Communication in Mathematical Modeling and Applications
http://ntmsci.com/cmma

Some characterizations of constant breadth timelike curves in Minkowski 4-space E_1^4

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Received: 13 February 2016, Revised: 15 February 2016, Accepted: 1 March 2016
Published online: 29 April 2016.

Abstract: In this study, the differential equation characterizations of constant breadth timelike curves are given in Minkowski 4-space E_1^4. Furthermore, a criterion for a timelike curve to be a curve of constant breadth in E_1^4 is introduced. As an example, the obtained results are applied to the case that the curvatures k_1, k_2, k_3 and are discussed.

Keywords: Constant breadth curve, timelike curve, Frenet frame.

1 Introduction

Euler introduced the constant breadth curves in 1778 [6]. He considered these special curves in the plane. Later, many geometers have shown increased interest in the properties of plane convex curves. Struik published a brief review of the most important publications on this subject [19]. Also, Ball [1], Barbier [2], Blaschke [3,4] and Mellish [13] investigated the properties of plane curves of constant breadth. A space curve of constant breadth was obtained by Fujiwara by taking a closed curve whose normal plane at a point P has only one more point Q in common with the curve, and for which the distance $d(P,Q)$ is constant [7]. He also defined and studied constant breadth surfaces. Later, Smakal studied the constant breadth space curves [18]. Furthermore, Blaschke considered the notion of curve of constant breadth on the sphere [4]. Moreover, Reuleaux studied the curves of constant breadth and gave the method related to these curves for the kinematics of machinery [15]. Then, constant breadth curves had an importance for engineering sciences and by considering this fact Tanaka used the constant breadth curves in the kinematics design of Com follower systems [20].

Moreover, Köse has presented some concepts for space curves of constant breadth in Euclidean 3-space in [10] and Sezer has obtained the differential equations characterizing space curves of constant breadth and introduced a criterion for these curves [17]. Constant breadth curves in Euclidean 4-space were given by Mağden and Köse [11]. Moreover, constant breath curves have been studied in Minkowski space. Kazaz, Önder and Kocayiğit have studied spacelike curves of constant breadth in Minkowski 4-space [8]. Later, Önder, Kocayiğit and Candan have obtained and studied the differential equations characterizing constant breadth curves in Minkowski 3-space [14]. Furthermore, Kocayiğit and Önder have showed that constant breadth spacelike curves are normal curves, helices, and spherical curves in some special cases im Minkowski 3-space [9]. Moreover, in [12] Mağden and Yılmaz have given characterizations curves of constant breadth in four dimensional Galilean space in terms Frenet-Serret vector fields. Also, Yılmaz and Turgut have presented partially null curves of constant breadth in Semi-Riemannian space [21].

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In this paper, we study the differential equations characterizing constant breadth timelike curves in the Minkowski 4-space E_1^4. Moreover, we give a criterion characterizing these curves in E_1^4.

2 Differential equations characterizing constant breadth timelike curves in E_1^4

Let (C) be a unit speed regular timelike curve in the Minkowski 4-space E_1^4 with parametrization $\alpha(s) : I \subset \mathbb{R} \to E_1^4$. Denote by $\{\mathbf{T},\mathbf{N},\mathbf{B},\mathbf{E}\}$ the moving Frenet frame along the timelike curve (C) in E_1^4. Then, the following Frenet formulae are given,

$$\begin{bmatrix} \mathbf{T}' \\ \mathbf{N}' \\ \mathbf{B}' \\ \mathbf{E}' \end{bmatrix} = \begin{bmatrix} 0 & k_1 & 0 & 0 \\ k_1 & 0 & k_2 & 0 \\ 0 & -k_2 & 0 & k_3 \\ 0 & 0 & -k_3 & 0 \end{bmatrix} \begin{bmatrix} \mathbf{T} \\ \mathbf{N} \\ \mathbf{B} \\ \mathbf{E} \end{bmatrix}$$

where k_1, k_2 and k_3 are the first, second and third curvatures of the curve (C), respectively and $\{\mathbf{T},\mathbf{N},\mathbf{B},\mathbf{E}\}$ denote the tangent, the principal normal, the first binormal and the second binormal vector fields, respectively and they satisfy the following equalities:

$$< \mathbf{T},\mathbf{T} >= -1, < \mathbf{N},\mathbf{N} >=< \mathbf{B},\mathbf{B} >=< \mathbf{E},\mathbf{E} >= 1,$$

where $<,>$ is the Lorentzian inner product defined by

$$< \mathbf{a},\mathbf{b} >= -a_1b_1 + a_2b_2 + a_3b_3 + a_4b_4,$$

here $\mathbf{a} = (a_1,a_2,a_3,a_4)$, $\mathbf{b} = (b_1,b_2,b_3,b_4)$ are the vectors in E_1^4 [22].

Definition 1. *Let (C) be a unit speed regular timelike curve in E_1^4 with position vector $\alpha(s)$. If (C) has parallel tangents $\mathbf{T}$ and $\mathbf{T}^*$ in opposite direction at the opposite points α and α^* of the curve and if the distance between these points is always constant then (C) is called a timelike curve of constant breadth in E_1^4. Moreover, a pair of curves (C) and (C^*) for which the tangents at the corresponding points are parallel and in opposite directions and the distance between these points is always constant is called a timelike curve pair of constant breadth in E_1^4.*

Let now (C) and (C^*) be a pair of unit speed curves in E_1^4 with position vectors $\alpha(s)$ and $\alpha^*(s^*)$, where s and s^* are arc length parameters of the curves, respectively. Let (C) and (C^*) have parallel tangents in opposite directions at opposite points. Then the curve (C^*) may be represented by the equation

$$\alpha^*(s) = \alpha(s) + m_1(s)\mathbf{T}(s) + m_2(s)\mathbf{N}(s) + m_3(s)\mathbf{B}(s) + m_4(s)\mathbf{E}(s) \tag{1}$$

where $m_i(s)$, $(1 \leq i \leq 4)$ are the differentiable functions of s which is the arc length of (C). Differentiating this equation with respect to s and using the Frenet formulae we obtain

$$\begin{aligned} \frac{\alpha^*(s)}{ds} = \mathbf{T}^*\frac{ds^*}{ds} = \left(1 + \frac{dm_1}{ds} + m_2k_1\right)\mathbf{T} + \left(m_1k_1 + \frac{dm_2}{ds} - m_3k_2\right)\mathbf{N} \\ + \left(m_2k_2 + \frac{dm_3}{ds} - m_4k_3\right)\mathbf{B} + \left(m_3k_3 + \frac{dm_4}{ds}\right)\mathbf{E}. \end{aligned}$$

Since $\mathbf{T} = -\mathbf{T}^*$ at the corresponding points of (C) and (C^*), we have

$$\begin{cases} \left(1+\frac{dm_1}{ds}+m_2k_1\right) = -\frac{ds^*}{ds}, \\ \left(m_1k_1+\frac{dm_2}{ds}-m_3k_2\right) = 0, \\ \left(m_2k_2+\frac{dm_3}{ds}-m_4k_3\right) = 0, \\ \left(m_3k_3+\frac{dm_4}{ds}\right) = 0. \end{cases} \tag{2}$$

It is well known that the curvature of (C) is $lim(\Delta\varphi/\Delta s) = (d\varphi/ds) = k_1(s)$, where $\varphi = \int_0^s k_1(s)ds$ is the angle between the tangent of the curve (C) and a given fixed direction at the point $\alpha(s)$. Then from (2) we have the following system

$$m_1' = -m_2 - f(\varphi),\ m_2' = m_3\rho k_2 - m_1,\ m_3' = m_4\rho k_3 - m_2\rho k_2,\ m_4' = -m_3\rho k_3. \tag{3}$$

Here and after we will use $(')$ to show the differentiation with respect to φ . In (3), $f(\varphi) = \rho+\rho^*$ and, $\rho = \frac{1}{k_1}$ and $\rho^* = \frac{1}{k_1^*}$ denote the radius of curvatures at the points α and α^*, respectively. From (3) eliminating m_2, m_3 and m_4 their derivatives we have the following differential equation

$$\frac{d}{d\varphi}\left[\frac{1}{\rho k_3}\frac{d}{d\varphi}\left[\frac{1}{\rho k_2}\left(\frac{d^2m_1}{d\varphi^2}-m_1\right)\right]-\frac{k_2}{k_3}\frac{dm_1}{d\varphi}\right]+\frac{k_3}{k_2}\left(\frac{d^2m_1}{d\varphi^2}-m_1\right)+\frac{d}{d\varphi}\left[\frac{1}{\rho k_3}\frac{d}{d\varphi}\left(\frac{1}{\rho k_2}\frac{df}{d\varphi}\right)-\frac{k_2}{k_3}f\right]+\frac{k_3}{k_2}\frac{df}{d\varphi}=0. \tag{4}$$

Then we can give the following theorem.

Theorem 1. *The general differential equation characterizing constant breadth timelike curves in E_1^4 is given by* (4).

Let now consider the system (3) again. The distance d between the opposite points α and α^* is the breadth of the curves and is constant, that is,

$$d^2 = \|\mathbf{d}\|^2 = \|\alpha^* - \alpha\|^2 = -m_1^2+m_2^2+m_3^2+m_4^2 = const. \tag{5}$$

Then the system (3) may be written as follows:

$$m_2 = -f(\varphi),\ m_2' = m_3\rho k_2,\ m_3' = m_4\rho k_3 - m_2\rho k_2,\ m_4' = -m_3\rho k_3,\ m_1 = 0, \tag{6}$$

or

$$m_1' = -m_2,\ m_2' = -m_1 + m_3\rho k_2,\ m_3' = m_4\rho k_3 - m_2\rho k_2,\ m_4' = -m_3\rho k_3, \tag{7}$$

which are the systems describing the curve (1).

Let us consider the system (7) with special chosen $m_1 = const.$. Here, eliminating first m_1, m_2, m_3 and their derivatives, and then m_1, m_2, m_4 and their derivatives, respectively, we obtain the following linear differential equations of second order

$$\begin{cases} (\rho k_3)m_4'' - (\rho k_3)' m_4' + (\rho k_3)^3 m_4 = 0,\ \ \rho k_2 \neq 0, \\ (\rho k_3)m_3'' - (\rho k_3)' m_3' + (\rho k_3)^3 m_3 = 0,\ \ \rho k_3 \neq 0. \end{cases} \tag{8}$$

By changing the variable φ of the form $\xi = \int_0^\varphi \rho(t)k_3(t)dt$, these equations can be transformed into the following differential equations with constant coefficients,

$$\frac{d^2m_4}{d\xi^2}+m_4 = 0 \ \text{ and } \ \frac{d^2m_3}{d\xi^2}+m_3 = 0, \tag{9}$$

respectively. Then, the general solutions of the differential equations in (9) are

$$\begin{cases} m_3 = A\cos\left(\int_0^{\varphi}\rho k_3 dt\right) + B\sin\left(\int_0^{\varphi}\rho k_3 dt\right), \\ m_4 = C\cos\left(\int_0^{\varphi}\rho k_3 dt\right) + D\sin\left(\int_0^{\varphi}\rho k_3 dt\right). \end{cases} \tag{10}$$

respectively, where A, B, C and D are real constants. Substituting (10) into (7), we obtain $A = -D$, $B = C$, and so, the set of the solutions of the system (7), in the form

$$\left\{ \begin{array}{c} m_1 = c = const.,\ m_2 = 0, \\ m_3 = A\cos\int_0^{\varphi}\rho k_3 dt + B\sin\int_0^{\varphi}\rho k_3 dt, \\ m_4 = B\cos\int_0^{\varphi}\rho k_3 dt - A\sin\int_0^{\varphi}\rho k_3 dt. \end{array} \right\} \tag{11}$$

Thus the equation (1) is described and since $d^2 = \|\alpha^* - \alpha\|^2 = const.$, from (11) the breadth of the curve is $d^2 = -c^2 + A^2 + B^2$.

Now, let us return to the system (6) with $m_1 = 0$. By changing the variable φ of the form $u = \int_0^{\varphi}\mu(t)dt$, $\mu = \rho k_3$ and eliminating m_1, m_2, m_4 and their derivatives we have the linear differential equation

$$\frac{d^2 m_3}{du^2} + m_3 = -\frac{d}{du}\left(\frac{k_2}{k_3} m_2\right), \tag{12}$$

which has the following solution

$$m_3 = A_1\cos\int_0^{\varphi}\rho k_3 dt + B_1\sin\int_0^{\varphi}\rho k_3 dt - \int_0^{\varphi}\cos[u(\varphi) - u(t)]\rho k_2 f(t)dt. \tag{13}$$

Then, the general solution of the system (6) is

$$\begin{cases} m_1 = 0, \\ m_2 = f(\varphi), \\ m_3 = A_1\cos\int_0^{\varphi}\rho k_3 dt + B_1\sin\int_0^{\varphi}\rho k_3 dt + \int_0^{\varphi}\cos[u(\varphi) - u(t)]\rho k_2 f(t)dt, \\ m_4 = B_1\cos\int_0^{\varphi}\rho k_3 dt - A_1\sin\int_0^{\varphi}\rho k_3 dt - \int_0^{\varphi}\sin[u(\varphi) - u(t)]\rho k_2 f(t)dt, \end{cases} \tag{14}$$

which determines the constant breadth timelike curve in (1) where A_1, B_1 are real constants.

Furthermore, in this case, i.e., $m_1 = 0$, from (4) we have the following differential equation

$$\frac{d}{d\varphi}\left[\frac{1}{\rho k_3}\frac{d}{d\varphi}\left(\frac{1}{\rho k_2}\frac{df}{d\varphi}\right) - \frac{k_2}{k_3} f\right] + \frac{k_2}{k_3}\frac{df}{d\varphi} = 0. \tag{15}$$

By changing the variable φ of the form $w = \int_0^{\varphi}\rho k_2 d\varphi$, (15) becomes

$$\frac{d}{dw}\left[\frac{k_2}{k_3}\left(\frac{d^2 f}{dw^2} - f\right)\right] + \frac{k_3}{k_2}\frac{df}{dw} = 0, \tag{16}$$

which also determines the constant breadth curve in (1).

So far we have dealt with a pair of timelike space curves having parallel tangents in opposite directions at corresponding points. Now let us consider a simple closed unit speed timelike space curve (C) in E_1^4 for which the normal plane of every point P on the curve meets the curve of a single opposite point Q other than P. Then, we may give the following theorem concerning the constant breadth timelike space curves in E_1^4.

Theorem 2. *Let (C) be a closed timelike space curve in E_1^4 having parallel tangents in opposite directions at the opposite points of the curve. If the chord joining the opposite points of (C) is a double-normal if and only if (C) is a timelike curve of constant breadth in E_1^4.*

Proof. Let the vector $\mathbf{d} = \alpha^* - \alpha = m_1\mathbf{T} + m_2\mathbf{N} + m_3\mathbf{B} + m_4\mathbf{E}$ be a double-normal of (C) where m_1, m_2, m_3 and m_4 are the functions of s, the arc length parameter of the curve. Then we get $\langle \mathbf{d}, \mathbf{T}^* \rangle = -\langle \mathbf{d}, \mathbf{T} \rangle = m_1 = 0$. Thus from (2) we have

$$m_2 \frac{dm_2}{ds} + m_3 \frac{dm_3}{ds} + m_4 \frac{dm_4}{ds} = 0. \tag{17}$$

It follows that $m_2^2 + m_3^2 + m_4^2 = constant$, i.e., the breadth of (C) is constant, i.e., (C) is a constant breadth timelike curve in E_1^4.

Conversely, if (C) is a constant breadth timelike curve in E_1^4 then $\|\mathbf{d}\|^2 = -m_1^2 + m_2^2 + m_3^2 + m_4^2 = constant$. Then as shown, $m_1 = 0$. This means that $\mathbf{d}$ is perpendicular to both $\mathbf{T}$ and $\mathbf{T}^*$. So, $\mathbf{d}$ is the double-normal of (C).

A simple closed timelike curve having parallel tangents in opposite directions at opposite points may be represented by the system (14). In this case a pair of opposite points of the curve is $(\alpha^*(\varphi), \alpha(\varphi))$ for φ, where $0 \le \varphi \le 2\pi$. Since (C) is a simple closed timelike curve we get $\alpha^*(0) = \alpha^*(2\pi)$. Hence from (14) we have

$$\int_0^{2\pi} \rho k_3 dt = 2n\pi, \quad (n \in \mathbb{Z}). \tag{18}$$

Using the equality $ds = \rho d\varphi$, this formula may be given as $\int_C k_3 ds = 2n\pi$, $(n \in \mathbb{Z})$. This says that the integral third curvature of (C) is zero. So, we can give the following corollary.

Corollary 1. *The total third curvature of a simple closed timelike curve (C) of constant breadth is $2n\pi$, $n \in \mathbb{Z}$.*

Furthermore, if we take $\frac{k_2}{k_3} = a = constant$, then from (16) we have

$$\frac{d^3 f}{dw^3} + K \frac{df}{dw} = 0. \tag{19}$$

where $K = -1 + \frac{1}{a^2}$. If we assume $K \neq \pm 1$, the general solution of (19) is

$$f = A_2 \sin \int_0^{\varphi} K\rho k_2 dt + B_2 \cos \int_0^{\varphi} K\rho k_2 dt + C_1, \tag{20}$$

where A_2, B_2 and C_1 are real constants. Since (C) is a simple closed timelike curve, i.e., $\alpha^*(0) = \alpha^*(2\pi)$, from (20) it follows,

$$\int_0^{\varphi} K\rho k_2 dt = 2n\pi, \quad (n \in \mathbb{Z}). \tag{21}$$

Using the equality $ds = \rho d\varphi$, this formula may be given as $\int_C k_2 ds = 2\frac{n}{K}\pi$, $(K, n \in \mathbb{Z})$. This says that the integral second curvature of (C) is $2\frac{n}{K}\pi$, $(K, n \in \mathbb{Z})$. So, we can give the following corollary.

Corollary 2. *The total second curvature of a simple closed constant breadth timelike curve (C) with $a = k_2/k_3 = constant$ is $2\frac{n}{K}\pi$, where $n \in \mathbb{Z}$ and $K = -1 + \frac{1}{a^2}$.*

3 A criterion for constant breadth timelike curves in E_1^4

Let us assume that (C) is a constant breadth timelike curve in E_1^4 and $\alpha(s)$ denotes the position vector of a generic point of the curve. If (C) is a closed curve, the position vector $\alpha(s)$ must be a periodic function of period $\omega = 2\pi$, where ω is the total length of (C). Then the curvatures $k_1(s)$, $k_2(s)$ and $k_3(s)$ are also periodic of the same period. However, periodicity of the curvatures and closeness of the curve are not sufficient to guarantee that a timelike space curve is a constant breadth curve in E_1^4. That is, if a timelike curve is closed curve (periodic), it may be the constant breadth curve or not. Therefore, to guarantee that a timelike curve is a constant breadth curve, we may use the system (7) characterizing a constant breadth timelike curve and follow the similar way given in [5].

For this purpose, first let us consider the following Frenet formulas at a generic point on the curve (C),

$$\frac{d\mathbf{T}}{ds} = k_1\mathbf{N}, \frac{d\mathbf{N}}{ds} = k_1\mathbf{T} + k_2\mathbf{B}, \frac{d\mathbf{B}}{ds} = -k_2\mathbf{N} + k_3\mathbf{E}, \frac{d\mathbf{E}}{ds} = -k_3\mathbf{B}. \tag{22}$$

Writing the formulas (22) in terms of φ and allowing for $\frac{d\varphi}{ds} = k_1 = \frac{1}{\rho}$ we have

$$\frac{d\mathbf{T}}{d\varphi} = \mathbf{N}, \frac{d\mathbf{N}}{d\varphi} = \mathbf{T} + \rho k_2\mathbf{B}, \frac{d\mathbf{B}}{d\varphi} = -\rho k_2\mathbf{N} + \rho k_3\mathbf{E}, \frac{d\mathbf{E}}{d\varphi} = -\rho k_3\mathbf{B}. \tag{23}$$

Furthermore we can write the Frenet vectors **T**, **N**, **B**, **E** in the coordinate forms as follows

$$\mathbf{T} = \sum_{i=1}^{4} t_i\mathbf{e_i}, \mathbf{N} = \sum_{i=1}^{4} n_i\mathbf{e_i}, \mathbf{B} = \sum_{i=1}^{4} b_i\mathbf{e_i}, \mathbf{E} = \sum_{i=1}^{4} \varepsilon_i\mathbf{e_i}. \tag{24}$$

Since $\{\mathbf{T},\mathbf{N},\mathbf{B},\mathbf{E}\}$ is the orthonormal base in E_1^4, putting (24) and their derivatives into (23), we have the systems of linear differential equations

$$\left\{\begin{array}{llll} \frac{dt_1}{d\varphi} = n_1, & \frac{dt_2}{d\varphi} = n_2, & \frac{dt_3}{d\varphi} = n_3, & \frac{dt_4}{d\varphi} = n_4 \\ \frac{dn_1}{d\varphi} = t_1 + \rho k_2 b_1, & \frac{dn_2}{d\varphi} = t_2 + \rho k_2 b_2, & \frac{dn_3}{d\varphi} = t_3 + \rho k_2 b_3, & \frac{dn_4}{d\varphi} = t_4 + \rho k_2 b_4 \\ \frac{db_1}{d\varphi} = \rho k_3\varepsilon_1 - \rho k_2 n_1, & \frac{db_2}{d\varphi} = \rho k_3\varepsilon_2 - \rho k_2 n_2, & \frac{db_3}{d\varphi} = \rho k_3\varepsilon_3 - \rho k_2 n_3, & \frac{db_4}{d\varphi} = \rho k_3\varepsilon_4 - \rho k_2 n_4 \\ \frac{d\varepsilon_1}{d\varphi} = -\rho k_3 b_1, & \frac{d\varepsilon_2}{d\varphi} = -\rho k_3 b_2, & \frac{d\varepsilon_3}{d\varphi} = -\rho k_3 b_3, & \frac{d\varepsilon_4}{d\varphi} = -\rho k_3 b_4. \end{array}\right\} \tag{25}$$

From (25), we find that $\{t_1,n_1,b_1,\varepsilon_1\}$, $\{t_2,n_2,b_2,\varepsilon_2\}$, $\{t_3,n_3,b_3,\varepsilon_3\}$ and
$\{t_4,n_4,b_4,\varepsilon_4\}$ are four independent solutions of the following system of differential equations:

$$\frac{d\psi_1}{d\varphi} = \psi_2, \frac{d\psi_2}{d\varphi} = \psi_1 + \rho k_2\psi_3, \frac{d\psi_3}{d\varphi} = \rho k_3\psi_4 - \rho k_2\psi_2, \frac{d\psi_4}{d\varphi} = -\rho k_3\psi_3. \tag{26}$$

If the curve (C) is the constant breadth timelike curve, then the systems (7) and (26) must be the same system. So, we observe that $\psi_1 = m_1$, $\psi_2 = m_2$, $\psi_3 = m_3$, $\psi_4 = m_4$. For brevity, we can write (7) or (26) in the form

$$\frac{d\psi}{d\varphi} = A(\varphi)\psi, \tag{27}$$

where

$$\psi = \begin{bmatrix} m_1 \\ m_2 \\ m_3 \\ m_4 \end{bmatrix}, A(\varphi) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \rho k_2 & 0 \\ 0 & -\rho k_2 & 0 & \rho k_3 \\ 0 & 0 & -\rho k_3 & 0 \end{bmatrix}.$$

Obviously, (27) is a special case of the general linear differential equations abbreviated to the form

$$\begin{cases} \frac{d\psi}{dt} = A(t)\psi, \\ \varphi = \begin{bmatrix} m_1 \\ m_2 \\ \vdots \\ m_n \end{bmatrix}, A(t) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}, (4 \leq n) \end{cases} \tag{28}$$

where $a_{ij}(t)$ are assumed to be continuous and periodic of period ω (See [5,16]). Let the initial conditions be $\psi_i(0) = x_i$, $(i = 1,2,...,n)$. Let us take $x = \left[x_1,\, x_2,\, ...,\, x_n\right]^T$ and

$$\psi(t,x) = \left[m_1(t,x),\, m_2(t,x),\, ...\, ,m_n(t,x)\right]^T.$$

Then the equation (28) may be written in the form $\frac{d\psi}{dt} = A(t)\psi$, $\psi(0) = x$ as is well known from [5], the solution $\psi(t,x)$ of this equation is periodic of period ω , if

$$\int_0^{\omega} A(\xi)\psi(\xi,x)d\xi = 0,$$

and

$$\begin{cases} \psi(t,x) = \{E + M(t)\}x, \ (E = \text{ unit matrix}), \\ M(t) = IA(t) + I^{(2)}A(t) + ... + I^{(n)}A(t) + ..., \\ (IA)(t) = I^{(I)}A(t) = \int_0^t A(\xi)d\xi, \\ (I^{(n)}A)(t) = \int_0^t A(\xi)(I^{(n-1)}A)(\xi)d\xi, \ n > 1. \end{cases} \tag{29}$$

Furthermore, the following theorem is given in [5].

Theorem 3. *The equations $\frac{d\psi}{dt} = A(t)\psi$ possess a non-vanishing periodic solution of period ω , if and only if $\det(M(\omega)) = 0$. In particular, in order that the equations $\frac{d\psi}{dt} = A(t)\psi$ possess n linearly independent periodic solutions of period ω, the necessary and sufficient condition is that $M(\omega)$ be a zero matrix.*

Now, let us apply this theorem to the system (27). If $M(\omega) = 0$, there exist the unit vector functions **T**, **N**, **B**, **E** of period ω, such that each set of functions $\{t_i, n_i, b_i, \varepsilon_i\}$, $(i = 1,2,3,4)$ form a solution of the equation (27) corresponding to the initial conditions (A_i, B_i, C_i, D_i). The curve (C) can be described as follows

$$\alpha(s) = \int_0^s \mathbf{T}(s)ds \quad \text{or} \quad \alpha(\varphi) = \int_0^{\varphi} \rho(\varphi)\mathbf{T}(\varphi)d(\varphi).$$

Here, to find **T**, we can make use of the equation

$$\begin{bmatrix} t_i \\ n_i \\ b_i \\ \varepsilon_i \end{bmatrix} = \{E + M(\varphi)\} \begin{bmatrix} A_i \\ B_i \\ C_i \\ D_i \end{bmatrix}, \ (i = 1,2,3,4), \tag{30}$$

which is established by (29). If we take the initial conditions as $t_i(0) = A_i$, $n_i(0) = B_i$, $b_i(0) = C_i$, $\varepsilon_i(0) = D_i$, $(i = 1,2,3,4)$ such that (A_1, A_2, A_3, A_4), (B_1, B_2, B_3, B_4), (C_1, C_2, C_3, C_4), (D_1, D_2, D_3, D_4) form an orthonormal frame, then from (30) we obtain

$$t_i = (1 + m_{11})A_i + m_{12}B_i + m_{13}C_i + m_{14}D_i; \ \ (i = 1,2,3,4). \tag{31}$$

When the timelike curve (C) is a constant breadth curve, which is also periodic of period ω, it is clear that

$$\int_0^{\omega} \rho t_i d\varphi = 0. \tag{32}$$

Hence, form (31) and (32), we have

$$A_i \int_0^{\omega} \rho(1+m_{11})d\varphi + B_i \int_0^{\omega} \rho m_{12} d\varphi + C_i \int_0^{\omega} \rho m_{13} d\varphi + D_i \int_0^{\omega} \rho m_{14} d\varphi = 0; \ (i = 1,2,3,4).$$

Since the coefficient determinant $\Delta \neq 0$ in this system, we obtain the equalities

$$\int_0^{\omega} \rho(1+m_{11})d\varphi = 0 = \int_0^{\omega} \rho m_{12} d\varphi = \int_0^{\omega} \rho m_{13} d\varphi = \int_0^{\omega} \rho m_{14} d\varphi, \tag{33}$$

which are the conditions for a timelike curve to be constant breadth curve in E_1^4. Here, we can take the period $\omega = 2\pi$ because of $0 \leq \varphi \leq 2\pi$. Thus we establish the following corollary.

Corollary 3. *Let (C) be a regular curve in E_1^4 such that $\rho(\varphi) > 0$, $k_2(\varphi)$ and $k_3(\varphi)$ are continuous periodic functions of period ω. Then (C) is a constant breadth timelike curve and also periodic of period ω, if and only if*

$$M(\omega) = 0, \ \int_0^{\omega} \rho(1+m_{11})d\varphi = 0 = \int_0^{\omega} \rho m_{12} d\varphi = \int_0^{\omega} \rho m_{13} d\varphi = \int_0^{\omega} \rho m_{14} d\varphi, \tag{34}$$

holds, where

$$\begin{cases} M(t) = IA(t) + I^{(2)}A(t) + \ldots + I^{(n)}A(t) + \ldots, \\ A(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & \rho k_2 & 0 \\ 0 & -\rho k_2 & 0 & \rho k_3 \\ 0 & 0 & -\rho k_3 & 0 \end{bmatrix} \end{cases} \tag{35}$$

and $m_{ij}(t)$ are the entries of the matrix $M(t)$.

By means of (29) and (35), the matrix $M(t)$ can be constructed and each m_{ij} involves infinitely many integrations. Hence, we can write the conditions (34) in the following forms:

$$\begin{cases} \int_0^{\omega} \rho(\varphi)d\varphi + \int_0^{\omega}\int_0^{r}\int_0^{s} \rho(\varphi)dsdtd\varphi \\ + \int_0^{\omega}\int_0^{\phi}\int_0^{p}\int_0^{r}\int_0^{s} \rho(\varphi)[1-\lambda(p)\lambda(s)]dtdsdrdpd\varphi + \ldots = 0 \\ \int_0^{\omega}\int_0^{s} \rho(\varphi)dtd\varphi + \int_0^{\omega}\int_0^{p}\int_0^{r}\int_0^{s} \rho(\varphi)[1-\lambda(t)\lambda(s)]dtdsdrd\varphi + \ldots = 0 \\ \int_0^{\omega}\int_0^{r}\int_0^{s} \rho(\varphi)\lambda(t)dtdsd\varphi \\ + \int_0^{\omega}\int_0^{\phi}\int_0^{p}\int_0^{r}\int_0^{s} \rho(\varphi)[\lambda(t)-\lambda(p)\{\lambda(t)\lambda(s)+\mu(t)\mu(s)\}]dtdsdrdpd\varphi + \ldots = 0 \\ \int_0^{\omega}\int_0^{p}\int_0^{r}\int_0^{s} \rho(\varphi)\lambda(s)\mu(t)dtdsdrd\varphi \\ + \int_0^{\omega}\int_0^{q}\int_0^{\phi}\int_0^{p}\int_0^{r}\int_0^{s} \rho(\varphi)\lambda(p)\mu(t)[1-\lambda(t)\lambda(s)-\mu(t)\mu(s)]dtdsdpd\phi d\varphi + \ldots = 0, \end{cases} \tag{36}$$

where $\lambda(\xi) = p(\xi)k_2(\xi)$, $\mu(\xi) = p(\xi)k_3(\xi)$.

*Example 1.*Let us consider the special case $\rho = const.$, $k_2 = const.$ and $k_3 = const.$ In this case, from (33), we have

$$\begin{cases} \omega + \frac{\omega^3}{3!} + (1-\rho^2k_2^2)\frac{\omega^5}{5!} + (1-\rho^2k_2^2)^2\frac{\omega^7}{7!} + ... = 0 \\ \frac{\omega^2}{2!} + (1-\rho^2k_2^2)\frac{\omega^4}{4!} + (1-\rho^2k_2^2)^2\frac{\omega^6}{6!} + ... = 0 \\ k_2[\frac{\omega^3}{3!} + (1-\rho^2k_2^2-\rho^2k_3^2)\frac{\omega^5}{5!} + (1-\rho^2k_2^2-\rho^2k_3^2)^2\frac{\omega^7}{7!} + ...] = 0 \\ k_2k_3[\frac{\omega^4}{4!} + (1-\rho^2k_2^2-\rho^2k_3^2)\frac{\omega^6}{6!} + ...] = 0, \end{cases} \tag{37}$$

or

$$\begin{cases} \rho^2k_2^2(1-\rho^2k_2^2)^{\frac{1}{2}}\omega - \sinh[(1-\rho^2k_2^2)^{\frac{1}{2}}\omega] = 0, \\ \cosh[(1-\rho^2k_2^2)^{\frac{1}{2}}\omega] = 1 \text{ or } (1-\rho^2k_2^2)^{\frac{1}{2}}\omega = 2k\pi, \ (k \in \mathbb{Z}), \\ k_2[(1-\rho^2k_2^2-\rho^2k_3^2)^{\frac{1}{2}}\omega - \sinh[(1-\rho^2k_2^2-\rho^2k_3^2)^{\frac{1}{2}}\omega]] = 0, \\ k_2k_3[(1-\rho^2k_2^2-\rho^2k_3^2)\frac{\omega^2}{4} - \sinh^2[(1-\rho^2k_2^2-\rho^2k_3^2)^{\frac{1}{2}}\omega]] = 0, \end{cases} \tag{38}$$

where $\omega = 2k\pi$. It is seen that all of the equalities (37) or (38) are satisfied simultaneously, if and only if $\rho k_2 = 0$, $\rho k_3 = 0$ that is, $\rho = const. > 0$ and $k_2, k_3 = 0$. Therefore, only ones with $\rho = const. > 0$ and $k_2, k_3 = 0$ of the curves with $\rho = const. > 0$ and $k_2, k_3 = const.$ are curves of constant breadth, which are Lorentzian circles in E_1^4.

Now let us construct the relation characterizing these circles. Since $\rho k_2 = 1, \rho k_3 = 0$, system (7) becomes

$$m_1' = -m_2, \ m_2' = m_3 - m_1, \ m_3' = -m_2, \ m_4' = 0. \tag{39}$$

The general solution of (39), is

$$\begin{cases} m_1 = \frac{c_1}{2}\varphi^2 + c_2\varphi + c_3 \\ m_2 = -c_1\varphi - c_2 \\ m_3 = \frac{c_1}{2}\varphi^2 + c_2\varphi + c_3 - c_1 \\ m_3 = c_4. \end{cases} \tag{40}$$

Consequently, replacing (40) into (1), we obtain the equation

$$\alpha^*(\varphi) = \alpha(\varphi) + (\frac{c_1}{2}\varphi^2 + c_2\varphi + c_3)\mathbf{T} + (-c_1\varphi - c_2)\mathbf{N} + (\frac{c_1}{2}\varphi^2 + c_2\varphi + c_3 - c_1)\mathbf{B} + c_4\mathbf{E},$$

which represents the Lorentzian circles with the diameter

$$d = \|\alpha^* - \alpha\| = (c_1^2 + c_2^2 + c_4^2 - 2c_1c_3)^{\frac{1}{2}}.$$

In this case, a pair of opposite points of the curve is $(\alpha^*(\varphi), \alpha(\varphi))$ for φ in $0 \leq \varphi \leq 2\pi$.

4 Conclusion

In the characterizations and determinations of the special curves and curve pairs are important in the curve theory. A differential equation or a system of differential equations with respect to the curvatures can determinate the special curves or curve pairs. In this paper, the differential equations characterizing the constant breadth timelike curves in are studied E_1^4. Furthermore, a criterion for a timelike space curve to be the curve of constant breadth in E_1^4 is given.

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