Communication in Mathematical Modeling and Applications

http://ntmsci.com/cmma

Characteristic properties of the parallel ruled surfaces with Darboux frame in Euclidean 3- space

Yasin Unluturk[1], *Muradiye Cimdiker*[1] *and Cumali Ekici*[2]

[1]Department of Mathematics, Kirklareli University, Kirklareli,Turkey
[2]Department of Mathematics-Computer, Eskisehir Osmangazi University, Eskisehir, Turkey

Received: 22 February 2016, Revised: 7 April 2016, Accepted: 12 April 2016
Published online: 17 April 2016

Abstract: In this paper, the parallel ruled surfaces with Darboux frame are introduced in Euclidean 3-space. Then some characteristic properties such as developability, striction point and distribution parameter of the parallel ruled surfaces with Darboux frame are given in Euclidean 3-space. Then we obtain Steiner rotation vector of this kind of surfaces Euclidean 3-space. By using this rotation vector, we compute pitch length and pitch angle of the parallel ruled surfaces with Darboux frame.

Keywords: Parallel ruled surface, Darboux frame, developability, Steiner rotation vector, pitch length, pitch angle.

1 Introduction

A parallel ruled surface is a surface represented by points which are at a constant distance along the normal vector of a ruled surface. As it is well known, both ruled surfaces and parallel surfaces have some applications in many industrial processes such as manufacturing for toolpath generation in sculptured surface machining and also in rapid prototyping, to fabricate additively a solid object or assembly from CAD models by using 3D printing technologies such as laser sintering, stereolithography, and laminated object manufacturing. Ruled surfaces have been widely applied in designing cars, ships, manufacturing of products and many other areas such as motion analysis and simulation of rigid body and model-based object recognition systems. Modern surface modeling systems include ruled surfaces. The geometry of ruled surfaces is essential for studying kinematical and positional mechanisms in E^3 and E_1^3(Aydemir & Kasap, 2005, Çöken et al., 2008; Karadağ et al., 2014; Ravani & Ku, 1991; Şentürk & Yüce; 2015; Şentürk & Yüce; 2015a; Ünlütürk, Y. & Ekici, C. 2014).

In differential geometry, ruled surface is a special type of surface which can be defined by choosing a curve and a line along that curve. The ruled surfaces are one of the easiest of all surfaces to parametrize. That surface was found and investigated by Gaspard Monge who established the partial differential equation that satisfies all ruled surfaces. Hlavaty (1945) and Hoschek (1973) also investigated ruled surfaces which are formed by one parameter set of lines. In addition, H.R. Müller (1978) showed that the pitch of closed ruled surfaces are integral invariants.

One of the another most important subjects of the differential geometry is the Darboux frame which is a natural moving frame constructed on a surface. It is the version of the Frenet frame that applied to surface geometry. A Darboux frame exists on a surface in a Euclidean or non-Euclidean spaces. It is named after the French mathematician Jean Gaston Darboux in the four volume collection of studies he published between 1887 and 1896. Since that time, there have been

* Corresponding author e-mail: yasinunluturk@klu.edu.tr

many important repercussions of Darboux frame, having been examined for example, (O'Neill, 1996; Darboux, 1896). Some characterizations and integral invariants of the ruled surfaces with Darboux frame was studied in E^3(Şentürk & Yüce, 2015). The same authors gave some characterizations of the involute-evolute offsets of the ruled surfaces with geodesic Frenet frame and also studied integral invariants of these kind of surfaces (Şentürk & Yüce, 2015a).

In this study, parallel ruled surfaces are studied according to Darboux frame. Regarding to the invariants of Darboux frame such as geodesic curvature, normal curvature and geodesic torsion, some properties of parallel ruled surfaces such as developability, striction point and distribution parameter are given in Euclidean 3-space. Steiner rotation vector of these kinds of surfaces are obtained in Euclidean 3-space. Using this rotation vector, pitch length and pitch angle of the parallel ruled surfaces with Darboux frame are computed in terms of magnitudes of the main surface.

2 Preliminaries

Let α be a curve and X be a generator vector. Then the ruled surface $\phi(s,v)$ has the following parametric representation:

$$\phi(s,v) = \alpha(s) + vX(s),$$

that is, the ruled surface is a surface generated by the motion of a straight line X along the curve α(Hacısalihoğlu, 1983).

The striction point on the ruled surface is the foot of the common perpendicular line successive rulings on the main ruling. The set of the striction points of the ruled surface generates its striction curve (Gray et al., 2006). It is given as

$$c(s) = \alpha(s) - \frac{\langle \alpha_s, X_s \rangle}{\langle X_s, X_s \rangle} X(s).$$

A unit direction vector of straight line X is stretched by the system $\{T, g\}$. So it can be written as

$$X = T cos\varphi + g sin\varphi,$$

where φ is the angle between the vectors T and X(Şentürk & Yüce, 2015). Furthermore, if α is a closed curve, then this surface is called closed ruled surface. The apex angle $\lambda(x)$ and the pitch $L(x)$ of the closed ruled surface are defined by

$$\lambda(x) = \langle D, X \rangle, L(x) = \langle V, X \rangle,$$

respectively (Hacısalihoğlu, 1983). Here D and V are Steiner rotation vector and Steiner translation vector, respectively. Steiner rotation vector $D$and Steiner translation vector $V$are given as follows:

$$D =_{(\alpha)} w = T_{(\alpha)} \tau ds + b_{(\alpha)} \kappa ds, \; V =_{(\alpha)} dx = T_{(\alpha)} ds,$$

where $w = n \wedge n' = \tau T + \kappa b$ is called Darboux vector (Hacısalihoğlu, 1983). If the Frenet vectors are the straight lines of the closed ruled surface, then we have

$$\lambda_T =_{(\alpha)} \tau ds, \; \lambda_n = 0, \; \lambda_b =_{(\alpha)} \kappa ds \quad \text{and} \quad L_T =_{(\alpha)} ds, \; l_n = 0, \; l_b = 0.$$

Theorem 1. *(Şentürk & Yüce, 2015) If successive rulings intersect, the ruled surface is called developable. The unit tangent vector of the striction curve of a developable ruled surface is the unit vector with direction X.*

The distribution of the ruled surface is identified by

$$P_X = \frac{det(\alpha_s, X, X_s)}{\langle X_s, X_s \rangle}.$$

Theorem 2. *The ruled surface is developable if and only if* $P_X = 0$ *(Ravani & Ku, 1991).*

The ruled surface is said to be a non-cylindrical ruled surface provided that $\langle X_s, X_s \rangle \neq 0$.

Theorem 3. *(Gray et al., 2006) Let M be a non-cylindrical ruled surface and defined by its striction curve. The Gaussian curvature of M is given with respect to its distribution parameter as*

$$K = -\frac{P_X^2}{(P_X^2 + v^2)^2}.$$

Theorem 4. *Let M be a non-cylindrical ruled surface and defined by its striction curve. If* P_X *never vanishes, then K is continuous and* $|K|$ *assumes its maximum value* $\frac{1}{P_X^2}$ *at* $v = 0$ *(Gray et al., 2006).*

Let $\alpha : I \to E^3$ be a closed space curve and $H/H^{'}$ be a closed space motion which is defined by α. The angle of pitch of the ruled surface with Darboux frame, which is drawn by a fixed line in $\{T, n, b\}$ during the motion $H/H^{'}$ in the fixed space H is

$$\lambda_X = \lambda_T cos\varphi + \lambda_b sin\varphi sin\theta, \tag{1}$$

where λ_T and λ_b are the angles of pitch of the ruled surfaces with Darboux frame which are drawn by the vectors T and b, respectively. If the ruled surface with Darboux frame whose ruling is X in $\{T, n, b\}$ during the motion $H/H^{'}$ is developable, then the harmonic curvature is constant. Besides, the harmonic curvature of the closed curve $\alpha(s)$ of the ruled surface with Darboux frame, during the space motion $H/H^{'}$, is calculated as follows:

$$\left(\frac{\kappa}{\tau}\right)^2 = \frac{P_b}{P_n} - 1 = \left(\frac{sin^2\varphi}{sin\varphi cos\varphi sin\theta}\right)^2 - 1 \tag{2}$$

where P_b and P_n are the distribution parameters of the ruled surface with Darboux frame which are drawn by b and n (Şentürk and Yüce, 2015).

Theorem 5. *(Şentürk and Yüce, 2015)*

(i) *The ruled surface with Darboux frame which is drawn by a line X in a normal plane in* $\{T, n, b\}$ *during the motion* $H/H^{'}$ *is developable if and only if* $\alpha(s)$ *is a plane curve.*

(ii) *The ruled surface with Darboux frame which is drawn by a line X in an osculator plane in* $\{T, n, b\}$ *during the motion* $H/H^{'}$ *is developable if and only if* $\alpha(s)$ *is a plane curve or* $\varphi = 0$

(iii) *The ruled surface with Darboux frame which is drawn by a line X in a rectifian plane in* $\{T, n, b\}$ *during the motion* $H/H^{'}$ *is developable if and only if* $\frac{\kappa}{\tau} = \tan\varphi$ *or* $\varphi = 0$.

Theorem 6. *The ruling of the ruled surface with Darboux frame drawn by a line X in* $\{T, n, b\}$ *during the motion* $H/H^{'}$ *in the fixed space is always in the rectifian plane of the striction curve (Hacısalihoğlu, 1983).*

Theorem 7. *Since the ruled surface with Darboux frame drawn by a line X in* $\{T, n, b\}$ *during the motion* $H/H^{'}$ *is developable, then the surface is tangent developable ruled surface (Şentürk and Yüce, 2015).*

Let M be an oriented surface in E^3 and α be a unit speed curve on M. If T is the unit tangent vector of α, N is the unit normal vector of M and $g = N \wedge T$ at the point $\alpha(s)$ of curve α, then $\{T, g, N\}$ is called the Darboux frame of α at that point. Thus the Darboux derivative formulas are:

$$\frac{d}{ds}\begin{bmatrix} T \\ g \\ N \end{bmatrix} = \begin{bmatrix} 0 & \kappa_g & \kappa_n \\ -\kappa_g & 0 & \tau_g \\ -\kappa_n & -\tau_g & 0 \end{bmatrix} \begin{bmatrix} T \\ g \\ N \end{bmatrix},$$

where $\kappa_n = \kappa cos\theta$, $\kappa_g = \kappa sin\theta$ and $\tau_g = \tau + \frac{d\theta}{ds}$ are called the normal curvature, geodesic curvature and the geodesic torsion of α, and also θ is the angle between N and the unit principal normal n of α.respectively (Kühnel, 2002).

Definition 1. *Let M and $\overline{M}$ be two surfaces in Euclidean space. The function*

$$\begin{aligned} f : M &\to \overline{M} \\ P &\to f(P) = P + rN_P \end{aligned} \tag{3}$$

is called the parallelization function between M and $\overline{M}$ and furthermore $\overline{M}$ is called parallel surface to M where N is the unit normal vector field on M and r is a given real number (Hacısalihoğlu, 1983).

Definition 2. *(Savcı, 2011) Let M and $\overline{M}$ be two parallel surfaces. Let α be a unit-speed curve on M and image of α stands on $\overline{M}$ which $(f \circ \alpha) = \beta$. If β is non-unit speed curve then $\|\beta'\| = \|f_*(T)\| = v \neq 1$. Darboux frame of the curve β on $\overline{M}$ is*

$$\left\{\overline{T} = \frac{f_*(T)}{v}, \overline{g} = \overline{T} \wedge \overline{N}, \overline{N} = N\right\} \tag{4}$$

where $\overline{N}$ is the unit normal vector of $\overline{M}$. And also the norm of β' is

$$\|\beta'\| = \sqrt{(1 - r\kappa_n)^2 + r^2\tau_g^2} = v \tag{5}$$

Theorem 8. *(Savcı, 2011) Let $\{\overline{T}, \overline{g}, \overline{N}\}$ be Darboux frame of curve β at $\overline{\alpha}(s) = f(P)$ on $\overline{M}$. Then the vectors $\{\overline{T}, \overline{g}, \overline{N}\}$ are*

$$\begin{aligned} &\overline{T} = \tfrac{1}{v}\left[(1 - r\kappa_n)T - r\tau_g g\right], \\ &\overline{g} = \tfrac{1}{v}\left[(1 - r\kappa_n)g + r\tau_g T\right], \\ &\overline{N} = N. \end{aligned} \tag{6}$$

Theorem 9. *Let α be a regular curve on the surface M. Then the geodesic curvature, the normal curvature and the geodesic torsion of the curve $(f \circ \alpha) = \beta$ are*

$$\begin{aligned} &\overline{\kappa}_g = \tfrac{\kappa_g}{v^3} - \tfrac{r}{v^3}\left[\left(\tau_g^{+} r(\tau_g \kappa_n' - \tau_g^{\kappa_n})\right)\right], \\ &\overline{\kappa}_n = \tfrac{1}{v^2}\left[\kappa_n - r(\kappa_n^2 + \tau_g^2)\right], \\ &\overline{\tau}_g = \tfrac{\tau_g}{v^2}. \end{aligned} \tag{7}$$

at the point $\overline{\alpha}(s)$ on the parallel surface $\overline{M}$ (Savcı, 2011).

Theorem 10. *(Savcı, 2011) Let $\{\overline{T}, \overline{n}, \overline{b}\}$ be Frenet frame of curve β at $f(P)$ on the surface $\overline{M}$ and$\overline{n}$ be principal normal vector of curve β, $\overline{N}$ be unit normal of surface $\overline{M}$, Φ^r be the angle between the vectors $\overline{n}$ and $\overline{N}$ and $\overline{\kappa}$ be curvature of curve β at $f(P)$ on $\overline{M}$, then*

$$(\overline{\kappa})^2 = (\overline{\kappa}_n)^2 + (\overline{\kappa}_g)^2, \; cos\Phi^r = \frac{\overline{\kappa}_n}{\sqrt{(\overline{\kappa}_n)^2 + (\overline{\kappa}_g)^2}}. \tag{8}$$

3 Parallel ruled surfaces with Darboux frame

A parallel ruled surface can be given as in the following parametrization:

$$\overline{\varphi}(s,\overline{v}) = \overline{\alpha}(s) + \overline{v}\overline{X}(s), \tag{9}$$

where $\overline{\alpha}(s)$ is a curve which is called the base curve of the parallel ruled surface, $\overline{X}$ is the ruling.

Using (6), Darboux derivative formulae of the parallel ruled surface are computed as

$$\frac{d}{ds}\begin{bmatrix}\overline{T}\\ \overline{g}\\ \overline{N}\end{bmatrix} = \begin{bmatrix}0 & \overline{\kappa}_g & \overline{\kappa}_n\\ -\overline{\kappa}_g & 0 & \overline{\tau}_g\\ -\overline{\kappa}_n & -\overline{\tau}_g & 0\end{bmatrix}\begin{bmatrix}\overline{T}\\ \overline{g}\\ \overline{N}\end{bmatrix}. \tag{10}$$

Darboux frame is obtained by rotating Frenet frame around $\overline{T}$ as far as $\overline{\theta} = \overline{\theta}(s)$ while the angle $\overline{\theta}$ is between the unit vector $\overline{g}$ and the normal vector field $\overline{N}$ of $\overline{\alpha}$.

Using (8), the relations between Frenet and Darboux frames of parallel ruled surfaces are computed in terms of $\overline{\theta}$ as follows:

The unit vector $\overline{g}$ is written as

$$\overline{g} = \overline{n}cos\overline{\theta} + \overline{b}sin\overline{\theta}.$$

By using the equations

$$\langle\overline{g},\overline{n}\rangle = cos\overline{\theta} = \frac{\overline{\kappa}_g}{\sqrt{\overline{\kappa}_n^2+\overline{\kappa}_g^2}} = sin\Phi^r,\ \langle\overline{g},\overline{b}\rangle = sin\overline{\theta} = -\frac{\overline{\kappa}_n}{\sqrt{\overline{\kappa}_n^2+\overline{\kappa}_g^2}} = -cos\Phi^r, \tag{11}$$

we have

$$\overline{g} = \overline{n}sin\Phi^r - \overline{b}cos\Phi^r. \tag{12}$$

The unit normal vector of the surface is

$$\overline{N} = -\overline{n}sin\overline{\theta} + \overline{b}cos\overline{\theta}.$$

By straightforward calculation, we find

$$\overline{N} = \overline{n}cos\Phi^r + \overline{b}sin\Phi^r. \tag{13}$$

From (12) and (13), the matrix representation of the relation between the frames is obtained as

$$\begin{bmatrix}\overline{T}\\ \overline{g}\\ \overline{N}\end{bmatrix} = \begin{bmatrix}1 & 0 & 0\\ 0 & sin\Phi^r & -cos\Phi^r\\ 0 & cos\Phi^r & sin\Phi^r\end{bmatrix}\begin{bmatrix}\overline{T}\\ \overline{n}\\ \overline{b}\end{bmatrix}, \tag{14}$$

and also transposing (14), we have

$$\overline{n} = \overline{g}sin\Phi^r + \overline{N}cos\Phi^r \tag{15}$$

and

$$\overline{b} = -\overline{g}cos\Phi^r + \overline{N}sin\Phi^r, \tag{16}$$

where Φ^r is the angle between the vectors $\overline{n}$ and $\overline{N}$.

Differentiating (15), (16) and using (14), we have Frenet derivative formulae of the parallel ruled surface as

$$\overline{T}^{'}=\overline{\kappa}\overline{n},\ \overline{n}^{'}=-\overline{\kappa}\overline{T}+\overline{\tau}\overline{b},\ \overline{b}^{'}=\overline{\tau}\overline{n}. \tag{17}$$

Let $\overline{\varphi}$ be the angle between direction vector $\overline{X}$ and tangent vector $\overline{T}$ at $\overline{\alpha}\in\overline{\phi}(s,\overline{v})$. If we choose the direction vector $\overline{X}$ as

$$\overline{X}=\overline{T}cos\overline{\varphi}+\overline{g}sin\overline{\varphi}, \tag{18}$$

then we get

$$\left\langle \overline{X},\overline{T}\right\rangle = cos\overline{\varphi}=cos\varphi,\ \left\langle \overline{X},\overline{g}\right\rangle = sin\overline{\varphi}=sin\varphi, \tag{19}$$

where $\left\|\overline{X}\right\|=1,$ and ϕ is the angle between the vectors T and X on the ruled surface. Thus we can give the following corollary.

Corollary 1. *For the parallel ruled surfaces with Darboux frame, the angle $\overline{\varphi}$ is an invariant since*

$$\overline{\varphi}=\varphi. \tag{20}$$

Since (20), the equation (18) becomes

$$\overline{X}=\overline{T}cos\varphi+\overline{g}sin\varphi. \tag{21}$$

Differentiating (21) and using (10), we find

$$\overline{X}^{'}=-(\varphi^{'}+\overline{\kappa}_g)sin\varphi\overline{T}+(\varphi^{'}+\overline{\kappa}_g)cos\varphi\overline{g}+(\overline{\kappa}_n cos\varphi+\overline{\tau}_g sin\varphi)\overline{N}. \tag{22}$$

Holding $\overline{v}$ =constant, we obtain the curve

$$\beta'=(f(\alpha(u)))^{'}+\overline{v}(f_*(\alpha(u)))^{'}=\overline{T}+\overline{v}\overline{X}^{'}=\overline{T}^{*} \tag{23}$$

on the parallel ruled surface whose vector field

$$\overline{T}^{*}=(1-\overline{v}(\varphi^{'}+\overline{\kappa}_g)sin\varphi)\overline{T}+\overline{v}(\varphi^{'}+\overline{\kappa}_g)cos\varphi\overline{g}+\overline{v}(\overline{\kappa}_n cos\varphi+\overline{\tau}_g sin\varphi)\overline{N}. \tag{24}$$

Using the equations (7), (21) and (22), the distribution parameter of the parallel ruled surface with Darboux frame is calculated by

$$\overline{P}_{\overline{X}}=\frac{v^4 sin\varphi((\kappa_n-r(\kappa_n^2+\tau_g^2))cos\varphi+\tau_g sin\varphi)}{(\kappa_g v^2+\varphi'-r\tau_g^2\left(\frac{(r\kappa_n-1)}{\tau_g}\right)^{'})^{2}+v^2(\kappa_n cos\varphi+\tau_g sin\varphi-r(\kappa_n^2+\tau_g^2)cos\varphi)^{2}}. \tag{25}$$

Theorem 11. *The parallel ruled surface with Darboux frame is a developable surface if and only if*

$$(\kappa_n-r(\kappa_n^2+\tau_g^2))cos\varphi+\tau_g sin\varphi=0. \tag{26}$$

Proof. Substituting (7) into (24) gives the vector field $\overline{T}^*$ as

$$\begin{aligned}\overline{T}^* &= \left(1 - \tfrac{\overline{v}}{v^3}\left(v^3\varphi' + \kappa_g v^2 - r\tau_g^2\left(\tfrac{(r\kappa_n-1)}{\tau_g}\right)'\right) sin\varphi\right)\overline{T} \\ &+ \tfrac{\overline{v}}{v^3}\left(\left(v^3\varphi' + \kappa_g v^2 - r\tau_g^2\left(\tfrac{(r\kappa_n-1)}{\tau_g}\right)'\right) cos\varphi\right)\overline{g} \\ &+ \tfrac{\overline{v}}{v^2}\left((\kappa_n - r(\kappa_n^2+\tau_g^2))cos\varphi + \tau_g sin\varphi\right)\overline{N}.\end{aligned} \tag{27}$$

Suppose that the parallel ruled surface with Darboux frame is developable surface, then $\overline{P}_{\overline{X}} = 0$, that is,

$$v^4 sin\varphi((\kappa_n - r(\kappa_n^2+\tau_g^2))cos\varphi + \tau_g sin\varphi) = 0 \tag{28}$$

In this case, let us study the following subcases related to the equation (28) vanishing:

(i) If $v^4 = 0$, then from (5), we obtain the result that is a contradiction.
(ii) If $sin\varphi = 0$, then from (21), we obtain $\overline{X} = \overline{T}cos\varphi$. So $\overline{T}^* = \overline{T}$. It means that tangent plane is constant along the main ruling.
(iii) If $(\kappa_n - r(\kappa_n^2+\tau_g^2))cos\varphi + \tau_g sin\varphi = 0$, then from (27), it is seen that the tangent plane and the normal vector of parallel ruled surface with Darboux frame are orthogonal vectors. Therefore the parallel ruled surface with Darboux frame is a developable surface.

Conversely, if we put

$$(\kappa_n - r(\kappa_n^2+\tau_g^2))cos\varphi + \tau_g sin\varphi = 0,$$

into (25), the distribution parameter becomes as $\overline{P}_{\overline{X}} = 0$.

Using (7), the striction curve of the parallel ruled surface with Darboux frame is calculated as follows:

$$\bar{c}(s) = \bar{\alpha}(s) + \frac{v^3 sin\varphi\left(v^3\varphi' + \kappa_g v^2 - r\tau_g^2\left(\frac{(r\kappa_n-1)}{\tau_g}\right)'\right)\overline{X}}{\left(v^6\varphi' + \kappa_g v^2 - r\tau_g^2\left(\frac{(r\kappa_n-1)}{\tau_g}\right)'\right)^2 + v^2((\kappa_n - r(\kappa_n^2+\tau_g^2))cos\varphi + \tau_g sin\varphi)^2}. \tag{29}$$

Thus we can mention the following corollary.

Corollary 2. *If the parallel ruled surface with Darboux frame is a developable ruled surface, then striction curve of the parallel ruled surface with Darboux frame is*

$$\overline{c}(s) = \overline{\alpha}(s) + \frac{v^3 sin\varphi\left(v^3\varphi' + \kappa_g v^2 - r\tau_g^2\left(\frac{(r\kappa_n-1)}{\tau_g}\right)'\right)}{\left(v^3\varphi' + \kappa_g v^2 - r\tau_g^2\left(\frac{(r\kappa_n-1)}{\tau_g}\right)'\right)}\overline{X}. \tag{30}$$

Theorem 12. *Let $\overline{M}$ be a parallel ruled surface with Darboux frame, then the shortest distance between the rulings of $\overline{M}$ along the orthogonal trajectories is*

$$\overline{v} = \frac{v^3 sin\varphi\left(v^3\varphi' + \kappa_g v^2 - r\tau_g^2\left(\frac{(r\kappa_n-1)}{\tau_g}\right)'\right)}{\left(v^6\varphi' + \kappa_g v^2 - r\tau_g^2\left(\frac{(r\kappa_n-1)}{\tau_g}\right)'\right)^2 + v^2((\kappa_n - r(\kappa_n^2+\tau_g^2))cos\varphi + \tau_g sin\varphi)^2}. \tag{31}$$

Proof. Supposing that the two rulings pass the points $\overline{\alpha}_{s_1}$ and $\overline{\alpha}_{s_2}$ where $s_1 < s_2$, the distance between these rulings along an orthogonal trajectory is given as

$$\overline{J}(\overline{v}) = \int_{s_1}^{s_2} \left\|\overline{T}^*\right\| ds. \tag{32}$$

Substituting (24) into (32) gives

$$\overline{J}(\overline{v}) = \int_{s_1}^{s_2} \left(1 - 2\overline{v}(\varphi' + \overline{\kappa}_g)sin\varphi + \overline{v}^2(\varphi' + \overline{\kappa}_g)^2 + \overline{v}^2(\overline{\kappa}_n cos\varphi + \overline{\tau}_g sin\varphi)^2\right)^{\frac{1}{2}} ds. \tag{33}$$

Differentiating (33) to the parameter $\overline{v}$ which gives the minimal value of $\overline{J}(\overline{v})$, we get

$$\overline{J}'(\overline{v}) = \int_{s_1}^{s_2} \frac{-2\overline{v}(\varphi'+\overline{\kappa}_g)sin\varphi + 2\overline{v}(\varphi'+\overline{\kappa}_g)^2 + 2\overline{v}(\overline{\kappa}_n cos\varphi + \overline{\tau}_g sin\varphi)^2}{\left(1-2\overline{v}(\varphi'+\overline{\kappa}_g)sin\varphi + \overline{v}^2(\varphi'+\overline{\kappa}_g)^2 + \overline{v}^2(\overline{\kappa}_n cos\varphi + \overline{\tau}_g sin\varphi)^2\right)^{\frac{1}{2}}} ds = 0 \tag{34}$$

which satisfies

$$\overline{v} = \frac{sin\varphi\left(\varphi' + \overline{\kappa}_g\right)}{\left(\varphi'+\overline{\kappa}_g\right)^2 + (\overline{\kappa}_n cos\varphi + \overline{\tau}_g sin\varphi)^2}. \tag{35}$$

Using (7) in (35), the parameter $\overline{v}$ turns into

$$\overline{v} = \frac{v^3 sin\varphi\left(v^3\varphi' + \kappa_g v^2 - r\tau_g^2\left(\frac{(r\kappa_n - 1)}{\tau_g}\right)'\right)}{\left(v^6\varphi' + \kappa_g v^2 - r\tau_g^2\left(\frac{(r\kappa_n-1)}{\tau_g}\right)'\right)^2 + v^2((\kappa_n - r(\kappa_n^2+\tau_g^2))cos\varphi + \tau_g sin\varphi)^2}.$$

Thus we can give the following corollary.

Corollary 3. *If the parallel ruled surface with Darboux frame is a developable ruled surface, then the parameter $\overline{v}$ of an orthogonal trajectory of parallel ruled surface with Darboux frame is*

$$\overline{v} = \frac{v^3 sin\varphi}{\left(v^3\varphi' + \kappa_g v^2 - r\tau_g^2\left(\frac{(r\kappa_n-1)}{\tau_g}\right)'\right)}.$$

Theorem 13. *Let $\overline{M}$ be a parallel ruled surface with Darboux frame. Moreover, the point $\overline{\phi}(s,\overline{v}_0)$, $\overline{v}_0 \in E$, on the main ruling which passes through the point $\overline{\alpha}(s)$, is a striction point if and only if X' is the unit normal vector field of tangent plane in the point $\overline{\phi}(s,\overline{v}_0)$.*

Proof. Suppose that the point $\overline{\phi}(s,\overline{v}_0)$ on the main ruling which passes through the point $\overline{\alpha}(s)$ is a striction point. We have to show that $\langle\overline{X},\overline{T}^*\rangle = \left\langle\overline{X},\overline{X}'\right\rangle = 0$. We know that $\left\|\overline{X}\right\| = 1$. By differentiating this equation, we obtain $\left\langle\overline{X},\overline{X}'\right\rangle = 0$. So we can find

$$\left\langle\overline{X}',\overline{T}^*\right\rangle = -sin\varphi(\varphi' + \overline{\kappa}_g) + \overline{v}(\varphi' + \overline{\kappa}_g) + \overline{v}(\overline{\kappa}_n cos\varphi + \overline{\tau}_g sin\varphi)^2. \tag{36}$$

Also, if we calculate the value $\overline{v}_0$ into (36), then we get $\left\langle\overline{X}',\overline{T}^*\right\rangle = 0$. So it means that $\overline{X}'$ is normal to $\overline{X}$ and $\overline{T}^*$.

Conversely, suppose that $\overline{X}'$ is a unit normal vector field of the tangent plane at the point $\overline{\phi}(s,\overline{v}_0)$. When holding $\overline{v}$ constant as $\overline{v}_0$, the tangent vector field of $\overline{\phi}(s,\overline{v}_0)$ is

$$\overline{T}^* = (1 - \overline{v}_0(\varphi' + \overline{\kappa}_g)sin\varphi)\overline{T} + \overline{v}_0(\varphi' + \overline{\kappa}_g)cos\varphi\overline{g} + \overline{v}_0(\overline{\kappa}_n cos\varphi + \overline{\tau}_g sin\varphi)\overline{N}.$$

Since $\overline{X}'$ is a unit normal to the tangent plane at the point $\overline{\phi}(s,\overline{v}_0)$, we can write that $\left\langle \overline{X}', \overline{T}^* \right\rangle = 0$. Thus we get

$$\overline{v}_0 = \frac{v^3 sin\varphi\left(v^3\varphi' + \kappa_g v^2 - r\tau_g^2\left(\frac{(r\kappa_n - 1)}{\tau_g}\right)'\right)}{\left(v^6\varphi' + \kappa_g v^2 - r\tau_g^2\left(\frac{(r\kappa_n - 1)}{\tau_g}\right)'\right)^2 + v^2((\kappa_n - r(\kappa_n^2 + \tau_g^2))cos\varphi + \tau_g sin\varphi)^2}$$

Hence the point $\overline{\phi}(s,\overline{v}_0)$ is a striction point on the parallel ruled surface with Darboux frame.

Corollary 4. *If the parallel ruled surface with Darboux frame is a developable ruled surface, we get $\overline{v}_0$ as follows.*

$$\overline{v}_0 = \frac{v^3 sin\varphi\left(v^3\varphi' + \kappa_g v^2 - r\tau_g^2\left(\frac{(r\kappa_n - 1)}{\tau_g}\right)'\right)}{\left(v^3\varphi' + \kappa_g v^2 - r\tau_g^2\left(\frac{(r\kappa_n - 1)}{\tau_g}\right)'\right)}.$$

Theorem 14. *Let $\overline{M}$ be a parallel ruled surface with Darboux frame. The absolute value of the Gaussian curvature $\overline{K}$ of the parallel ruled surface $\overline{M}$ along a ruling takes the maximum value at the striction point on that ruling.*

Proof. If we calculate the Gauss curvature of the parallel ruled surface with Darboux frame, we get

$$\overline{K}(s,\overline{v}) = \frac{(\overline{\kappa}_n cos\varphi + \overline{\tau}_g sin\varphi)^2 \sin^2\varphi}{\left(1 - 2\overline{v}(\varphi' + \overline{\kappa}_g)sin\varphi + \overline{v}^2(\varphi' + \overline{\kappa}_g)^2 + \overline{v}^2(\overline{\kappa}_n cos\varphi + \overline{\tau}_g sin\varphi)^2 - cos^2\varphi\right)^2}. \quad (37)$$

If we differentiate (37) with respect to $\bar{v}$, and using $\frac{\partial K}{\partial \bar{v}} = 0$, we get

$$\overline{v} = \frac{sin\varphi\left(\varphi' + \overline{\kappa}_g\right)}{\left(\varphi' + \overline{\kappa}_g\right)^2 + (\overline{\kappa}_n cos\varphi + \overline{\tau}_g sin\varphi)^2}. \quad (38)$$

Therefore the absolute value of Gauss curvature $\overline{K}$ of the parallel ruled surface $\overline{M}$ along a ruling takes the maximum value at the striction point on that ruling.

Example 1. Let us establish a parallel ruled surface by using the following tangent developable ruled surface of

$$\alpha(s) = \left(\tfrac{\sqrt{3}}{3}coss, 1 - sins, -\tfrac{\sqrt{6}}{3}coss\right).$$

The tangent developable parallel ruled surface is as follows:

$$\overline{\phi}(s,v) = \phi(s,v) + r\overline{N}(s,v).$$

Firstly, we find the normal vector of the parallel ruled surface with Darboux frame as

$$\overline{N}(s,v) = \left(\tfrac{\sqrt{6}}{3}, 0, \tfrac{\sqrt{3}}{3}\right).$$

We get the parallel tangent developable ruled surface as follows

$$\overline{\phi}(s,v) = \left(\tfrac{\sqrt{3}}{3}coss, 1 - sins, -\tfrac{\sqrt{6}}{3}coss\right) + v\left(-\tfrac{\sqrt{3}}{3}sins, -coss, \tfrac{\sqrt{6}}{3}sins\right) + r\left(\tfrac{\sqrt{6}}{3}, 0, \tfrac{\sqrt{3}}{3}\right).$$

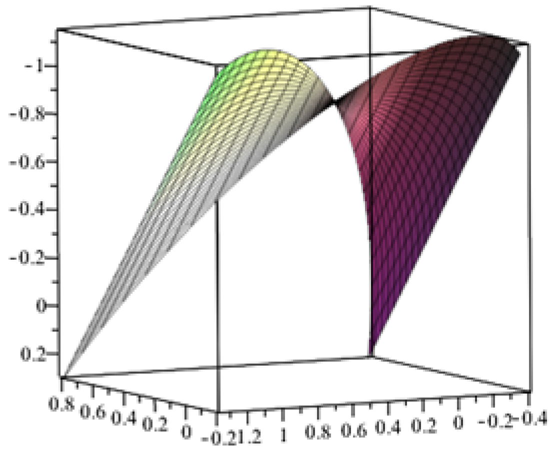

Fig. 1: The main surface.

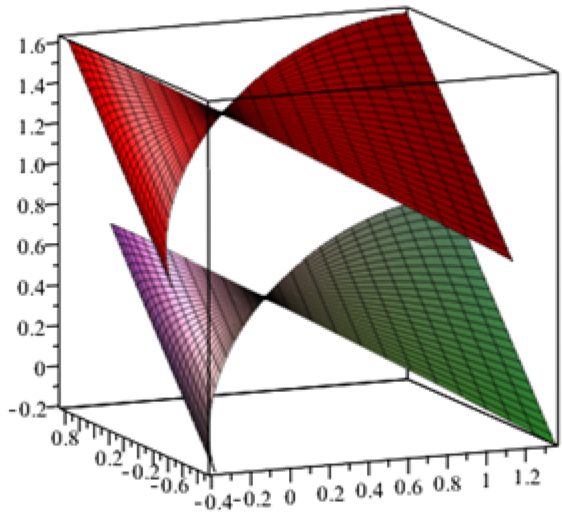

Fig. 2: The main surface and its parallel surface (red one).

The Darboux frame of the tangent developable parallel ruled surface is

$$\begin{aligned}\overline{T}(s) &= \left(-\tfrac{\sqrt{3}}{3}sins, -coss, \tfrac{\sqrt{6}}{3}sins\right)\\ \overline{g}(s) &= \left(\tfrac{\sqrt{3}}{3}coss, -sins, -\tfrac{\sqrt{6}}{3}coss\right)\\ \overline{N}(s) &= \left(\tfrac{\sqrt{6}}{3}, 0, \tfrac{\sqrt{3}}{3}\right).\end{aligned}$$

We have the distribution parameter of the tangent developable parallel ruled surface with Darboux frame as the equation (25). Since

$$sin\varphi = 0$$

in (25), then we get

$$\overline{P}_{\overline{X}} = 0.$$

The striction curve of the tangent developable parallel ruled surface with Darboux frame is defined in the equation (30). From here, since $sin\varphi = 0$, we find $\overline{c}(s) = \overline{\alpha}(s)$.

Example 2. Let

$$\phi(s,v) = (coss - vsins, sins + vcoss, v)$$

be a ruled surface. We find the normal vector of this surface as

$$\overline{N}(s,v) = \frac{1}{\sqrt{1+2v^2}}(coss - vsins, sins + vcoss, -v).$$

Using the normal vector, we get the parallel ruled surface with Darboux frame as follows

$$\begin{aligned}\overline{\phi}(s,v) = & (coss, sins, 0) \\ & +v\left(-sins\left(1+\frac{r}{\sqrt{1+2v^2}}\right), coss\left(1+\frac{r}{\sqrt{1+2v^2}}\right), \left(1-\frac{r}{\sqrt{1+2v^2}}\right)\right) \\ & +\frac{r}{\sqrt{1+2v^2}}(coss, sins, 0).\end{aligned}$$

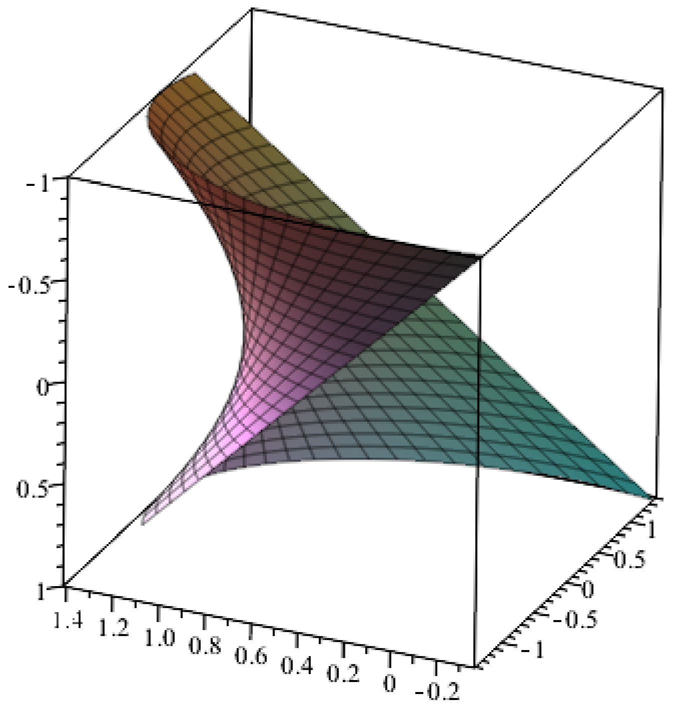

Fig. 3: The main surface.

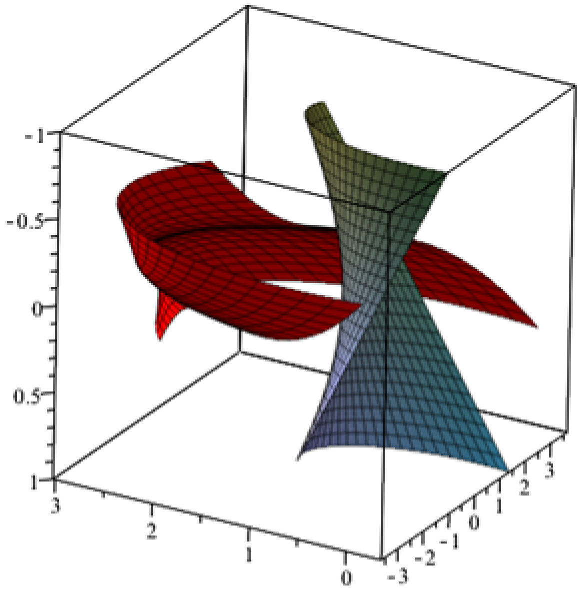

Fig. 4: The main surface and its parallel surface (red one).

The Darboux frame of the parallel ruled surface is

$$\begin{aligned}\overline{T}(s) &= (-sins, coss, 0)\\ \overline{g}(s) &= \frac{1}{\sqrt{1+2v^2}}(vcoss, vsins, 1)\\ \overline{N}(s) &= \frac{1}{\sqrt{1+2v^2}}(coss - vsins, sins + vcoss, -v).\end{aligned}$$

Since we find $\overline{P}_{\overline{X}} = -1$, this surface is not a developable ruled surface.

4 Integral invariants of parallel ruled surfaces with Darboux frame

Firstly, we calculate the Darboux vector of the parallel ruled surfaces with Darboux frame in E^3.Using (17), the Darboux vector is obtained as

$$\overline{w} = \overline{\tau}\overline{T} + \overline{\kappa}\overline{b}. \tag{39}$$

From (39), we get the Steiner rotation vector of the parallel ruled surface with Darboux frame as

$$\overline{D} = (\overline{\tau}\overline{T} + \overline{\kappa}\overline{b})ds. \tag{40}$$

Hence, the angle of pitch of the closed parallel ruled surface with Darboux frame is defined as

$$\overline{D} = \overline{\lambda}_{\overline{T}}\overline{T} + \overline{\lambda}_{\overline{b}}\overline{b},$$

such that

$$\langle \overline{D}, \overline{T} \rangle = \overline{\tau}ds = \overline{\lambda}_{\overline{T}},\ \langle \overline{D}, \overline{n} \rangle = \overline{\lambda}_{\overline{n}} = 0,\ \langle \overline{D}, \overline{b} \rangle = \overline{\kappa}ds = \overline{\lambda}_{\overline{b}}.$$

Additionally, the pitch length of closed parallel ruled surface is obtained as

$$\overline{L}_{\overline{T}} = \left\langle d\overline{\alpha}, \overline{T} \right\rangle = \left\langle \overline{T} ds, \overline{T} \right\rangle = ds,\ \overline{L}_{\overline{n}} = 0,\ \overline{L}_{\overline{b}} = 0.$$

An orthogonal trajectory of the parallel ruled surface with Darboux frame is defined as

$$cos\varphi ds = -dv.$$

From (21), we get

$$\overline{L}_{\overline{X}} = -\left\langle \overline{X}, d\overline{\alpha} \right\rangle = dv = cos\varphi ds. \tag{41}$$

Substituting (12) into (21), we obtain the ruling of the parallel ruled surface with Darboux frame as

$$\overline{X} = \overline{T}cos\varphi + \overline{n}sin\varphi sin\Phi^r - \overline{b}sin\varphi cos\Phi^r,\quad \left\|\overline{X}\right\| = 1, \tag{42}$$

where Φ^r be the angle between the vectors $\overline{n}$ and $\overline{N}$ and φ be the angle between the vectors $\overline{T}$ and $\overline{X}$.

Theorem 15. *Let $\overline{\alpha} : I \to E^3$ be a closed space curve and $H/H^{'}$ be a closed space motion which is defined by $\overline{\alpha}$. The angle of pitch of the parallel ruled surface with Darboux frame, which is drawn by a fixed line in $\{\overline{T}, \overline{n}, \overline{b}\}$ during the motion H/H' in the fixed space H is*

$$\overline{\lambda}_{\overline{X}} = \overline{\lambda}_{\overline{T}}cos\varphi - \overline{\lambda}_{\overline{b}}sin\varphi cos\Phi^r, \tag{43}$$

where $\overline{\lambda}_{\overline{T}}$ and $\overline{\lambda}_{\overline{b}}$ are the angles of pitch of the parallel ruled surfaces with Darboux frame which are drawn by the vectors $\overline{T}$ and $\overline{b}$, respectively.

Proof. From the definition of the angle of the pitch of the ruling, the inner product of the vectors $\overline{D}$ and $\overline{X}$ gives us the result (43).

Remark. From (11) we have $cos\Phi^r = -sin\overline{\theta}$, therefore (43) turns into (1). Here the angle $\overline{\theta}$ is between the unit vector $\overline{g}$ and the normal vector $\overline{N}$ of $\overline{\alpha}$ on the parallel ruled surfaces with Darboux frame.

Theorem 16. *The pitch of the parallel ruled surface with Darboux frame, which is drawn by a fixed line in $\{\overline{T}, \overline{n}, \overline{b}\}$ during the motion $H/H^{'}$ in the fixed space H is equal to*

$$\overline{L}_{\overline{X}} = cos\varphi \overline{L}_{\overline{T}}. \tag{44}$$

Proof. Substituting (42) into (41), we have

$$\overline{L}_{\overline{X}} = -\oint_{\overline{\alpha}} \left\langle \overline{X}, d\overline{\alpha} \right\rangle = cos\varphi \oint_{\overline{\alpha}} ds = cos\varphi \overline{L}_{\overline{T}}. \tag{45}$$

As a result, we obtain (44) for the parallel ruled surface which is similar to the one of the ruled surface with Darboux frame.

Theorem 17. *If the parallel ruled surface with Darboux frame whose ruling is $\overline{X}$ in $\{\overline{T}, \overline{n}, \overline{b}\}$ during the motion $H/H^{'}$ is developable, then the harmonic curvature is constant and given as*

$$\overline{h} = \frac{\overline{\kappa}}{\overline{\tau}} = -\frac{sin^2\varphi}{sin\varphi cos\varphi cos\Phi^r} = \frac{(\overline{L}^2_{\overline{T}} - \overline{L}^2_{\overline{X}})\overline{\lambda}_{\overline{b}}}{(\overline{\lambda}_{\overline{X}}\overline{L}_{\overline{T}} - \overline{L}_{\overline{X}}\overline{\lambda}_{\overline{T}})\overline{L}_{\overline{X}}}.$$

Let $\overline{X}$ draw a developable parallel ruled surface with Darboux frame. Then, the distribution parameter of the parallel ruled surface is zero. Differentiating (42) with respect to the parameter s gives

$$\frac{d\overline{X}}{ds}=\overline{T}(-\overline{\kappa}sin\varphi sin\Phi^{r})+\overline{n}(\overline{\kappa}cos\varphi+\overline{\tau}sin\varphi cos\Phi^{r})+\overline{b}(\overline{\tau}sin\varphi sin\Phi^{r}).$$

Also the following determinant is calculated as

$$det(\overline{T},\overline{X},\overline{X}')=\overline{\tau}\,sin^{2}\varphi+\overline{\kappa}sin\varphi cos\varphi cos\Phi^{r}=0 \tag{46}$$

From (46), we have the harmonic curvature as

$$\frac{\overline{\kappa}}{\overline{\tau}}=-\frac{sin^{2}\varphi}{sin\varphi cos\varphi cos\Phi^{r}}. \tag{47}$$

From (44), we get

$$cos\varphi=\frac{\overline{L}_{\overline{X}}}{\overline{L}_{\overline{T}}}, \tag{48}$$

and also if we put (44) into (43), we have

$$sin\varphi cos\Phi^{r}=-\frac{(\overline{\lambda}_{\overline{X}}\overline{L}_{\overline{T}}-\overline{L}_{\overline{X}}\overline{\lambda}_{\overline{T}})}{\overline{L}_{\overline{T}}\overline{\lambda}_{\overline{b}}}, \tag{49}$$

and finally using (49) in (47), we find

$$\overline{h}=\frac{\overline{\kappa}}{\overline{\tau}}=\frac{(\overline{L}^{2}_{\overline{T}}-\overline{L}^{2}_{\overline{X}})\overline{\lambda}_{\overline{b}}}{(\overline{\lambda}_{\overline{X}}\overline{L}_{\overline{T}}-\overline{L}_{\overline{X}}\overline{\lambda}_{\overline{T}})\overline{L}_{\overline{X}}}.$$

Remark. We get the following ratio of curvature functions in terms of the angles which belong to parallel ruled surfaces with Darboux frame as

$$\frac{\kappa}{\tau}=\frac{sin^{2}\varphi}{sin\varphi cos\varphi sin\overline{\theta}}. \tag{50}$$

The equation (50) means that the harmonic curvature of the closed space curve $\overline{\alpha}$ on the parallel ruled surface with Darboux frame is equal to the one of the curve on the original surface.

Theorem 18. *The harmonic curvature of the closed curve $\overline{\alpha}(s)$ on the parallel ruled surface with Darboux frame, during the space motion $H/H^{'}$, is calculated as follows:*

$$\frac{\overline{P}_{\overline{b}}}{\overline{P}_{\overline{n}}}-1=\left(\frac{\overline{\kappa}}{\overline{\tau}}\right)^{2}=\left(\frac{sin^{2}\varphi}{sin\varphi cos\varphi cos\Phi^{r}}\right)^{2}-1 \tag{51}$$

where $\overline{P}_{\overline{b}}$ and $\overline{P}_{\overline{n}}$ are the distribution parameters of the parallel ruled surface with Darboux frame with respect to the vectors $\overline{b}$ and $\overline{n}$, respectively.

Proof. The distribution parameter of the closed parallel ruled surface which is drawn by tangent line $\overline{T}$ is calculated as

$$\overline{P}_{\overline{T}}=\frac{det(\overline{T},\overline{T},\overline{\kappa}\overline{n})}{\langle\overline{\kappa}\overline{n},\overline{\kappa}\overline{n}\rangle}=0.$$

In similar way, we get

$$\overline{P}_{\overline{n}}=\frac{det(\overline{T},\overline{n},(-\overline{\kappa}\overline{T}+\overline{\tau}\overline{b}))}{\big\langle(-\overline{\kappa}\overline{T}+\overline{\tau}\overline{b}),(-\overline{\kappa}\overline{T}+\overline{\tau}\overline{b})\big\rangle}=\frac{\overline{\tau}}{\overline{\kappa}^{2}+\overline{\tau}^{2}} \tag{52}$$

and

$$\overline{P}_{\overline{b}} = \frac{det(\overline{T}, \overline{b}, -\overline{\tau}\overline{n})}{\langle -\overline{\tau}\overline{n}, -\overline{\tau}\overline{n} \rangle} = \frac{1}{\overline{\tau}}. \tag{53}$$

(52) can also be written as

$$\overline{P}_{\overline{n}} = \frac{\frac{1}{\overline{\tau}}}{\left(\frac{\overline{\kappa}}{\overline{\tau}}\right)^2 + 1}. \tag{54}$$

Using (53) and (54), we obtain (51).

Remark. From (11) we have $cos\Phi^r = -sin\overline{\theta}$, therefore (51) becomes (2).

Theorem 19. (i) *The parallel ruled surface with Darboux frame which is drawn by $\overline{X}$ is a line in a normal plane in $\{\overline{T}, \overline{n}, \overline{b}\}$ during the motion H/H' is developable if and only if $\overline{\alpha}(s)$ is a plane curve.*

(ii) *The parallel ruled surface with Darboux frame which is drawn by $\overline{X}$ is a line in an osculator plane in $\{\overline{T}, \overline{n}, \overline{b}\}$ during the motion H/H' is developable if and only if $\overline{\alpha}(s)$ is a plane curve or the parallel ruled surface with Darboux frame is tangent developable surface since $\varphi = 0$.*

(iii) *The parallel ruled surface with Darboux frame which is drawn by $\overline{X}$ is a line in a rectifian plane in $\{\overline{T}, \overline{n}, \overline{b}\}$ during the motion H/H' is developable if and only if $\frac{\overline{\kappa}}{\overline{\tau}} = tan\varphi$ or the parallel ruled surface with Darboux frame is tangent developable surface since $\varphi = 0$.*

Proof. (i) $\overline{X}$ is a line in a normal plane, therefore from (42), we have

$$cos\varphi = 0. \tag{55}$$

If the parallel ruled surface with Darboux frame is developable, since (46) we find

$$\overline{\tau} sin^2 \varphi = 0. \tag{56}$$

From (55) and (56), $\overline{\alpha}(s)$ is a plane curve.

Conversely, if $\overline{\alpha}(s)$ is a plane curve, thus from (55) and vanishing of torsion in (46), we obtain that the parallel ruled surface with Darboux frame is developable.

(ii) $\overline{X}$ is a line in an osculator plane, hence from (42), we obtain

$$sin\varphi cos\Phi^r = 0. \tag{57}$$

If the parallel ruled surface with Darboux frame is developable, then the subcases of $sin\varphi cos\Phi^r = 0$ can be examined as follows:

Case 1. If $cos\Phi^r = 0$, then from (57), the parallel ruled surface with Darboux frame is tangent developable surface since $\varphi = 0$ or $\overline{\alpha}(s)$ is a plane curve.

Case 2. If $sin\varphi = 0$, then from (57), the parallel ruled surface with Darboux frame is tangent developable surface since $\varphi = 0$ or $\overline{\alpha}(s)$ is a plane curve.

Conversely, if $\overline{\alpha}(s)$ is a plane curve or $\varphi = 0$, then from (56) and (46), it is shown that the parallel ruled surface with Darboux frame is developable.

(iii) $\overline{X}$ is a line in a rectifian plane, so from (42), we have

$$sin\varphi sin\Phi^r = 0. \tag{58}$$

If the parallel ruled surface with Darboux frame is developable, then the subcases of (58) are examined according to (46) as follows:

Case 1. If $sin\varphi = 0$ and $sin\Phi^r \neq 0$, we get $\varphi = 0$. Hence the parallel ruled surface with Darboux frame is tangent developable surface.

Case 2. If $sin\Phi^r = 0$ and $sin\varphi \neq 0$, then $\frac{\overline{\kappa}}{\overline{\tau}} = tan\varphi$. Conversely, if $\frac{\overline{\kappa}}{\overline{\tau}} = tan\varphi$ or $\varphi = 0$, then from (46), it is shown that the parallel ruled surface with Darboux frame is developable surface.

Theorem 20. *The parallel ruled surface with Darboux frame which is drawn by $\overline{X}$ is a line in a rectifian plane in $\{\overline{T}, \overline{n}, \overline{b}\}$ during the motion H/H' is developable, then the relation between $\overline{P}_{\overline{b}}$ and $\overline{P}_{\overline{n}}$ is calculated as follows:*

$$\frac{\overline{P}_{\overline{b}}}{\overline{P}_{\overline{n}}} = \frac{(\overline{L}^2{}_{\overline{T}} - \overline{L}^2{}_{\overline{X}})^2 \overline{\lambda}^2_{\overline{b}}}{(\overline{L}_{\overline{T}}\overline{\lambda}_{\overline{X}} - \overline{L}_{\overline{X}}\overline{\lambda}_{\overline{T}})^2 \overline{L}^2{}_{\overline{X}}} + 1 = \left(\frac{sin\varphi cos\Phi^r}{cos\varphi}\right)^2 + 1.$$

Proof. Since $\overline{X}$ is a line in a rectifian plane from (42), we have the equation (58). Hence $sin\varphi = 0$ or $sin\Phi^r = 0$. If the parallel ruled surface with Darboux frame is developable, then substituting (49) into (51) gives

$$\left(\frac{\overline{\kappa}}{\overline{\tau}}\right)^2 = \left(\frac{(\overline{L}^2{}_{\overline{T}} - \overline{L}^2{}_{\overline{X}})\overline{\lambda}_{\overline{b}}}{(\overline{\lambda}_{\overline{X}}\overline{L}_{\overline{T}} - \overline{L}_{\overline{X}}\overline{\lambda}_{\overline{T}})\overline{L}_{\overline{X}}}\right)^2 = \frac{\overline{P}_{\overline{b}}}{\overline{P}_{\overline{n}}} - 1.$$

By (48) and (49), we get

$$sin\varphi = \frac{\sqrt{\overline{L}^2{}_{\overline{T}} - \overline{L}^2{}_{\overline{X}}}}{\overline{L}_{\overline{T}}}. \tag{59}$$

Using (48), (49) and (59), we find

$$\frac{\overline{\kappa}}{\overline{\tau}} = -\frac{sin\varphi cos\Phi^r}{cos\varphi} = \frac{\overline{L}_{\overline{T}}\overline{\lambda}_{\overline{X}} - \overline{L}_{\overline{X}}\overline{\lambda}_{\overline{T}}}{\overline{L}_{\overline{X}}\overline{\lambda}_{\overline{b}}}.$$

Therefore, we have

$$\frac{\overline{P}_{\overline{b}}}{\overline{P}_{\overline{n}}} = \frac{(\overline{L}^2{}_{\overline{T}} - \overline{L}^2{}_{\overline{X}})^2 \overline{\lambda}^2_{\overline{b}}}{(\overline{L}_{\overline{T}}\overline{\lambda}_{\overline{X}} - \overline{L}_{\overline{X}}\overline{\lambda}_{\overline{T}})^2 \overline{L}^2{}_{\overline{X}}} + 1 = \left(\frac{sin\varphi cos\Phi^r}{cos\varphi}\right)^2 + 1. \tag{60}$$

Now, by studying special cases of (58), let us interpret (60):

Case 1. If $sin\varphi sin\Phi^r = 0$ while $sin\Phi^r \neq 0$ and $sin\varphi = 0$, then we get $\frac{\overline{P}_{\overline{b}}}{\overline{P}_{\overline{n}}} = 1$.

Case 2. If $sin\varphi sin\Phi^r = 0$ while $sin\Phi^r = 0$ and $sin\varphi \neq 0$, then we find $\frac{\overline{P}_{\overline{b}}}{\overline{P}_{\overline{n}}} = sec^2\varphi$.

Case 3. If $sin\varphi sin\Phi^r = 0$ while $sin\Phi^r = 0$ and $sin\varphi = 0$, then we have $\frac{\overline{P}_{\overline{b}}}{\overline{P}_{\overline{n}}} = 1$.

Theorem 21. *The parallel ruled surface with Darboux frame which is drawn by $\overline{X}$ is a line in a rectifian plane in $\{\overline{T}, \overline{n}, \overline{b}\}$ during the motion H/H' in the fixed space is always in the rectifian plane of the striction curve.*

Proof. In order that the base curve of the parallel ruled surface is the striction curve, we have

$$\left\langle \overline{X}', \overline{T} \right\rangle = \overline{\kappa} sin\varphi sin\Phi^r = 0$$

Since $\overline{\kappa} \neq 0$, we have $sin\varphi sin\Phi^r = 0$. Therefore, we obtain

$$\overline{X} = \overline{T}cos\varphi - \overline{b}sin\varphi cos\Phi^r. \tag{61}$$

(61) means that the fixed space is always in the rectifian plane of the striction curve.

Theorem 22. *The parallel ruled surface with Darboux frame which is drawn by $\overline{X}$ is a line in a rectifian plane in $\{\overline{T},\overline{n},\overline{b}\}$ during the motion H/H' is developable, then the surface is tangent developable ruled surface.*

Proof. We calculate the distribution parameter according to $\overline{X}$ as

$$\overline{P}_{\overline{X}} = \frac{det(\overline{T},\overline{T}cos\varphi - \overline{b}sin\varphi cos\Phi^r, \overline{n}(\overline{\kappa}cos\varphi + \overline{\tau}sin\varphi cos\Phi^r))}{\|\overline{n}(\overline{\kappa}cos\varphi + \overline{\tau}sin\varphi cos\Phi^r)\|^2}.$$

Since $\overline{X}$ is a line in a rectifian plane, from (58), we get

$$\overline{P}_{\overline{X}} = \frac{sin\varphi cos\Phi^r}{(\overline{\kappa}cos\varphi + \overline{\tau}sin\varphi cos\Phi^r)}.$$

If the parallel ruled surface with Darboux frame is developable, that is, $\overline{P}_{\overline{X}} = 0$, then using $sin\varphi sin\Phi^r = 0$ and $sin\varphi cos\Phi^r = 0$, we obtain $sin\varphi = 0$. Hence (42) turns into $\overline{X} = \overline{T}cos\varphi$, it means that the parallel ruled surface is a tangent developable surface.

Theorem 23. *The parallel ruled surface with Darboux frame whose striction curve is a closed spherical one is developable if and only if the angles of the pitch are zero.*

Proof. Assume that the parallel ruled surface with Darboux frame whose striction curve is a closed spherical one is developable. Since the total curvature of the closed spherical curve is zero, we find

$$\overline{\lambda}_{\overline{X}} = \overline{\lambda}_{\overline{T}} = \overline{\tau}ds = 0.$$

Conversely, given the angles of the pitch are zero, and also the striction curve of the parallel ruled surface is a closed spherical curve, we can write

$$\overline{\lambda}_{\overline{X}} = 0. \tag{62}$$

Hence, using (62) into (43), we have

$$-sin\varphi cos\Phi^r \overline{\kappa}ds = 0. \tag{63}$$

Based on (63), we have either $sin\varphi cos\Phi^r = 0$ or $\overline{\kappa}ds = 0$. We can examine these cases as follows:

(i) $cos\varphi = 1$ and $\overline{X} = \overline{T}$ from $sin\varphi cos\Phi^r = 0$ and Theorem 2.2.

(ii) $\overline{\kappa}ds = 0$ while $\overline{\kappa} > 0$, $ds \geq 0$, also since $\overline{\kappa} \neq 0$, ds must vanish, it means that s is a constant. For this case, the parallel ruled surface with Darboux frame is a cone because of the striction curve turns into a point.

5 Conclusion

In this paper, parallel ruled surfaces have been studied according to Darboux frame in Euclidean 3-space. Regarding to the invariants of Darboux frame such as geodesic curvature, normal curvature and geodesic torsion, some properties of parallel ruled surfaces such as developability, striction point and distribution parameter have been given in Euclidean

3-space. Steiner rotation vector of these kinds of surfaces have been also obtained in Euclidean 3-space. Using this rotation vector, pitch length and pitch angle of the parallel ruled surfaces with Darboux frame have been computed in terms of magnitudes of the main surface. Reserschers can try to see correspondings of these results obtained in this work at ambient space of Euclidean 3-space such as Galilean 3-space.

Acknowledgement. This work was supported by Scientific Research Projects Coordination Unit of Kırklareli University. Project number KLUBAP/054.

References

[1] Aydemir, İ. & Kasap, E. 2005. Timelike ruled surface with spacelike rulings in R. Kuwait Journal of Science & Engineering 32(2): 13-24.

[2] Çöken, A.C., Çiftçi, Ü. & Ekici, C. 2008. On parallel timelike ruled surfaces with timelike rulings. Kuwait Journal of Science & Engineering 35(1): 21-31.

[3] Darboux, G. 1896. Leçons sur la theorie generale des surfaces I-II-III-IV., Gauthier-Villars, Paris.

[4] Gray, A., Salamon, S. & Abbena, E. 2006. Modern differential geometry of curves and surfaces with Mathematica. Chapman and Hall/CRC.

[5] Hacısalihoğlu, H. H. 1983. Diferensiyel geometri. İnönü Üniv. Fen Edebiyat Fak. Yay. No.2.

[6] Hlavaty, V. 1945. Differentielle linien geometrie. Uitg P. Noorfhoff, Groningen.

[7] Hoschek, J. 1973. Integral invarianten von regelflachen. Arch. Math, XXIV.

[8] Karadağ, H. B., Kılıç, E. & Karadağ, M. 2014. On the developable ruled surfaces kinematically generated in Minkowski 3-Space. Kuwait Journal of Science 41(1): 21-34.

[9] Kühnel, W. 2002. Differential geometry, curves-surfaces-manifolds. American Mathematical Society.

[10] Müller, H. R. 1978. Verallgemeinerung einer formelvon Steiner. Abh. Braunschweig Wiss. Ges. 31:107-113, 1978.

[11] O'Neill, B. 1996. Elementary differential geometry. Academic Press Inc, New York.

[12] Ravani, T. & Ku S. 1991. Bertrand offsets of ruled surface and developable surface. Comp Aided Geom Design 23(2):145-152.

[13] Savcı, Z. 2011. On parallel ruled Weingarten surfaces in 3-dimensional Euclidean space. (in Turkish) PhD Thesis, Eskişehir Osmangazi Univ. Grad. Sch. Nat. Sci., Eskisehir.

[14] Şentürk, G. Y. & Yüce, S. 2015. Characteristic properties of the ruled surface with Darboux frame in E^3.Kuwait Journal of Science, 42(2):14-33.

[15] Şentürk, G. Y. & Yüce, S. 2015a. Properties of integral invariants of the involute-evolute offsets of ruled surfaces. Int. J. Pure Appl. Math. 102(4):757-768.

[16] Ünlütürk, Y. & Ekici, C. 2014. Parallel surfaces satisying the properties of ruled surfaces in Minkowski 3-space. Global Journal of Science Frontier Research F, 14(1):78-95.