof.

For brevity, we shall use the following notations

$$Lu := -u''(x) + q(x)\,u$$

$$P_1(u) := u(-1) - u(1)\,, P_2(u) := u'(-1) - u'(1)$$

$$P_3(u) := u(0+) - u'(0-), P_4(u) := u'(0+) - \alpha u'(0-) + u(0-)$$

Let $u(x,\lambda)$ be an eigenfunction belonging to the eigenvalue λ, that is

$$Lu(.,\lambda) = \lambda\ u(.,\lambda) \tag{2.1}$$

$$P_i(u(.,\lambda)) = 0, \quad i = 1,2,3,4 \tag{2.2}$$

Then, by taking the complex-conjugates of the (2.1)-(2.2) and keeping in mind that the function $q(x)$ is real valued and the coefficient α is a real number, we see that

$$L\overline{u(.,\lambda)} = \overline{\lambda}\ \overline{u(.,\lambda)} \tag{2.3}$$

$$P_i\left(\overline{u(.,\lambda)}\right) = 0, \quad i = 1,2,3,4$$

It means that the function $\overline{u(x,\lambda)}$ is an eigenfunction belonging to the eigenvalue $\overline{\lambda}$. Using the above Equalities (2.1) and (2.3) we have

$$\overline{u(x,\lambda)}Lu(x,\lambda) - u(x,\lambda)\,L\overline{u(x,\lambda)} = \left(\lambda - \overline{\lambda}\right) u(x,\lambda)\,\overline{u(x,\lambda)} \tag{2.4}$$

Integrating (2.4) over $[-1,0)$ and $(0,1]$, we arrive at

$$\begin{aligned}\left(\lambda - \overline{\lambda}\right)\left(\int_{-1}^{0-} |\, u(x,\lambda)\,|^2 dx + \int_{0+}^{1} |\, u(x,\lambda)\,|^2 dx\right) &= W\left(u(.,\lambda),\ \overline{u(.,\lambda)};1\right) - W\left(u(.,\lambda),\ \overline{u(.,\lambda)};-1\right) \\ &\quad + W\left(u(.,\lambda),\ \overline{u(.,\lambda)};0-\right) \\ &\quad + W\left(u(.,\lambda),\ \overline{u(.,\lambda)};0+\right)\end{aligned}$$

where $W\left(u(.,\lambda),\ \overline{u(.,\lambda)};x\right)$ is the Wronskian of $u(x,\lambda)$ and $\overline{u(x,\lambda)}$, that is,

$$W\left(u(.,\lambda),\ \overline{u(.,\lambda)};x\right) = u(x,\lambda)\ \overline{u'(x,\lambda)} - \overline{u(x,\lambda)}u'(x,\lambda)$$

Using boundary and transmission conditions, we get

$$W\left(u(.,\lambda),\ \overline{u(.,\lambda)};1\right) = W\left(u(.,\lambda),\ \overline{u(.,\lambda)};-1\right) \tag{2.5}$$

and

$$\begin{aligned}W\left(u(.,\lambda),\ \overline{u(.,\lambda)};0+\right) &= u(0+,\lambda),\ \overline{u'(0+,\lambda)} - \overline{u(0+,\lambda)}u'(0+,\lambda) \\ &= u'(0-,\lambda)\left(\alpha\overline{u'(0-,\lambda)} - \overline{u(0-,\lambda)}\right) - \overline{u'(0-,\lambda)}(\alpha u'(0-,\lambda) - u(0-,\lambda)) \\ &= W\left(u(.,\lambda),\ \overline{u(.,\lambda)};0-\right) \qquad (2.6)\end{aligned}$$

respectively.

Equalities (2.5) and (2.6) permit us to restate Equality (2.5) as

$$\left(\lambda - \overline{\lambda}\right)\left(\int_{-1}^{0-} |\, u(x,\lambda)\,|^2 dx + \int_{0+}^{1} |\, u(x,\lambda)\,|^2 dx\right) = 0 \tag{2.7}$$

Since the eigenfunction $u(x,\lambda)$ is a non-trivial solution of the problem (1.1)-(1.5), the last Equality (2.7) implies that $\lambda = \overline{\lambda}$. Thus the eigenvalue λ is real. □

Theorem 2.2. For every given eigenvalue λ, there is a real-valued eigenfunction corresponding to the given eigenvalue λ.

Proof.

Let $u(x,\lambda) = \eta(x,\lambda) + i\zeta(x,\lambda)$ be an eigenfunction corresponding to the eigenvalue λ, where η and ζ are real-valued functions. If we make the substitution $u = \eta + i\zeta$ in the problem (1.1)-(1.5) and separate real and imaginary parts, we have that both real-valued functions $\eta(x,\lambda)$ and $\zeta(x,\lambda)$ are themselves eigenfunctions corresponding to the same eigenvalue λ. $\square$

Definition 2.3. Let $L_2(-1,0)\oplus L_2(0,1)$ be the space consisting of all square-integrable functions on each of the intervals $(-1,0)$ and $(0,1)$. Then the number $\langle f,g\rangle$ defined by

$$\langle f,g\rangle := \int_{-1}^{0-} f\left(x\right)\overline{g\left(x\right)}dx + \int_{0+}^{1} f\left(x\right)\overline{g\left(x\right)}dx$$

is said to be the inner product of the functions $f\left(x\right)$ and $g\left(x\right)$.

Definition 2.4. Two eigenfunctions $u(x)$ and $v(x)$ are called orthogonal on the two-interval $[-1,0)\cup(0,1]$, if

$$\int_{-1}^{0-} u\left(x\right)\overline{v\left(x\right)}dx + \int_{0+}^{1} u\left(x\right)\overline{v\left(x\right)}dx = 0$$

Theorem 2.5. If λ_1 and λ_2 are distinct eigenvalues of the problem $(1.1)-(1.5)$, then the corresponding eigenfunctions $u\left(x,\lambda_1\right)$ and $u\left(x,\lambda_2\right)$ are orthogonal.

Proof.

Multiplying the identities $Lu\left(x,\lambda_1\right) = \lambda_1 u\left(x,\lambda_1\right)$ and $L\overline{u\left(x,\lambda_2\right)} = \lambda_2\overline{u\left(x,\lambda_2\right)}$ by $u\left(x,\lambda_2\right)$ and $u\left(x,\lambda_1\right)$, respectively, and then subtracting one from another, then using the well-known Lagrange identity [3], we have

$$W\left(u\left(.,\lambda_1\right),\overline{u\left(.,\lambda_2\right)};x\right) = \left(\lambda_1-\lambda_2\right)u\left(x,\lambda_1\right)u\left(x,\lambda_2\right)$$

Integrating this equality over the intervals $[-1,0)\cup(0,1]$, yields

$$\left(\lambda_1-\lambda_2\right)\quad\left(\int_{-1}^{0-} u\left(x,\lambda_1\right)\overline{u\left(x,\lambda_2\right)}dx + \int_{0+}^{1} v\left(x,\lambda_1\right)\ \overline{v(x,\lambda_2)}dx\right)$$

$$= \left(Wu\left(.,\lambda_1\right),u\left(.,\lambda_2\right);x\right)|_{-1}^{1} + \left(Wu\left(.,\lambda_1\right),u\left(.,\lambda_2\right);x\right)|_{0-}^{0+}$$

As in the proof of the previous theorem, we can show that the right side of the last equality is equal to zero. Hence, the left side of this equality is also zero. Consequently, $\lambda_1\neq\lambda_2$. $\square$

Theorem 2.6. The periodic Sturm-Liouville boundary value transmission problem (1.1)-(1.5) is self-adjoint.

Proof.

Let ω and $\theta\in L^2[-1,0)\oplus L^2(0,1]$ that satisfies the given problem $(1.1)-(1.5)$. Let it be

$$\Phi[\omega] := -\omega''(x)+q(x)\omega(x)$$

and

$$\Phi[\theta] := -\theta''(x)+q(x)\theta(x)$$

Multiplying the first by θ and the second by w and then subtracting yields,

$$\omega\Phi[\theta]-\theta\Phi[\omega] = \theta\omega''-\omega\theta'' = \frac{d}{dx}\left(\theta\omega'-\omega\theta'\right)$$

By using, well known integral form of Lagrange's identity (see, for example, [3]), we obtain

$$\langle\omega,\Phi[\theta]\rangle-\langle\Phi[\omega],\theta\rangle = \left(\theta\frac{d\omega}{dx}-\omega\frac{d\theta}{dx}\right)_{-1}^{0-} + \left(\theta\frac{d\omega}{dx}\omega-\omega\frac{d\theta}{dx}\right)_{0+}^{1}$$

That is,

$$\int_{-1}^{0-} (\omega\Phi[\theta] - \theta\Phi[\omega])\, dx + \int_{0+}^{1} (\omega\Phi[\theta] - \theta\Phi[\omega])\, dx = \left(\theta\frac{d\omega}{dx} - \omega\frac{d\theta}{dx}\right)_{-1}^{0-} + \left(\theta\frac{d\omega}{dx}\omega - \omega\frac{d\theta}{dx})\right)_{0+}^{1} \tag{2.8}$$

Since ω and θ satisfy the boundary transmission conditions $(1.4) - (1.5)$, we have

$$\left(\theta\frac{d\omega}{dx} - \omega\frac{d\theta}{dx}\right)_{-1}^{0-} + \left(\theta\frac{d\omega}{dx} - \omega\frac{d\theta}{dx}\right)_{0+}^{1} = 0$$

Consequently, by Equality (2.8), we get

$$\langle\omega, \Phi[\theta]\rangle = \langle\Phi[\omega], \theta\rangle$$

□

3. Asymptotic Behaviours of the Left-Hand and Right-Hand Solutions

Let $v_1(x,\lambda)$ and $w_1(x,\lambda)$ be solutions of Equation (1.1) on the left interval $[-1,0)$ satisfying the initial conditions $v_1(-1,\lambda) = 1, v_1'(-1,\lambda) = 0$, and $w_1(-1,\lambda) = 0$, $w_1'(-1,\lambda) = 1$, respectively. Similarly, let $v_2(x,\lambda)$ and $w_2(x,\lambda)$ be solutions of Equation (1.1) on the right interval $(0,1]$, satisfying the initial conditions $v_2(1,\lambda) = 1$, $v_2'(1,\lambda) = 0$, and $w_2(1,\lambda) = 0, w_2'(1,\lambda) = 1$, respectively. We know that the functions $v_i(x,\lambda)$ and $w_i(x,\lambda)$ are entire functions of complex variable λ for each fixed x (see, [5]).

Theorem 3.1. Let $\lambda = z^2, z = t + is; t, s \in \mathbb{R}$. The following asymptotic formulas are valid as $|\lambda|$ tends to infinity.

$$v_1(x,\lambda) = \cos((1+x)z) + O\left(\frac{1}{|z|}\exp((1+x)|s|)\right) \tag{3.1}$$

$$v_1'(x,\lambda) = -z\sin((1+x)z) + O\left(\exp((1+x)|s|)\right) \tag{3.2}$$

$$w_1(x,\lambda) = \frac{1}{z}\sin((1+x)z) + O\left(\frac{1}{|z|^2}\exp((1+x)|s|)\right) \tag{3.3}$$

$$w_1'(x,\lambda) = \cos((1+x)z) + O\left(\frac{1}{|z|}\exp((1+x)|s|)\right) \tag{3.4}$$

$$v_2(x,\lambda) = \cos((1-x)z) + O\left(\frac{1}{|z|}\exp((1-x)|s|)\right) \tag{3.5}$$

$$v_2'(x,\lambda) = z\sin((1-x)z) + O\left(\exp((1-x)|s|)\right) \tag{3.6}$$

$$w_2(x,\lambda) = -\frac{1}{z}\sin((1-x)z) + O\left(\frac{1}{|z|^2}\exp((1-x)|s|)\right) \tag{3.7}$$

$$w_2'(x,\lambda) = \cos((1-x)z) + O\left(\frac{1}{|z|}\exp((1-x)|s|)\right) \tag{3.8}$$

Proof.

The proof is similar to the Lemma 1.7. in [4]. □

4. The Transfer-Characteristic Function

This section defines a new concept for the problem (1.1)-(1.5), which we call the transfer-characteristic functions.

Definition 4.1. The determinant

$$A(\lambda) := \begin{vmatrix} v_1'(0-,\lambda) - v_2(0+,\lambda) & w_1'(0-,\lambda) - w_2(0+,\lambda) \\ \alpha v_1'(0-,\lambda) - v_1(0-,\lambda) - v_2'(0+,\lambda) & \alpha w_1'(0-,\lambda) - w_1(0-,\lambda) - w_2'(0+,\lambda) \end{vmatrix}$$

is called the transfer-characteristic function. Here v_1, v_2, w_1, w_2 are functions defined as in section 3.

Theorem 4.2. The number λ is an eigenvalue of the problem (1.1)-(1.5) if and only if $A(\lambda) = 0$.

Proof.

Since

$$W\left(v_1\left(.,\lambda\right),\ w_1\left(.,\lambda\right);-1\right) = W\left(v_2\left(.,\lambda\right),\ w_2\left(.,\lambda\right);1\right) = 1 \tag{4.1}$$

the left hand solutions v_1 and w_1 are linearly independent in the left interval $[-1, 0)$ and the right hand solutions v_2 and w_2 are linearly independent in the right interval $(0, 1]$. Therefore, the general solution of the two-interval differential equation (1.1) can be represented in the form

$$u(x,\lambda) = \begin{cases} c_1 v_1(x,\lambda) + c_2 w_1(x,\lambda), x \in [-1,0) \\ c_3 v_2(x,\lambda) + c_4 w_2(x,\lambda), x \in (0,1] \end{cases}$$

If we try to satisfy the periodic boundary conditions (1.2)-(1.3), then we have $c_1 = c_3$ and $c_2 = c_4$. The requirement for validity of the transmission conditions (1.4)-(1.5) gives the following system of equations

$$\begin{cases} c_1 v_1'(0-,\lambda) - v_2(0+,\lambda) + c_2(w_1'(0-,\lambda) - w_2(0+,\lambda)) = 0 \\ c_1\left(\alpha v_1'(0-,\lambda) - v_1(0-,\lambda) - v_2'(0+,\lambda)\right) + c_2\left(\alpha w_1'(0-,\lambda) - w_1(0-,\lambda) - w_2'(0+,\lambda)\right) = 0 \end{cases}$$

For this system of linear equations to have a non-trivial solution (with respect to the variables c_1, c_2). We would need the determinant of the coefficient matrix $A(\lambda)$ would need to be zero. □

Theorem 4.3. Let $\lambda = z^2, z = t + is : t, s \in \mathbb{R}$. Then, the transfer-characteristic function $A(\lambda)$ satisfies the asymptotic formula

$$A(\lambda) = z\sin(2z) + O\left(\exp|2s|\right),\ \text{as } |\lambda| \to \infty \tag{4.2}$$

Proof.

By using asymptotic formulas (3.1)-(3.8) we can show that

$$v_1(0-,\lambda) = \cos(z) + O\left(\frac{1}{|z|}\exp|s|\right)$$

$$v_1'(0-,\lambda) = -z\sin(z) + O\left(\exp|s|\right)$$

$$w_1(0-,\lambda) = \frac{1}{z}\sin(z) + O\left(\frac{1}{|z|^2}\exp|s|\right)$$

$$w_1'(0-,\lambda) = \cos(z) + O\left(\frac{1}{|z|}\exp|s|\right)$$

$$v_2(0+,\lambda) = \cos(z) + O\left(\frac{1}{|z|}\exp|s|\right)$$

$$v_2'(0+,\lambda) = z\sin(z) + O\left(\exp|s|\right)$$

$$w_2(0+,\lambda) = -\frac{1}{z}\sin(z) + O\left(\frac{1}{|z|^2}\exp|s|\right)$$

$$w_2'(0+,\lambda) = \cos(z) + \left(\frac{1}{|z|}\exp|s|\right)$$

as $|\lambda| \to \infty$. Substituting these into ($A(\lambda) = 0$) and simplifying, we arrive at the wanted formula (4.2). □

Theorem 4.4. The problem (1.1)-(1.5) has a countable number of eigenvalues $\lambda_1, \lambda_2, ...$ for which the following asymptotic formula holds

$$z_n = \frac{n\pi}{2} + O\left(\frac{1}{n}\right).\lambda_n = z_n^2$$

Proof.

It is easy to see that the transfer-characteristic function $A(\lambda)$ is entire function and has countable many zeros. Take a circle S_n of the radius $r_n = \frac{n\pi}{2} + \frac{\pi}{4}$ in the z-plane, where n is a sufficiently large natural number. By using the well-known Rouche Theorem (see, for example, [3]), we can show that there are as many zeros of $A(\lambda)$ inside S_n as of the function $A_0(z^2) = z\sin(2z)$, i.e. $2n + 2$. Consequently,

$$z_n = \frac{n\pi}{2} + a_n,\ n = 1, 2, ...$$

where a_n is a bounded sequence and $\mid a_n \mid< \frac{\pi}{4}$ for each n. The equation $A(\lambda_n) = 0$ then takes the form

$$\left(\frac{n\pi}{2} + a_n\right)\sin\left(2\left(\frac{n\pi}{2} + a_n\right)\right) = 0$$

From this, it follows that $a_n = O(\frac{1}{n})$. Then, we have

$$z_n = \frac{n\pi}{2} + O\left(\frac{1}{n}\right)$$

□

5. Conclusion

In this paper, we study a new type of periodic boundary-value-transmission problem and generalize some results of classical Sturm-Lioville problems. This new approach may be further developed in the future by adding parameters to the boundary conditions of problems of the type described herein. Additionally, the properties of eigenvalues and eigenfunctions can be investigated for these problems that involve parameters in their boundary conditions.

Author Contributions

All the authors equally contributed to this work. They all read and approved the final version of the paper.

Conflicts of Interest

All the authors declare no conflict of interest.

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