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A new computational approach for solving a boundary-value problem for DEPCAG

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Abstract

In this paper, a new computational approach is presented to solve a boundary-value problem for a differential equation with piecewise constant argument of generalized type (DEPCAG). The presented technique is based on the Dzhumabaev parametrization method. A useful numerical algorithm is developed to obtain the numerical values from the problem. Numerical experiments are conducted to demonstrate the accuracy and efficiency.

Keywords: two-point boundary value problem differential equation piecewise-constant argument of generalized type Dzhumabaev parametrization method numerical solution.
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1. Introduction and Motivation

Differential equations with piecewise constant arguments (abbreviated DEPCA) arise in an attempt to extend the theory of functional differential equations with continuous arguments to differential equations with discontinuous arguments. Applications of DEPCA are hybrid equations, which combine the properties of both continuous systems and discrete equations [1]. These equations have been studied by many researchers in diverse fields such as biomedicine, chemistry, biology, physics, population dynamics, and mechanical engineering [2]-[4].

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M.Akhmet [1] considered the equation

$$\frac{dx}{dt} = f(t, x(t), x(\beta(t))),$$

where $\beta(t) = \theta_i$ if $\theta_i \leq t < \theta_{i+1}$, i are integers, is an identification function, θ_i is a strictly increasing sequence of real numbers. These equations are called differential equations with piecewise constant argument of generalized type (abbreviated DEPCAG). DEPCAG have been studied in [1], [5] and have attracted the attention of many scientists [6]-[9]. DEPCAG are closely related to impulse and loaded equations [10]-[12] and, especially, to difference equations of a discrete argument. The theory of DEPCAG is useful while investigating dynamic behavior of the real life problems. Various models in biology, mechanics, and electronics were developed by using these systems [13], [14].

In the present paper, we consider the following boundary-value problem for system of differential equations with piecewise constant argument of generalized type

$$\frac{dz}{dt} = A_0(t)z + K_0(t)z(\gamma(t)) + f(t), \quad z \in R^n, \quad t \in [0, T], \tag{1}$$

$$B_0 z(0) + C_0 z(T) = d_0, \quad d_0 \in R^n, \tag{2}$$

where $(n \times n)$ -matrices $A_0(t)$, $K_0(t)$ are continuous on $[0, T]$, and the n-vector-function $f(t)$ are piecewise continuous on $[0, T]$ with possible discontinuities of the first kind at the points $t = t_i$, $(i = \overline{1, m})$; $(n \times n)$ -matrices B_0 and C_0 and n -vector d_0 are constant, $\|x\| = \max\limits_{i=\overline{1,n}} |x_i|$. The argument $\gamma(t)$ is a step function defined as $\gamma(t) = \chi_j$ if $t \in [t_j, t_{j+1})$, $j = \overline{0, m}$; $t_j \leq \chi_j \leq t_{j+1}$ for all $j = \overline{0, m}$; where $0 = t_0 < t_1 < ... < t_m < t_{m+1} = T$.

A function $z(t)$ is called a solution to problem (1), (2) if:

(i) $z(t)$ is continuous on $[0, T]$;

(ii) $z(t)$ is differentiable on $[0, T]$ with the possible exception of the points t_j, $j = \overline{0, m}$, where the one-sided derivatives exist;

(iii) $z(t)$ satisfies (1) on each interval (t_j, t_{j+1}), $j = \overline{0, m}$; at the points t_j, Eq. (1) is satisfied by the right-hand derivatives of $z(t)$;

(iv) $z(t)$ satisfies the boundary condition (2).

Our main goal in this paper is to developed another approach to the investigation of boundary-value problem for a system of differential equation with piecewise constant argument of generalized type, different from what was proposed by the founder of the DEPCAG theory [1].

The rest of this paper is organized as follows. In Section 2, some necessary notations and the scheme of the Dzhumabaev parametrization method [15] for solving two-point boundary-value problem for a differential equation with piecewise constant argument of generalized type are given. Analytic solution of considering problem is also discussed. In Section 3, we propose numerical algorithm of solving two-point boundary-value problem for a system of DEPCAG. Numerical experiments and table values showing the advantage of the applied methods are given in Section 4.

2. Methodology of the Dzhumabaev parametrization method

We use the approach offered in [15]-[18] to solve the boundary-value problem for the system of DEPCAG (1), (2). This approach based on the algorithms of the Dzhumabaev parametrization method and numerical methods for solving initial value problems.

The interval $[0, T]$ is divided into subintervals by points:
$[0, T) = \bigcup\limits_{r=1}^{m+1} [t_{r-1}, t_r)$.

Let $C([0, T], \mathbb{R}^n)$ be the space of continuous functions $z : [0, T] \to \mathbb{R}^n$ with norm $\|z\|_1 = \max\limits_{t\in[0,T]} ||z(t)|| = \max\limits_{t\in[0,T]} \max\limits_{i=\overline{1,n}} |z_i(t)|$;

$C([0,T],t_r,\mathbb{R}^{n(m+1)})$ be the space of functions systems $z[t]=(z_1(t),z_2(t),\ldots,z_{m+1}(t))'$, where $z_r:[t_{r-1},t_r)\to\mathbb{R}^n$ are continuous and have finite left-hand side limits $\lim\limits_{t\to t_r-0} z_r(t)$ for all $r=\overline{1,m+1}$ with norm $\|z[\cdot]\|_2=\max\limits_{r=\overline{1,m+1}}\sup\limits_{t\in[t_{r-1},t_r)}|z_r(t)|$.

Denote by $z_r(t)$ a restriction of function $z(t)$ on r-th interval $[t_{r-1},t_r)$, i.e.

$z_r(t)=z(t)$ for $t\in[t_{r-1},t_r)$, $r=\overline{1,m+1}$.

Then the function system $z[t]=(z_1(t),z_2(t),\ldots,z_{m+1}(t))$ belongs to $C([0,T],t_r,R^{n(m+1)})$, and its elements $z_r(t)$, $r=\overline{1,m+1}$, satisfy the following boundary-value problem for the system of DEPCAG

$$\frac{dz_r}{dt}=A_0(t)z_r+K_0(t)z_r(\chi_{r-1})+f(t),\quad t\in[t_{r-1},t_r),\quad r=\overline{1,m+1}, \tag{3}$$

$$B_0z_1(0)+C_0\lim_{t\to T-0}z_{m+1}(t)=d_0, \tag{4}$$

$$\lim_{t\to t_p-0}z_p(t)=z_{p+1}(t_p),\quad p=\overline{1,m}. \tag{5}$$

In (3) we take into account that $\gamma(t)=\chi_j$ if $t\in[t_j,t_{j+1})$, $j=\overline{0,m}$.

The system of equations of the form (3) refers to loaded differential equations or equations with discrete memory effects.

Equations with discrete memory effects were investigated in [19], [20] and the references therewith. These equations are actively used in problems of mathematical modeling and control of groundwater level in the soil moisture. Various problems with discrete memory effects and methods for solving them are studied in the literature [21]-[31].

Introduce parameters $\mu_r=z_r(\chi_{r-1})$ for all $r=\overline{1,m+1}$. Making the substitution $w_r(t)=z_r(t)-\mu_r$ on every r-th interval $[t_{r-1},t_r)$, we obtain the boundary value problem with parameters:

$$\frac{dw_r}{dt}=A_0(t)(w_r+\mu_r)+K_0(t)\mu_r+f(t),\quad t\in[t_{r-1},t_r), \tag{6}$$

$$w_r(\chi_{r-1})=0,\qquad r=\overline{1,m+1}, \tag{7}$$

$$B_0w_1(0)+B_0\mu_1+C_0\mu_{m+1}+C_0\lim_{t\to T-0}w_{m+1}(t)=d_0, \tag{8}$$

$$\mu_p+\lim_{t\to t_p-0}w_p(t)=\mu_{p+1}+w_{p+1}(t_p),\quad p=\overline{1,m}. \tag{9}$$

The initial conditions of the Cauchy problem (6), (7) are given at the interior points of the interval $[t_{r-1},t_r)$, $r=\overline{1,m+1}$.

A solution to problem (6)–(9) is a pair $(\mu^*,w^*[t])$, with elements $\mu^*=(\mu_1^*,\mu_2^*,\ldots,\mu_{m+1}^*)\in R^{n(m+1)}$, $w^*[t]=\big(w_1^*(t),w_2^*(t),\ldots,w_{m+1}^*(t)\big)\in C([0,T],t_r,R^{n(m+1)})$, where $w_r^*(t)$ are continuously differentiable on $[t_{r-1},t_r)$, $r=\overline{1,m+1}$, and satisfying the system of ordinary differential equations (6), initial conditions (7) and conditions (8), (9) at the $\mu_r=\mu_r^*$, $j=\overline{1,m+1}$.

The problem (1), (2) and (6)–(9) are equivalent. Really, if a function $z^*(t)$ is a solution to problem (1), (2), then the pair $(\mu^*,w^*[t])$, where $\mu^*=\Big(z^*(\chi_0),z^*(\chi_1),\ldots,z^*(\chi_m)\Big)$, and $w^*[t]=\Big(z^*(t)-z^*(\chi_0),z^*(t)-z^*(\chi_1),\ldots,z^*(t)-z^*(\chi_m)\Big)$ is a solution to problem (6)–(9). Conversely, if a pair $(\widetilde{\mu},\widetilde{w}[t])$, with elements $\widetilde{\mu}=(\widetilde{\mu}_1,\widetilde{\mu}_2,\ldots,\widetilde{\mu}_{m+1})$, $\widetilde{w}[t]=\big(\widetilde{w}_1(t),\widetilde{w}_2(t),\ldots,\widetilde{w}_{m+1}(t)\big)$, is a solution to problem (6)–(9), then the function $\widetilde{z}(t)$ defined by the equalities

$\widetilde{z}(t)=\widetilde{w}_r(t)+\widetilde{\mu}_r$, $t\in[t_{r-1},t_r)$, $r=\overline{1,m+1}$, $\widetilde{z}(T)=\widetilde{\mu}_{m+1}+\lim\limits_{t\to T-0}\widetilde{w}_{m+1}(t)$,

will be the solution of the original problem (1), (2).

Let take $\mathbb{X}_r(t)$ a fundamental matrix of the differential equation

$\dfrac{dz_r}{dt}=A(t)z_r(t)$ on $[t_{r-1},t_r]$, $r=\overline{1,m+1}$.

Then the solution to the Cauchy problem (6), (7) can be written as follows

$$w_r(t) = \mathbb{X}_r(t) \int_{\chi_{r-1}}^{t} \mathbb{X}_r^{-1}(\tau)\Big[A_0(\tau) + K_0(\tau)\Big] d\tau \mu_r + \mathbb{X}_r(t) \int_{\chi_{r-1}}^{t} \mathbb{X}_r^{-1}(\tau) f(\tau) d\tau, \tag{10}$$

$t \in [t_{r-1}, t_r)$, $\quad r = \overline{1, m+1}$.
Consider the Cauchy problems on the subintervals

$$\frac{dy}{dt} = A_0(t)y + \mathbb{P}(t), \quad y(\chi_{r-1}) = 0, \quad t \in [t_{r-1}, t_r], \quad r = \overline{1, m+1}, \tag{11}$$

where $\mathbb{P}(t)$ is a square matrix or a vector of dimension n, piecewise continuous on $[0, T]$ with possible discontinuities of the first kind at the points $t = t_i$, $(i = \overline{1, m})$, $t_{r-1} \le \chi_{r-1} \le t_r$ for all $r = \overline{1, m+1}$. Denote by $\mathbb{S}_r(\mathbb{P}, t)$ a unique solution to the Cauchy problem (11) on each r-th interval. The uniqueness of the solution to the Cauchy problem for ordinary differential equations yields

$$\mathbb{S}_r(\mathbb{P}, t) = \mathbb{X}_r(t) \int_{\chi_{r-1}}^{t} \mathbb{X}_r^{-1}(\tau)\mathbb{P}(\tau) d\tau, \; t \in [t_{r-1}, t_r], \; r = \overline{1, m+1}. \tag{12}$$

Substituting the right-hand side of (10) using (12) into the boundary condition (8) and the continuity condition (9), we obtain the following system of linear algebraic equations:

$$B_0\mu_1 + B_0\mathbb{S}_1(A_0 + K_0, t_0)\mu_1 + C_0\mu_{m+1} + C_0\mathbb{S}_{m+1}(A_0 + K_0, T)\mu_{m+1} = \\ = d_0 - B_0\mathbb{S}_1(f, t_0) - C_0\mathbb{S}_{m+1}(f, T), \tag{13}$$

$$\mu_p + \mathbb{S}_p(A_0 + K_0, t_p)\mu_p - \mu_{p+1} - \mathbb{S}_{p+1}(A_0 + K_0, t_p)\mu_{p+1} = \mathbb{S}_{p+1}(f, t_p) - \mathbb{S}_p(f, t_p), \quad p = \overline{1, m}. \tag{14}$$

Let us rewrite the system (13), (14) as

$$Q(h)\mu = F(h), \qquad \mu \in R^{n(m+1)}, \tag{15}$$

where $F(h) = \big(d_0 - B_0\mathbb{S}_1(f, t_0) - C_0\mathbb{S}_{m+1}(f, T),\; \mathbb{S}_2(f, t_1) - \mathbb{S}_1(f, t_1),\; \mathbb{S}_3(f, t_2) - \mathbb{S}_2(f, t_2),\; \ldots,\; \mathbb{S}_{m+1}(f, t_m) - \mathbb{S}_m(f, t_m)\big) \in \mathbb{R}^{n(m+1)}$,

$$Q(h) = \begin{pmatrix} B_0[I + \mathbb{S}_1(A_0 + K_0, t_0)] & O & \ldots & C_0[I + \mathbb{S}_{m+1}(A_0 + K_0, T)] \\ I + \mathbb{S}_1(A_0 + K_0, t_1) & -I - \mathbb{S}_2(A_0 + K_0, t_1) & \ldots & O \\ O & I + \mathbb{S}_2(A_0 + K_0, t_2) & \ldots & O \\ \vdots & \vdots & \ddots & \vdots \\ O & O & \ldots & -I - \mathbb{S}_{m+1}(A_0 + K_0, t_m) \end{pmatrix}$$

where I and O will denote the identity and the zero matrix of size n.

It is simple to set that the (1), (2) boundary value problem's solvability is equivalent to the (15) system's solvability. The solution of the system (15) is a vector $\mu^* = \Big(\mu_1^*, \mu_2^*, \ldots, \mu_{m+1}^*\Big)$ consists of the values of the solutions of the original problem (1), (2) in the interior points of subintervals, i.e. $\mu_r^* = z^*(\chi_{r-1})$, $r = \overline{1, m+1}$.

If the fundamental matrices $\mathbb{X}_r(t)$, $r = \overline{1, m+1}$, are known, then we can construct the system (15). Let μ^* be a solution to (15) and define the solution to the boundary value problem (1), (2) by the equalities:

$$z^*(t) = \mathbb{X}_r(t)\mathbb{X}_r^{-1}(\chi_{r-1})\mu_r^* + \mathbb{S}_r(K_0, t)\mu_r^* + \mathbb{S}_r(f, t), \; t \in [t_{r-1}, t_r), \; r = \overline{1, m+1}. \tag{16}$$

So, the proposed method gives us a solution to the boundary-value problem for the system of DEPCAG (1), (2) in the form (16).

As a rule, the construction of fundamental matrix for the systems of ordinary differential equations with variable coefficients fails. Therefore, we offer the numerical algorithm for solving problem (1), (2) in the following Section 3.

3. An algorithm for solving problem (1), (2)

Step 1. Divide each r-th interval $[t_{r-1}, t_r]$, $r = \overline{1, m+1}$, into N_r parts. Define the approximate values of coefficients and right-hand side of (15) via solutions to the following Cauchy matrix and vector problems with initial conditions at the interior points:

$$\frac{dy}{dt} = A_0(t)y + A_0(t), \qquad y(\chi_{r-1}) = 0, \quad t \in [t_{r-1}, t_r], \quad r = \overline{1, m+1},$$

$$\frac{dy}{dt} = A_0(t)y + K_0(t), \qquad y(\chi_{r-1}) = 0, \qquad t \in [t_{r-1}, t_r], \quad r = \overline{1, m+1},$$

$$\frac{dy}{dt} = A_0(t)y + f(t), \qquad y(\chi_{r-1}) = 0, \qquad t \in [t_{r-1}, t_r], \quad r = \overline{1, m+1}.$$

Step 2. Then we obtain the following approximate system of algebraic equations with respect to parameters μ:

$$Q_*(h)\mu^* = F_*(h), \qquad \mu^* \in \mathbb{R}^{n(m+1)}. \tag{17}$$

Solve the composed system and find $\mu^* = (\mu_1^*, \mu_2^*, \dots, \mu_{m+1}^*)$. Note that the elements of μ^* are the values of the solution to problem (1), (2): $\mu_r^* = z^*(\chi_{r-1})$, $r = \overline{1, m+1}$.

Step 3. Solve the following Cauchy problems

$$\frac{dy}{dt} = A_0(t)y + K_0(t)\mu_r^* + f(t),$$

$$y(\chi_{r-1}) = \mu_r^*, \qquad t \in [t_{r-1}, t_r), \quad r = \overline{1, m+1},$$

and define the values of the solution $z^*(t)$ at the remaining points of the subintervals.

To solve the Cauchy problems, we use the Adams method, Runge-Kutta method of order four, Bulirsch-Stoer method and Runge-Kutta-Fehlberg method [32], [33]. Thus the algorithm offered provides us with the numerical solution to the problem (1), (2).

The Cauchy problems for ordinary differential equations are solved at each step of the proposed algorithm. As noted above, the initial conditions are set at the interior points of the subintervals. Therefore, the Cauchy problems are solved to the left from χ_{r-1} to t_{r-1}, $r = \overline{1, m+1}$ and to the right from χ_{r-1} to t_r, $r = \overline{1, m+1}$. This is the one difference of the proposed algorithm from the previously proposed algorithm [10]-[12], [23].

To illustrate the proposed approach of the numerical solving of boundary value problem for the system of differential equations with piecewise constant argument of generalized type (1), (2) based on the Dzhumabaev parametrization method, let us consider the following examples.

4. Illustrative Examples

In this section, several numerical examples are given to illustrate the effectiveness properties of the method and all of them were performed on the computer using a program written in MathCad 15.

Example 4.1. *We consider the following boundary-value problem for the system of DEPCAG:*

$$\frac{dz}{dt} = \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} z + \begin{pmatrix} t^2 & 0 \\ 2 & t \end{pmatrix} z(\gamma(t)) + \begin{pmatrix} -18t^2 - 3t - 20 \\ 2t^2 - 32t - 17 \end{pmatrix}, \ z \in \mathbb{R}^2, \quad t \in [0, T], \tag{18}$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} z(0) + \begin{pmatrix} -1 & 3 \\ 0 & -1 \end{pmatrix} z(T) = \begin{pmatrix} 32 \\ -8 \end{pmatrix}, \tag{19}$$

where $\quad t_0 = 0, \quad t_1 = T = 1, \quad \gamma(t) = \chi_0 = \dfrac{1}{2}.$

In Example 4.1 a fundamental matrix of differential part is

$$\mathbb{X}(t)=\begin{pmatrix} e^{2t} & te^{2t} \\ e^{2t} & (1+t)e^{2t} \end{pmatrix}.$$

Introduce the parameter $\mu=z\left(\frac{1}{2}\right)$ as a value of solution at the interior point of interval $[0,T]$ and make replacement $w(t)=z(t)-\mu$. Using $\mathbb{X}(t)$ and boundary condition (19), we obtain the equation with respect to parameter μ

$$\begin{pmatrix} \frac{17}{8e}+\frac{9e}{8}-\frac{25}{8} & \frac{21e}{4}-\frac{9}{8e}-1 \\ \frac{3}{2e}-\frac{7e}{8}+\frac{3}{4} & \frac{3}{8e}-\frac{15e}{8}+\frac{1}{4} \end{pmatrix}\begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix}=\begin{pmatrix} \frac{209}{8e}+\frac{219e}{4}-57 \\ \frac{213}{8e}-\frac{217e}{8}+\frac{55}{4} \end{pmatrix}. \tag{20}$$

From (20) we find $\mu_1^*=16, \quad \mu_2^*=7$. In accordance with (16) we find a unique solution to problem (18), (19)

$$z^*(t)=\begin{pmatrix} e^{2t} & te^{2t} \\ e^{2t} & (1+t)e^{2t} \end{pmatrix}\left\{\begin{pmatrix} \frac{3}{2e} & \frac{-1}{2e} \\ \frac{-1}{e} & \frac{1}{e} \end{pmatrix}\cdot\begin{pmatrix} 16 \\ 7 \end{pmatrix}+\right.$$
$$\left.+\int\limits_{\frac{1}{2}}^{t}\begin{pmatrix} -2e^{-2\tau}(2\tau^3-10\tau^2+19\tau+10) \\ e^{-2\tau}(4\tau^2-22\tau+35) \end{pmatrix}d\tau\right\}=\begin{pmatrix} 2t^2-t+16 \\ 8t+3 \end{pmatrix}, \quad t\in[0,1].$$

In Example 1 we were able to construct the fundamental matrix of differential part of considered differential equations. Dzhumabaev parameterization method allows us to construct the solution to problem (18), (19) explicitly.

Example 4.2. *We consider the following boundary-value problem for the system of DEPCAG:*

$$\frac{dz}{dt}=\begin{pmatrix} t+1 & 2t^3 \\ 3t^2 & 7 \end{pmatrix}z+\begin{pmatrix} t^2 & 8t \\ 2 & t-8 \end{pmatrix}z(\gamma(t))+f(t), \quad z\in R^2, \quad t\in[0,T], \tag{21}$$

$$\begin{pmatrix} 4 & -3 \\ -5 & -1 \end{pmatrix}z(0)+\begin{pmatrix} 8 & -2 \\ -9 & 3 \end{pmatrix}z(T)=\begin{pmatrix} 7 \\ 14 \end{pmatrix}, \tag{22}$$

where $t_0=0, \quad t_1=T=1.$

In Example 4.2 the matrix of differential part is variable and the construction of fundamental matrix breaks down. Here we use the numerical algorithm for solving problem (21), (22). Let's consider three problems.

Problem A.

$$\gamma(t)=\chi_0=t_0=0, \quad f(t)=\begin{pmatrix} 8t^3-8t^5-2t^6+t^2+47t-3 \\ 2t^3-6t^4-25t^2+13t-5 \end{pmatrix}, \quad t\in\Big[0,1\Big].$$

Applying the method scheme above, introduce an additional parameter $\mu=z(0)$. Making the substitution $w(t)=z(t)-\mu, \quad t\in[0,1]$, we obtain the boundary value problem with parameters:

$$\frac{dw}{dt}=\begin{pmatrix} t+1 & 2t^3 \\ 3t^2 & 7 \end{pmatrix}(w+\mu)+\begin{pmatrix} t^2 & 8t \\ 2 & t-8 \end{pmatrix}\mu+\begin{pmatrix} 8t^3-8t^5-2t^6+t^2+47t-3 \\ 2t^3-6t^4-25t^2+13t-5 \end{pmatrix}, \quad t\in[0,1], \tag{23}$$

$$w(0)=0, \tag{24}$$

$$\begin{pmatrix} 4 & -3 \\ -5 & -1 \end{pmatrix}\mu+\begin{pmatrix} 8 & -2 \\ -9 & 3 \end{pmatrix}\mu+\begin{pmatrix} 8 & -2 \\ -9 & 3 \end{pmatrix}\lim_{t\to 1-0}w(t)=\begin{pmatrix} 7 \\ 14 \end{pmatrix}. \tag{25}$$

Accuracy of solution depends on the accuracy of solving the Cauchy problem on $[0,1]$. We provide the results of the numerical implementation of algorithm $[0,1]$ with step $h=0.125$.

Using boundary value problem with parameters (23)-(25) and solving the relevant system of linear algebraic equations (17) we get $\mu^* = \begin{pmatrix} 0.00169901 \\ -5.00117997 \end{pmatrix}$.

Then, using the found value μ^* we solve the Cauchy problem for the system of ordinary differential equations by the Runge-Kutta method of order four

$$\frac{dy}{dt} = \begin{pmatrix} t+1 & 2t^3 \\ 3t^2 & 7 \end{pmatrix} y + \begin{pmatrix} t^2 & 8t \\ 2 & t-8 \end{pmatrix} \cdot \begin{pmatrix} 0.00169901 \\ -5.00117997 \end{pmatrix} + \\ + \begin{pmatrix} 8t^3 - 8t^5 - 2t^6 + t^2 + 47t - 3 \\ 2t^3 - 6t^4 - 25t^2 + 13t - 5 \end{pmatrix}, \quad y(0) = \begin{pmatrix} 0.00169901 \\ -5.00117997 \end{pmatrix}, \quad t \in [0,1],$$

and we find numerical solution to the Problem A.

Exact solution of the problem (21), (22) is $z^*(t) = \begin{pmatrix} 2t^2 - 3t \\ t^3 + 4t^2 - 5 \end{pmatrix}$.

Problem B.

$$\gamma(t) = \chi_0 = \frac{1}{4}, \quad f(t) = \begin{pmatrix} 8t^3 - 8t^5 - 2t^6 + \frac{13}{8}t^2 + \frac{359}{8}t - 3 \\ 2t^3 - 6t^4 - 25t^2 + \frac{815}{64}t - \frac{13}{8} \end{pmatrix}, \quad t \in [0,1].$$

Applying the method scheme above, introduce an additional parameter $\mu = z\left(\frac{1}{4}\right)$. Making the substitution $w(t) = z(t) - \mu, \quad t \in [0,1]$, we obtain the boundary value problem with parameters:

$$\frac{dw}{dt} = \begin{pmatrix} t+1 & 2t^3 \\ 3t^2 & 7 \end{pmatrix} (w+\mu) + \begin{pmatrix} t^2 & 8t \\ 2 & t-8 \end{pmatrix} \mu + \begin{pmatrix} 8t^3 - 8t^5 - 2t^6 + \frac{13}{8}t^2 + \frac{359}{8}t - 3 \\ 2t^3 - 6t^4 - 25t^2 + \frac{815}{64}t - \frac{13}{8} \end{pmatrix}, \quad t \in [0,1], \tag{26}$$

$$w\left(\frac{1}{4}\right) = 0, \tag{27}$$

$$\begin{pmatrix} 4 & -3 \\ -5 & -1 \end{pmatrix} w(0) + \begin{pmatrix} 4 & -3 \\ -5 & -1 \end{pmatrix} \mu + \begin{pmatrix} 8 & -2 \\ -9 & 3 \end{pmatrix} \mu + \begin{pmatrix} 8 & -2 \\ -9 & 3 \end{pmatrix} \lim_{t \to 1-0} w(t) = \begin{pmatrix} 7 \\ 14 \end{pmatrix}. \tag{28}$$

Accuracy of solution depends on the accuracy of solving the Cauchy problem on $[0,1]$. We provide the results of the numerical implementation of algorithm $[0,1]$ with step $h = 0.125$.

Using boundary value problem with parameters (26)-(28) and solving the relevant system of linear algebraic equations (17) we get $\mu^* = \begin{pmatrix} -0.62297939 \\ -4.7355604 \end{pmatrix}$.

Then, using the found value μ^* we solve the Cauchy problems for the system of ordinary differential equations by the Runge-Kutta method of order four

$$\frac{dy}{dt} = \begin{pmatrix} t+1 & 2t^3 \\ 3t^2 & 7 \end{pmatrix} y + \begin{pmatrix} t^2 & 8t \\ 2 & t-8 \end{pmatrix} \cdot \begin{pmatrix} -0.62297939 \\ -4.7355604 \end{pmatrix} + \\ + \begin{pmatrix} 8t^3 - 8t^5 - 2t^6 + \frac{13}{8}t^2 + \frac{359}{8}t - 3 \\ 2t^3 - 6t^4 - 25t^2 + \frac{815}{64}t - \frac{13}{8} \end{pmatrix}, \quad y\left(\frac{1}{4}\right) = \begin{pmatrix} -0.62297939 \\ -4.7355604 \end{pmatrix}, \ t \in [0,1],$$

and we find numerical solution to the Problem B.

Problem C.

$$\gamma(t) = \chi_0 = t_1 = 1, \quad f(t) = \begin{pmatrix} 8t^3 - 8t^5 - 2t^6 + 2t^2 + 7t - 3 \\ 2t^3 - 6t^4 - 25t^2 + 8t + 37 \end{pmatrix}, \quad t \in [0,1].$$

Applying the method scheme above, introduce an additional parameter $\mu = z(1)$. Making the substitution $w(t) = z(t) - \mu, \quad t \in [0,1]$, we obtain the boundary value problem with parameters:

$$\frac{dw}{dt} = \begin{pmatrix} t+1 & 2t^3 \\ 3t^2 & 7 \end{pmatrix} (w+\mu) + \begin{pmatrix} t^2 & 8t \\ 2 & t-8 \end{pmatrix} \mu + \begin{pmatrix} 8t^3 - 8t^5 - 2t^6 + 2t^2 + 7t - 3 \\ 2t^3 - 6t^4 - 25t^2 + 8t + 37 \end{pmatrix}, \quad t \in [0,1], \tag{29}$$

$$w(1) = 0, \tag{30}$$

$$\begin{pmatrix} 4 & -3 \\ -5 & -1 \end{pmatrix} w(0) + \begin{pmatrix} 4 & -3 \\ -5 & -1 \end{pmatrix} \mu + \begin{pmatrix} 8 & -2 \\ -9 & 3 \end{pmatrix} \mu = \begin{pmatrix} 7 \\ 14 \end{pmatrix}. \tag{31}$$

Accuracy of solution depends on the accuracy of solving the Cauchy problem on $[0, 1]$. We provide the results of the numerical implementation of algorithm $[0, 1]$ with step $h = 0.125$.

Using boundary value problem with parameters (29)-(31) and solving the relevant system of linear algebraic equations (17) we get $\mu^* = \begin{pmatrix} -1.0049829 \\ -0.00396959 \end{pmatrix}$.

Then, using the found value μ^* we solve the Cauchy problem for the system of ordinary differential equations by the Runge-Kutta method of order four

$$\frac{dy}{dt} = \begin{pmatrix} t+1 & 2t^3 \\ 3t^2 & 7 \end{pmatrix} y + \begin{pmatrix} t^2 & 8t \\ 2 & t-8 \end{pmatrix} \cdot \begin{pmatrix} -1.0049829 \\ -0.00396959 \end{pmatrix} + \\ + \begin{pmatrix} 8t^3 - 8t^5 - 2t^6 + 2t^2 + 7t - 3 \\ 2t^3 - 6t^4 - 25t^2 + 8t + 37 \end{pmatrix}, \quad y(1) = \begin{pmatrix} -1.0049829 \\ -0.00396959 \end{pmatrix}, \quad t \in [0, 1],$$

and we find numerical solution to the Problem C.

Table 1 gives an estimate of the difference between the exact and numerical solutions of Problems A-C by Adams method, Runge-Kutta method of order four and Bulirsch-Stoer method with different interval partitioning steps.

Table 1: Maximum Absolute Error for Example 1

	Adams method			Runge-Kutta method of order four				Bulirsch-Stoer method		
N	Problem A	Problem B	Problem C	Problem A	Problem B	Problem C	N	Problem A	Problem B	Problem C
8	5.4507E-3	1.4487E-3	4.3174E-7	5.7505E-3	2.0206E-3	7.9902E-3	8	2.8586E-2	3.2788E-3	9.3924E-5
20	3.5976E-3	1.4265E-3	7.0811E-7	1.9842E-4	1.7147E-4	1.5805E-4	20	6.3805E-4	2.0099E-5	2.2562E-6
40	3.7709E-3	1.3962E-3	9.2688E-7	1.3658E-5	1.3636E-5	9.0291E-6	40	8.8025E-6	4.0974E-7	1.3777E-7
80	2.5361E-3	1.3801E-3	9.2671E-7	8.9518E-7	9.4578E-7	5.3917E-7	80	2.2502E-6	2.1180E-8	8.5089E-9

Example 4.3. *We consider the following boundary-value problem for the system of DEPCAG:*

$$\frac{dz}{dt} = \begin{pmatrix} 2 & t+3 \\ 4t^2 & -1 \end{pmatrix} z + \begin{pmatrix} t^2 & 7 \\ 5 & t-6 \end{pmatrix} z(\gamma(t)) + f(t), \quad z \in R^2, \quad t \in (0, T), \tag{32}$$

$$\begin{pmatrix} 3 & -4 \\ 7 & 2 \end{pmatrix} z(0) + \begin{pmatrix} -2 & 1 \\ 0 & 8 \end{pmatrix} z(T) = \begin{pmatrix} 2e^2 - 30 \\ 16e^2 - 136 \end{pmatrix}, \tag{33}$$

where $t_0 = 0, \quad t_1 = 1, \quad t_2 = T = 2.$

Problem D.

$\gamma(t) = \chi_0 = t_0 = 0,$

$f(t) = \begin{pmatrix} 4t - 6e^t + 24t^2 + 4t^3 - 2t^4 - 2te^t - 29 \\ 4e^t - 14t - 20t^3 + 28t^4 - 4t^6 + 16 \end{pmatrix}, \quad t \in [0, 1),$

$\gamma(t) = \chi_1 = t_1 = 1,$

$f(t) = \begin{pmatrix} 4t - 14e - 6e^t + 25t^2 + 4t^3 - 2t^4 - 2te^t + 55 \\ 12e - 2t + 4e^t - 2te - 20t^3 + 28t^4 - 4t^6 - 51 \end{pmatrix}, \quad t \in [1, 2).$

In this problem, we have two subintervals: $[0, 1), \quad [1, 2)$. Applying the method scheme above, introduce an additional parameters $\mu_1 = z_1(0), \quad \mu_2 = z_2(1)$. Making the substitution

$$w_1(t) = z_1(t) - \mu_1, \quad t \in [0, 1), \quad w_2(t) = z_2(t) - \mu_2, \quad t \in [1, 2),$$

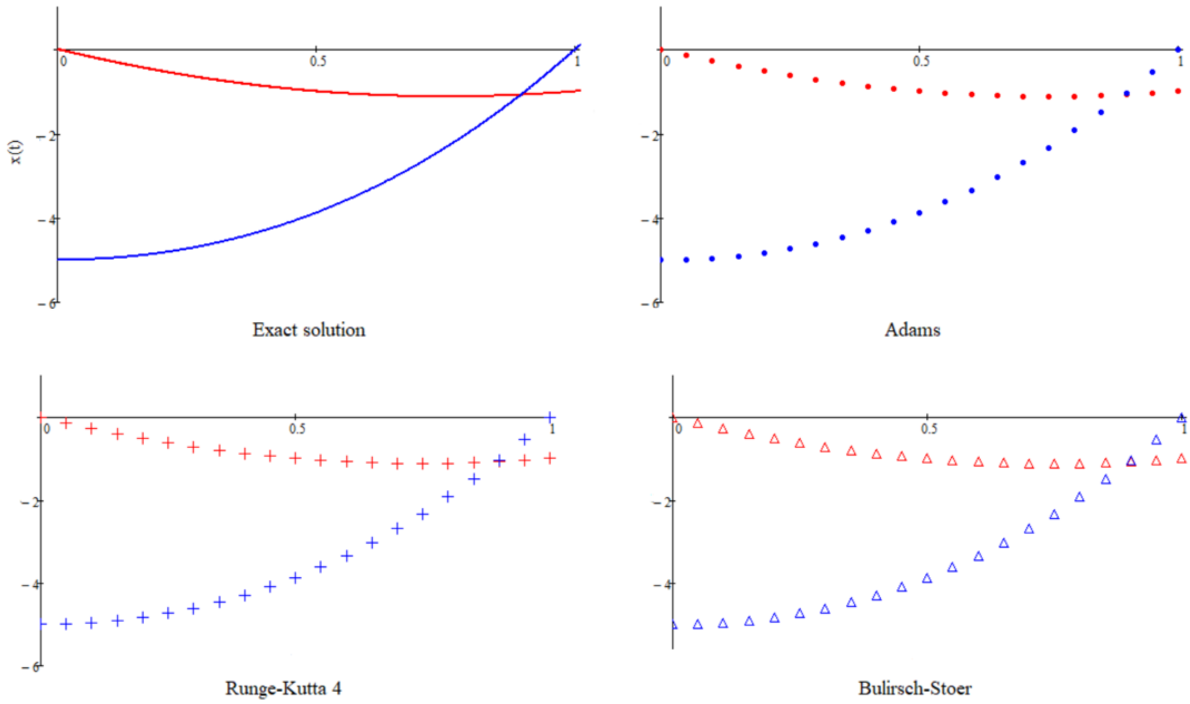

Figure 1: At h=0.05 in Problem A

we obtain the boundary value problem with parameters:

$$\frac{dw_1}{dt} = \begin{pmatrix} 2 & t+3 \\ 4t^2 & -1 \end{pmatrix}(w_1+\mu_1) + \begin{pmatrix} t^2 & 7 \\ 5 & t-6 \end{pmatrix}\mu_1 + \begin{pmatrix} 4t - 6e^t + 24t^2 + 4t^3 - 2t^4 - 2te^t - 29 \\ 4e^t - 14t - 20t^3 + 28t^4 - 4t^6 + 16 \end{pmatrix}, \tag{34}$$

$$w_1(0) = 0, \quad t \in [0,1), \tag{35}$$

$$\frac{dw_2}{dt} = \begin{pmatrix} 2 & t+3 \\ 4t^2 & -1 \end{pmatrix}(w_2+\mu_2) + \begin{pmatrix} t^2 & 7 \\ 5 & t-6 \end{pmatrix}\mu_2 + \begin{pmatrix} 4t - 14e - 6e^t + 25t^2 + 4t^3 - 2t^4 - 2te^t + 55 \\ 12e - 2t + 4e^t - 2te - 20t^3 + 28t^4 - 4t^6 - 51 \end{pmatrix}, \tag{36}$$

$$w_2(1) = 0, \quad t \in [1,2), \tag{37}$$

$$\begin{pmatrix} 3 & -4 \\ 7 & 2 \end{pmatrix}\mu_1 + \begin{pmatrix} -2 & 1 \\ 0 & 8 \end{pmatrix}\mu_2 + \begin{pmatrix} -2 & 1 \\ 0 & 8 \end{pmatrix}\lim_{t\to 2-0} w_2(t) = \begin{pmatrix} 2e^2 - 30 \\ 16e^2 - 136 \end{pmatrix}, \tag{38}$$

$$\mu_1 + \lim_{t\to 1-0} w_1(t) = \mu_2. \tag{39}$$

Accuracy of solution depends on the accuracy of solving the Cauchy problem on subintervals. We provide the results of the numerical implementation of algorithm by partitioning the subintervals $[0,1)$, $[1,2)$, with step $h = 0.05$.

Using boundary value problem with parameters (34)-(39) and solving the relevant system of linear algebraic equations (17) we get

$$\mu_1^* = \begin{pmatrix} 0.00000662 \\ 4.00028045 \end{pmatrix}, \quad \mu_2^* = \begin{pmatrix} -0.99941079 \\ -2.56419818 \end{pmatrix}.$$

Then, using the found values μ_r^*, $r=\overline{1,2}$, we solve the Cauchy problems for the system of ordinary differential equations by the Runge-Kutta method of order four

$$\frac{dy}{dt}=\begin{pmatrix}2 & t+3\\ 4t^2 & -1\end{pmatrix}y+\begin{pmatrix}t^2 & 7\\ 5 & t-6\end{pmatrix}\cdot\begin{pmatrix}0.00000662\\ 4.00028045\end{pmatrix}+$$
$$+\begin{pmatrix}4t-6e^t+24t^2+4t^3-2t^4-2te^t-29\\ 4e^t-14t-20t^3+28t^4-4t^6+16\end{pmatrix},\ y(0)=\begin{pmatrix}0.00000662\\ 4.00028045\end{pmatrix},\ t\in[0,1),$$

$$\frac{dy}{dt}=\begin{pmatrix}2 & t+3\\ 4t^2 & -1\end{pmatrix}y+\begin{pmatrix}t^2 & 7\\ 5 & t-6\end{pmatrix}\cdot\begin{pmatrix}-0.99941079\\ -2.56419818\end{pmatrix}+$$
$$+\begin{pmatrix}4t-14e-6e^t+25t^2+4t^3-2t^4-2te^t+55\\ 12e-2t+4e^t-2te-20t^3+28t^4-4t^6-51\end{pmatrix},$$
$$y(1)=\begin{pmatrix}-0.99941079\\ -2.56419818\end{pmatrix},\ t\in[1,2),$$

and we find numerical solution to the Problem D.

Exact solution of the problem (32), (33) is $z^*(t)=\begin{pmatrix}t^4-7t^2+5t\\ 2e^t-10t+2\end{pmatrix}$.

Problem E.

$\gamma(t)=\chi_0=\dfrac{1}{2}$,

$f(t)=\begin{pmatrix}4t-14\sqrt{e}-6e^t+\frac{371}{16}t^2+4t^3-2t^4-2te^t+20\\ 12\sqrt{e}-7t+4e^t-2t\sqrt{e}-20t^3+28t^4-4t^6-\frac{481}{16}\end{pmatrix},\quad t\in[0,1),$

$\gamma(t)=\chi_1=\dfrac{3}{2}$,

$f(t)=\begin{pmatrix}4t-14e^{\frac{3}{2}}-6e^t+\frac{435}{16}t^2+4t^3-2t^4-2te^t+90\\ 3t+12e^{\frac{3}{2}}+4e^t-2te^{\frac{3}{2}}-20t^3+28t^4-4t^6-\frac{1121}{16}\end{pmatrix},\quad t\in[1,2).$

In this problem, we have two subintervals: $[0,1)$, $[1,2)$. Applying the method scheme above, introduce an additional parameters

$\mu_1=z_1\Big(\dfrac{1}{2}\Big)$, $\mu_2=z_2\Big(\dfrac{3}{2}\Big)$. Making the substitution

$$w_1(t)=z_1(t)-\mu_1,\quad t\in[0,1),\quad w_2(t)=z_2(t)-\mu_2,\quad t\in[1,2),$$

we obtain the boundary value problem with parameters:

$$\frac{dw_1}{dt}=\begin{pmatrix}2 & t+3\\ 4t^2 & -1\end{pmatrix}(w_1+\mu_1)+\begin{pmatrix}t^2 & 7\\ 5 & t-6\end{pmatrix}\mu_1+$$
$$+\begin{pmatrix}4t-14\sqrt{e}-6e^t+\frac{371}{16}t^2+4t^3-2t^4-2te^t+20\\ 12\sqrt{e}-7t+4e^t-2t\sqrt{e}-20t^3+28t^4-4t^6-\frac{481}{16}\end{pmatrix},\tag{40}$$

$$w_1\Big(\frac{1}{2}\Big)=0,\quad t\in[0,1),\tag{41}$$

$$\frac{dw_2}{dt}=\begin{pmatrix}2 & t+3\\ 4t^2-1\end{pmatrix}(w_2+\mu_2)+\begin{pmatrix}t^2 & 7\\ 5 & t-6\end{pmatrix}\mu_2+$$
$$+\begin{pmatrix}4t-14e^{\frac{3}{2}}-6e^t+\frac{435}{16}t^2+4t^3-2t^4-2te^t+90\\ 3t+12e^{\frac{3}{2}}+4e^t-2te^{\frac{3}{2}}-20t^3+28t^4-4t^6-\frac{1121}{16}\end{pmatrix},\tag{42}$$

$$w_2\Big(\frac{3}{2}\Big) = 0, \quad t \in [1,2), \tag{43}$$

$$\begin{pmatrix} 3 & -4 \\ 7 & 2 \end{pmatrix} w_1(0) + \begin{pmatrix} 3 & -4 \\ 7 & 2 \end{pmatrix} \mu_1 + \begin{pmatrix} -2 & 1 \\ 0 & 8 \end{pmatrix} \mu_2 + \begin{pmatrix} -2 & 1 \\ 0 & 8 \end{pmatrix}, \lim_{t\to 1-0} w_2(t) = \begin{pmatrix} 2e^2 - 30 \\ 16e^2 - 136 \end{pmatrix}, \tag{44}$$

$$\mu_1 + \lim_{t\to 1-0} w_1(t) = \mu_2 + w_2(1). \tag{45}$$

Accuracy of solution depends on the accuracy of solving the Cauchy problem on subintervals. We provide the results of the numerical implementation of algorithm by partitioning the subintervals $[0,1)$, $[1,2)$, with step $h = 0.05$.

Using boundary value problem with parameters (40)-(45) and solving the relevant system of linear algebraic equations (17) we get

$$\mu_1^* = \begin{pmatrix} 0.812501116 \\ 0.297436967 \end{pmatrix}, \quad \mu_2^* = \begin{pmatrix} -3.187472096 \\ -4.036606684 \end{pmatrix}.$$

Then, using the found values μ_r^*, $r = \overline{1,2}$, we solve the Cauchy problems for the system of ordinary differential equations by the Runge-Kutta method of order four

$$\frac{dy}{dt} = \begin{pmatrix} 2 & t+3 \\ 4t^2 & -1 \end{pmatrix} y + \begin{pmatrix} t^2 & 7 \\ 5 & t-6 \end{pmatrix} \cdot \begin{pmatrix} 0.812501116 \\ 0.297436967 \end{pmatrix} + \\ + \begin{pmatrix} 4t - 14\sqrt{e} - 6e^t + \frac{371}{16}t^2 + 4t^3 - 2t^4 - 2te^t + 20 \\ 12\sqrt{e} - 7t + 4e^t - 2t\sqrt{e} - 20t^3 + 28t^4 - 4t^6 - \frac{481}{16} \end{pmatrix},$$

$$y\Big(\frac{1}{2}\Big) = \begin{pmatrix} 0.812501116 \\ 0.297436967 \end{pmatrix}, \quad t \in [0,1),$$

$$\frac{dy}{dt} = \begin{pmatrix} 2 & t+3 \\ 4t^2 & -1 \end{pmatrix} y + \begin{pmatrix} t^2 & 7 \\ 5 & t-6 \end{pmatrix} \cdot \begin{pmatrix} -3.187472096 \\ -4.036606684 \end{pmatrix} + \\ + \begin{pmatrix} 4t - 14e^{\frac{3}{2}} - 6e^t + \frac{435}{16}t^2 + 4t^3 - 2t^4 - 2te^t + 90 \\ 3t + 12e^{\frac{3}{2}} + 4e^t - 2te^{\frac{3}{2}} - 20t^3 + 28t^4 - 4t^6 - \frac{1121}{16} \end{pmatrix},$$

$$y\Big(\frac{3}{2}\Big) = \begin{pmatrix} -3.187472096 \\ -4.036606684 \end{pmatrix}, \quad t \in [1,2),$$

and we find numerical solution to the Problem E.

Table 2 gives an estimate of the difference between the exact and numerical solutions of Problems D, E by Runge-Kutta method of order four and Bulirsch-Stoer method with different interval partitioning steps.

From Tables 1 and 2, it can be observed that the errors obtained by the Bulirsch-Stoer method are better than those from Adams method and Runge-Kutta method of order four.

Table 2: Maximum Absolute Error for Example 2

	Runge-Kutta method of order four		Bulirsch-Stoer method	
N	Problem D	Problem E	Problem D	Problem E
80	6.8881E-5	6.7897E-6	1.0226E-6	1.0873E-7
100	2.7760E-5	2.7902E-6	4.1809E-7	4.4694E-8
200	1.6787E-5	1.7567E-7	2.5965E-8	2.8113E-9
400	1.0321E-9	1.1036E-8	2.3090E-9	1.7648E-10

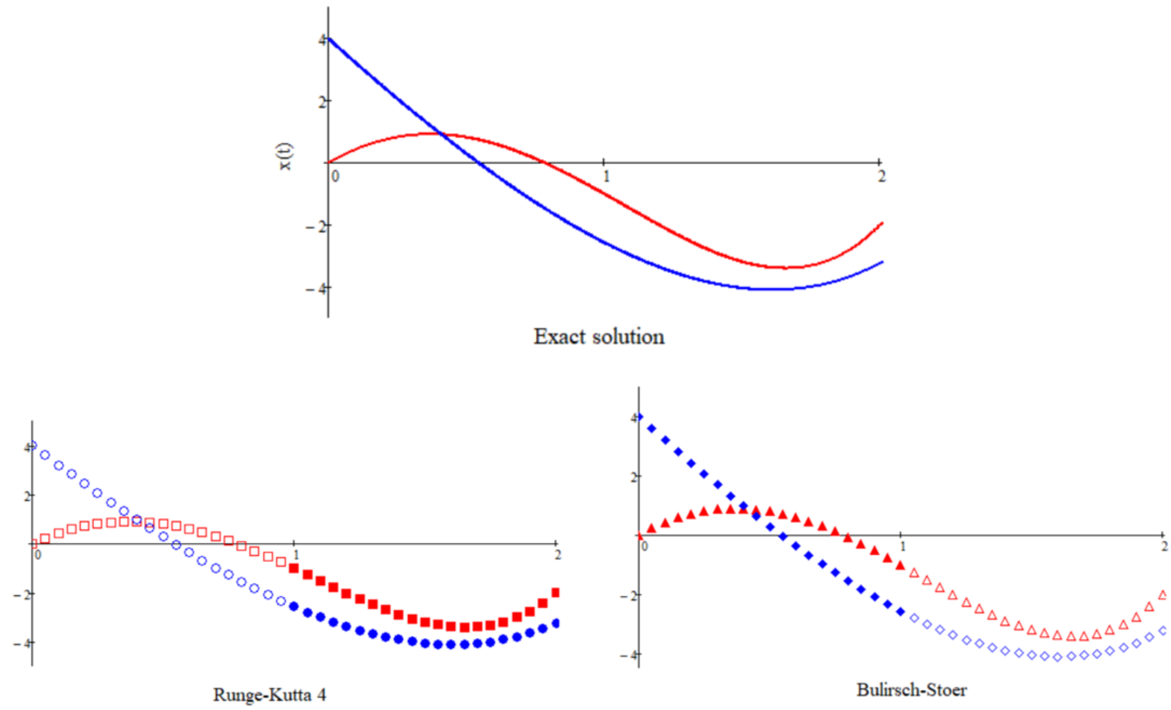

Figure 2: At h=0.05 in Problem D

Example 4.4. *We consider the following boundary-value problem for the system of DEPCAG:*

$$\frac{dz}{dt} = \begin{pmatrix} 5t & 3 \\ 1 & -2t^2 \end{pmatrix} z + \begin{pmatrix} t^2 & -2 \\ 7 & 0 \end{pmatrix} z(\gamma(t)) + f(t), \quad z \in R^2, \quad t \in (0,T), \tag{46}$$

$$\begin{pmatrix} 2 & -5 \\ -3 & 11 \end{pmatrix} z(0) + \begin{pmatrix} -1 & 0 \\ 6 & 9 \end{pmatrix} z(T) = \begin{pmatrix} 20 - 2\sin(8) \\ 12\sin(8) + 72\sin^2(2) - 44 \end{pmatrix}, \tag{47}$$

where $t_0 = 0, \quad t_1 = \frac{1}{2}, \quad t_2 = \frac{3}{4}, \quad t_3 = T = 1.$

$\gamma(t) = \chi_0 = \frac{1}{4}, \quad t \in \left[0, \frac{1}{2}\right):$

$f(t) = \begin{pmatrix} 12\cos(4t) - 22t + 16\cos(8t) - 8\cos(1) - 10t\sin(8t) + \frac{635}{64}t^2 - 25t^4 - 2t^2\sin(2) - \frac{1}{2} \\ 16\sin(4t) - 2\sin(8t) - 14\sin(2) - 7t^3 - 8t^2\cos(4t) - \frac{2659}{64} \end{pmatrix},$

$\gamma(t) = \chi_1 = t_1 = \frac{1}{2}, \quad t \in \left[\frac{1}{2}, \frac{3}{4}\right):$

$f(t) = \begin{pmatrix} 12\cos(4t) - 22t + 16\cos(8t) - 8\cos(2) - 10t\sin(8t) + \frac{75}{8}t^2 - 25t^4 - 2t^2\sin(4) - 1 \\ 16\sin(4t) - 2\sin(8t) - 14\sin(4) - 7t^3 - 8t^2\cos(4t) - \frac{363}{8} \end{pmatrix},$

$\gamma(t) = \chi = t_3 = 1,, \quad t \in \left[\frac{3}{4}, 1\right):$

$f(t) = \begin{pmatrix} 12\cos(4t) - 22t + 16\cos(8t) - 8\cos(4) - 10t\sin(8t) + 5t^2 - 25t^4 - 2t^2\sin(8) - 2 \\ 16\sin(4t) - 2\sin(8t) - 14\sin(8) - 7t^3 - 8t^2\cos(4t) - 76 \end{pmatrix}.$

In this Example 4.4, we have three subintervals: $\left[0, \frac{1}{2}\right), \quad \left[\frac{1}{2}, \frac{3}{4}\right), \quad \left[\frac{3}{4}, 1\right)$. Accuracy of solution depends on the accuracy of solving the Cauchy problem on subintervals. We provide the results of the numerical implementation of algorithm by partitioning the subintervals $\left[0, \frac{1}{2}\right), \left[\frac{1}{2}, \frac{3}{4}\right), \left[\frac{3}{4}, 1\right)$, with step $h = 0.05$.

Exact solution of the problem (46), (47) is $z^*(t) = \begin{pmatrix} 2\sin(8t) + 5t^3 + 5 \\ -4\cos(4t) - t \end{pmatrix}$.

Table 3 provides the difference between numerical and exact solutions of problem (46), (47).

Table 3: Error analysis in Example 4

t	Error in Runge-Kutta four		Error in Runge-Kutta Fehlberg	
	$\lvert z_1^*(t) - \widetilde{z}_1(t)\rvert$	$\lvert z_2^*(t) - \widetilde{z}_2(t)\rvert$	$\lvert z_1^*(t) - \widetilde{z}_1(t)\rvert$	$\lvert z_2^*(t) - \widetilde{z}_2(t)\rvert$
0	0.0734E-4	0.0250E-4	0.8742E-8	0.3299E-8
0.05	0.0329E-4	0.0147E-4	0.8885E-8	0.3442E-8
0.1	0.1205E-4	0.0436E-4	0.7933E-8	0.3611E-8
0.15	0.1704E-4	0.0659E-4	0.6189E-8	0.3787E-8
0.2	0.1739E-4	0.0866E-4	0.4098E-8	0.3977E-8
0.25	0.1345E-4	0.1107E-4	0.2160E-8	0.4216E-8
0.3	0.0703E-4	0.1442E-4	0.4003E-8	0.4533E-8
0.35	0.0014E-4	0.1872E-4	0.5063E-8	0.4595E-8
0.4	0.0502E-4	0.2375E-4	0.5224E-8	0.4353E-8
0.45	0.0684E-4	0.2907E-4	0.4586E-8	0.3809E-8
0.5	0.0472E-4	0.3407E-4	0.3442E-8	0.3035E-8
0.55	0.0252E-4	0.3179E-4	0.2375E-8	0.2607E-8
0.6	0.0021E-4	0.2814E-4	0.1735E-8	0.2264E-8
0.65	0.0026E-4	0.2293E-4	0.1901E-8	0.2246E-8
0.7	0.0497E-4	0.1627E-4	0.3104E-8	0.2764E-8
0.75	0.1571E-4	0.0863E-4	0.5365E-8	0.3952E-8
0.8	0.2425E-4	0.0323E-4	0.3984E-8	0.3116E-8
0.85	0.3195E-4	0.0078E-4	0.2348E-8	0.1778E-8
0.9	0.3412E-4	0.0259E-4	0.0968E-8	0.0243E-8
0.95	0.2561E-4	0.0167E-4	0.0371E-8	0.1086E-8
1	0.0221E-4	0.0207E-4	0.0985E-8	0.1776E-8

From Table 3 it can be seen that the Runge-Kutta Fehlberg method is able to produce lower error rates as compared with those from the Runge-Kutta method of order four.

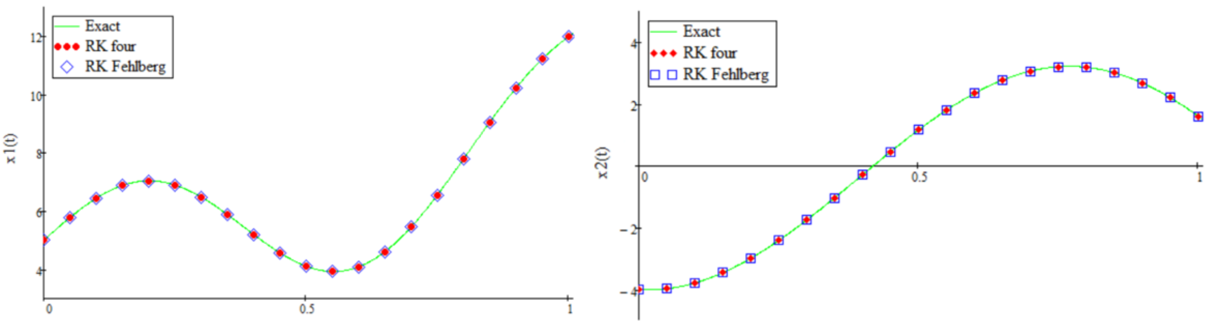

Figure 3: The exact solution and the numerical solutions of (46), (47)

5. Conclusion

Hereafter, the proposed numerical method will be applied to DEPCAG with impulse effects [32]. The main problem in the numerical solution of (1), (2) is the problem with initial conditions (6), (7). It is planned to use other methods for it in the future.

6. Acknowledgement

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7. Conflicts of interest

This work does not have any conflicts of interest.

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