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The shifted odd divisor functions and divisor leaves model

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Abstract

In this research, modelling of the leaves with the help of divisor functions are worked on. Given a positive integer k $(1 \leq k \leq 100)$, we investigate solutions of the equation $\sigma(n) = \sigma(n + 2k)$ with odd square-free integer n. Further, for a positive integer l and odd prime q, there are no results of the equation $\sigma_{2i}(n) = \sigma_{2i}(q)$. As an application, we pose the basic structure of the leaves model for real-time virtual ecosystem construction derived from the equation of shifted odd divisor functions. Also, the elliptic, flabellate and five-lobes leaves's area and the growth process of the leaves were made modelling with the help of divisor functions.

Keywords: *Modelling of the leaves, shifted odd divisor functions, modelling.*

Değiştirilmiş tek bölen fonksiyonları ve bölen yaprak modeli

Öz

Bu araştırmada yaprakların bölen fonksiyonları yardımıyla modellenmesi üzerinde çalışılmıştır. Bir pozitif tamsayı k $(1 \leq k \leq 100)$ verildiğinde, $\sigma(n) = \sigma(n + 2k)$ denkleminin tek tam kare olmayan tamsayı n ile çözümlerini araştırıyoruz. Ayrıca, pozitif bir tamsayı l ve tek asal q için, $\sigma_{2i}(n) = \sigma_{2i}(q)$ denkleminin hiçbir sonucu yoktur. Bir uygulama olarak, kaydırılmış tek bölen fonksiyonlarının denkleminden türetilen gerçek zamanlı sanal ekosistem inşası için yaprak modelinin temel yapısını oluşturuyoruz. Ayrıca eliptik,

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flabellate ve beş loblu yaprakların alanı ve yaprakların büyüme süreci bölen fonksiyonları yardımıyla modelleme yapılmıştır.

Anahtar kelimeler: *Yaprak modeli, değiştirilmiş tek bölen fonksiyonları, modelleme yöntemi.*

1. Introduction

In number theory, a divisor function

$$\sigma_k(n) := \sum_{d|n} d^k$$

is defined as the sum of the *kth* power of positive *d* divisors of *n*. If we take $k = 1$, an odd divisor function is defined as

$$\sigma(n) := \sum_{\substack{d|n \\ d\ odd}} d\,.$$

In 1964, W. Sierpinski [1] made the expression that "we do not know whether there exists infinitely many natural numbers *n* for which $\sigma(n) = \sigma(n + 1)$". A. Makowski [2] has listed nine solutions to $\sigma(n) = \sigma(n+1)$ for $n < 10000$, e.g. *n = 14, 206, 957, 1334, 1364, 1634, 2685, 2974, 4364*. W. E. Mientka and R. L. Vogt [3] have found fifteen solutions to $\sigma(n) = \sigma(n + 1)$ for *n = 14841, 18873, 19358, 20145, 24957, 33998, 36566, 42818, 56564, 64665, 74918, 79826, 79833, 84134, 92685*. One might also ask "whether for certain values of *k* there exist an infinite number of solutions to the equation $\sigma(n) = \sigma(n + k)$" ([4],[5]). For $n < 10000$ and *k = 2, 3, 4, 5* the numbers of solutions of $\sigma(n) = \sigma(n+k)$ are noted to be *19, 2, 14* and *6* respectively. In the book of R. Guy [6], B13 part, Paul Erdos doubts that "$\sigma_2(n) = \sigma_2(n + 2)$ has infinitely many solutions, and thinks that $\sigma_3(n) = \sigma_3(n + 2)$ has no solutions at all". In 2004, J. M. De Koninck [7] considered $\sigma_2(n) = \sigma_2(n + l)$, where *l* is a fixed positive integer.
This paper consists of three parts. Section 1 is the introduction. Section 2 is to find the solutions of the shifted odd divisor functions and to give the proof of the theorems. Section 3 is to model the leaves model for real-time virtual ecosystem construction using the formula of shifted divisor function.

We will now describe details below. Given a positive integer k $(1 \le k \le 100)$, *q* is a prime, we shall find all solutions of $\sigma(n) = \sigma(n + 2k)$ with odd square-free integer *n* (Theorem 1.1). Also, we ask whether *n* is an odd positive square-free integer with $q = n + 2k$ $(k \ge 1)$ and there exist solutions of the equation $\sigma_{2k}(n) = \sigma_{2k}(q)$. More precisely, in Section 2, we prove the following theorems.

Theorem 1. 1. For $1 \le k \le 100$, all the solutions of the equation

$$\sigma(n) = \sigma(n+2k) = \sigma(q) \tag{1.1}$$

for an odd square-free integer *n* and a prime integer *q* are

(2k, n, q) = (8, 3.5, 23), (10, 3.7, 31), (12, 5.7, 47), (14, 3.11, 47), (16, 5.11, 71), (18, 5.13, 83), (20, 3.17, 71), (22, 3.19, 79), (22, 5.17, 107), (24, 11.13, 167), (30, 7.23, 191), (30, 11.19, 239), (30, 13.17, 251), (34, 3.31, 127), (34, 5.29, 179), (36, 5.31, 191), (36, 7.29, 239), (36, 17.19, 359), (40, 3.37, 151), (40, 11.29, 359), (40, 17.23, 431), (42, 5.37, 227), (42, 11.31, 383), (42, 13.29, 419), (42, 19.23, 479), (44, 3.41, 167), (46, 5.41, 251), (48, 5.43, 263), (48, 19.29, 599), (50, 3.47, 191), (52, 11.41, 503), (52, 23.29, 719), (54, 7.47, 383), (54, 13.41, 587), (54, 17.37, 683), (60, 7.53, 431), (60, 19.41, 839), (60, 23.37, 911), (62, 3.59, 239), (64, 5.59, 359), (64, 11.53, 647), (64, 17.47, 863), (66, 7.59, 479), (70, 3.67, 271), (70, 11.59, 719), (70, 17.53, 971), (70, 23.47, 1151), (70, 29.41, 1259), (72, 11.61, 743), (72, 13.59, 839), (72, 29.43, 1319), (76, 5.71, 431), (76, 29.47, 1439), (78, 5.73, 443), (82, 11.71, 863), (82, 23.59, 1439), (82, 29.53, 1619), (84, 5.79, 479), (84, 11.73, 887), (84, 17.67, 1223), (84, 23.61, 1487), (84, 37.47, 1823), (84, 41.43, 1847), (86, 3.5.7, 191), (88, 5.83, 503), (90, 19.71, 1439), (90, 43.47, 2111), (92, 3.89, 359), (94, 41.53, 2267), (96, 7.89, 719), (96, 17.79, 1439), (96, 29.67, 2039), (100, 17.83, 1511), (100, 47.53, 2591), (102, 5.97, 587), (102, 13.89, 1259), (106, 17.89, 1619), (106, 47.59, 2879), (108, 29.79, 2399), (110, 3.107, 431), (112, 3.109, 439), (112, 5.107, 647), (112, 11.101, 1223), (112, 41.71, 3023), (114, 5.109, 659), (114, 7.107, 863), (114, 13.101, 1427), (114, 31.83, 2687), (114, 43.71, 3167), (114, 53.61, 3347), (118, 5.113, 683), (118, 29.89, 2699), (120, 7.113, 911), (120, 11.109, 1319), (120, 13.107, 1511), (120, 17.103, 1871), (120, 19.101, 2039), (120, 23.97, 2351), (120, 31.89, 2879), (120, 37.83, 3191), (120, 41.79, 3359), (120, 53.67, 3671), (120, 59.61, 3719), (124, 11.113, 1367), (124, 23.101, 2447), (124, 41.83, 3527), (126, 17.109, 1979), (126, 29.97, 2939), (126, 59.67, 4079), (130, 23.107, 2591), (130, 41.89, 3779), (132, 29.103, 3119), (132, 61.71, 4463), (138, 29.109, 3299), (138, 59.79, 4799), (142, 5.137, 827), (142, 11.131, 1583), (142, 41.101, 4283), (142, 59.83, 5039), (144, 5.139, 839), (144, 7.137, 1103), (144, 13.131, 1847), (144, 47.97, 4703), (148, 59.89, 5399), (150, 13.137, 1931), (150, 43.107, 4751), (150, 67.83, 5711), (152, 3.149, 599), (152, 3.7.11, 383), (154, 3.151, 607), (154, 23.131, 3167), (154, 41.113, 4787), (154, 53.101, 5507), (154, 71.83, 6047), (156, 5.151, 911), (156, 47.109, 5279), (156, 59.97, 5879), (160, 3.157, 631), (160, 47.113, 5471), (162, 5.157, 947), (162, 11.151, 1823), (162, 13.149, 2099), (162, 23.139, 3359), (162, 53.109, 5939), (162, 61.101, 6323), (162, 73.89, 6659), (162, 79.83, 6719), (166, 17.149, 2699), (166, 29.137, 4139), (168, 5.163, 983), (168, 19.149, 2999), (168, 59.109, 6599), (172, 83.89, 7559), (174, 17.157, 2843), (174, 43.131, 5807), (174, 47.127, 6143), (174, 71.103, 7487), (174, 73.101, 7547), (176, 3.5.17, 431), (180, 13.167, 2351), (180, 31.149, 4799), (180, 41.139, 5879), (180, 53.127, 6911), (180, 71.109, 7919), (180, 83.97, 8231), (182, 3.179, 719), (184, 3.181, 727), (184, 11.173, 2087), (184, 17.167, 3023), (184, 53.131, 7127), (186, 5.181, 1091), (186, 7.179, 1439), (186, 19.167, 3359), (186, 47.139, 6719), (186, 89.97, 8819), (190, 41.149, 6299), (190, 53.137, 7451), (190, 59.131, 7919), (192, 29.163, 4919), (192, 43.149, 6599), (192, 53.139, 7559), (192, 83.109, 9239), (194, 3.5.19, 479), (196, 5.191, 1151), (196, 29.167, 5039), (196, 89.107, 9719), (198, 5.193, 1163).

Theorem 1. 2. Let n be an odd positive square-free integer with $q = n + 2k$ $(k \geq 1)$. Then there does not exist $\sigma_{2k}(n) = \sigma_{2k}(q)$.

Furthermore, let $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$ and p_i, q are odd distinct positive primes $(1 \leq i \leq r)$. If

i) $\#\{i : \alpha_i \text{ is odd}, 1 \leq i \leq r\} \geq 2$ or

ii) $\alpha_1 \equiv 1 \pmod 4$ and $\alpha_2 = \alpha_3 = ... = \alpha_r = 0$ or

iii) $\alpha_i \equiv 3 \pmod 4$ and α_j is even $(i \neq j)$ *for all* $1 \leq j \leq r$,
then $\sigma_{2k}\left(p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}\right) \neq \sigma_{2k}(q)$.

Mathematics is a universal language to understand the science and nature. In 2013, J. Kim, D. Kim and H. Cho [8] proposed procedural modeling method using convolution sums of divisor functions to model a variety of natural trees in a virtual ecosystem efficiently. With a similar perspective, we think of modeling methods using the divisor function.

Main purpose of Section 3 is to model the basic structure of the leaves modeling method. In (1.1), we define *(2k, n, q)−Divisor Leaves Model (DLM)*. To introduce key idea of *DLM* easily, we put $n = p_1 p_2$ with p_1, p_2, q odd primes. There are some nice *(2k, n, q)−DLM* connecting the shifted divisor functions and it is proposed along with general natural number *n*. Firstly, we give it between prime numbers in (1.1) and areas, base and heights of leaf of *(2k, n, q)−DLM*. Considering this, we create elliptic, flabellate and five-lobed divisor leaves model. Leaves have different growing sizes (depending on time) when they live on the earth. For animation model, we suggest four-steps growing patterns. Finally, we give real examples of $(2k, p_1p_2, q)-DLM$. Real leaf samples are used in this article.

2. Proofs of theorem 1.1 and theorem 1.2

To prove Theorem 1.1, we need the following three lemmas.

Lemma 2.1. Let *p* be positive prime integer, $q = p + 2k$ with $k \geq 1$ is fixed positive integer and *q* is prime. Then there does not exist *p* and *q* satisfying $\sigma(p) = \sigma(q)$.

Proof. We assume that *p* and *q* satisfying $\sigma(p) = \sigma(q)$. Then $\sigma(p) = 1 + p$, $\sigma(q) = 1 + q$ and $p = q$. It is a contradiction. This completes the Lemma 2.1.

Lemma 2.2. Let p_1, p_2, q be odd distinct positive prime integers and *k* be fixed positive integer with $q = p_1 p_2 + 2k$, $(1 \leq k \leq 100)$. Then, there does not exist *k = 1, 2, 3, 13, 14, 16, 19, 28, 29, 34, 37, 40, 43, 49, 52, 58, 61, 64, 67, 68, 70, 73, 79, 82, 85, 88, 89, 94, 97, 100* satisfying $\sigma(p_1 p_2) = \sigma(q)$.

Proof. We note that

$$\sigma(p_1 p_2) = (1 + p_1)(1 + p_2) = (1 + q) = \sigma(q) \tag{2.1}$$

and

$$p_1 + p_2 = 2k. \tag{2.2}$$

It is well known that $p_1 \geq 3$ and $p_2 \geq 5$, with $p_1 < p_2$. By (2.2) there does not exist $2k = 2, 4, 6$ satisfying (2.1). Thus, we consider the case of $2k \geq 8$ in (2.1) and (2.2). First, we get $p_1 + p_2 = 8$, $p_1 = 3$ and $p_2 = 5$ with $p_1 < p_2$ and $q = 23$. In a similar way, when working with $p_1 + p_2 = 2k$ $(5 \leq k \leq 100)$, we derive Lemma 2.2.

Lemma 2.3. Let p_1, p_2, p_3, q be odd distinct positive prime integers, k be fixed positive integer with $q = p_1p_2p_3 + 2k$. For $1 \le k \le 100$, all the solutions of the equation

$$\sigma(p_1p_2p_3) = \sigma(q)$$

are
$(2k, p_1p_2p_3, q)$ = (86, 3.5.7, 191), (152, 3.7.11, 383), (176, 3.5.17, 431), (194, 3.5.19, 479).

Proof. We assume that p and q satisfy $\sigma(p_1p_2p_3) = \sigma(q)$. Then

$$\sigma(p_1p_2p_3) = (1+p_1)(1+p_2)(1+p_3) = (1+q) = \sigma(q),$$
$$(1+p_1)(1+p_2)(1+p_3) = (1+q) = 1 + p_1p_2p_3 + 2k$$

and

$$p_1 + p_2 + p_3 + p_1p_2 + p_1p_3 + p_2p_3 = 2k. \qquad (2.3)$$

Consider the lower bound of $2k$ satisfying (2.3), it is well known that $p_1 \ge 3$, $p_2 \ge 5$, $p_3 \ge 7$ and $2k \ge 86$. So we do not consider $2 \le 2k \le 86$. By (2.3), we consider the case of $2k \ge 86$, that is, $q = p_1p_2p_3 + 86 \ge 3.5.7 + 86 = 191$. Since 191 is a prime number, we choose $(2k, p_1p_2p_3, q) = (86,\ 3.5.7,\ 191)$. Likely, with the same method of Lemma 2.2, we check all numbers $88 \le 2k \le 200$. This completes Lemma 2.3.

Remark 2.4. We ask a general question as follows:
(Question) For fixed $2k$, does there exist n satisfying $\sigma(n) = \sigma(n+2k)$ with an odd n?
If n is an odd square-free integer and q prime number, then our (Question) is false by Theorem 1.1.

Proof of the Theorem 1.1. Assume that n is an odd square-free integer, that is, $n = p_1p_2 \ldots p_r$ with p_i odd distinct prime integers. The cases of $n = p_1$ or $n = p_1p_2$ or $n = p_1p_2p_3$ are considered Lemma 2.1, Lemma 2.2, Lemma 2.3. Let $n = p_1p_2p_3p_4$ and assume

$$\sigma(n) = \sigma(n+2k) = \sigma(q). \qquad (2.4)$$

Hence, we have

$$\sigma(n) = \sigma(p_1p_2p_3p_4) = (1+p_1)(1+p_2)(1+p_3)(1+p_4) = (1+q)$$

and

$$(1+p_1)(1+p_2)(1+p_3)(1+p_4) - (1+p_1p_2p_3p_4) = 2k. \qquad (2.5)$$

We know that $p_1 \ge 3$, $p_2 \ge 5$, $p_3 \ge 7$ and $p_4 \ge 11$ with $p_1 < p_2 < p_3 < p_4$. The lower bound of $2k$ in (2.5) is

$$2k \ge (1+3)(1+5)(1+7)(1+11) - (1+3.5.7.11) \ge 1148.$$

This is not a case of $2k \leq 200$. If $r = 4$, then the lower bound of $2k$ in (2.5) is bigger than 200. Consider $\sigma(p_1 p_2 ... p_r) = (1+p_1)(1+p_2)...(1+p_r)$ and

$$2k = (1+p_1)(1+p_2)...(1+p_r) - (1+p_1 p_2 ... p_r) \tag{2.6}$$

with $r > 4$. Similarly, the lower bound of $2k$ in (2.6) is bigger than 200. So this completes the proof of the theorem.

Proof of the theorem 1.2. Assume $n = p_1 p_2 ... p_r$. Then $\sigma_{2l}(p_1 p_2 ... p_r) = (p_1^{2l}+1)(p_2^{2l}+1)...(p_r^{2l}+1)$ and $\sigma_{2l}(q) = 1+q^{2l}$. Thus $2^r | (p_1^{2l}+1)(p_2^{2l}+1)...(p_r^{2l}+1)$ and $2 || (1+q^{2l})$, where $p^a | n$ and $p^{a+1} \nmid n$ is $p^a || n$.

If $r \geq 2$, then

$$\sigma_{2l}(p_1 p_2 ... p_r) \equiv 0 \pmod{4}$$

and

$$\sigma_{2l}(q) \equiv 2 \pmod{4}. \tag{2.7}$$

Thus, we get $\sigma_{2l}(p_1 p_2 ... p_r) \neq \sigma_{2l}(q)$.

If $r = 1$, then $(p_1^{2l}+1) = (q^{2l}+1)$ and $p_1 = q$. This contradicts to $l > 0$. So $\sigma_{2l}(p_1) \neq \sigma_{2l}(q)$. Therefore, n is an odd square-free integer, then $\sigma_{2l}(n) \neq \sigma_{2l}(q)$ with $n \neq q$. Furthermore, we assume $n = p_1^{\alpha_1} p_2^{\alpha_2} ... p_r^{\alpha_r}$ with $\#\{i : \alpha_i \text{ is odd}, 1 \leq i \leq r\} \geq 2$. Assume $\alpha_1 \equiv \alpha_2 \equiv 1 \pmod{2}$. Then

$$\sigma_{2l}(p_1^{\alpha_1}) = 1 + p_1^{2l} + ... + p_1^{2l(\alpha_1)} \equiv 0 \pmod{2} \ (i = 1, 2).$$

By (2.7) and $4 | \sigma_{2l}(n)$, we derive that $\sigma_{2l}(n) \neq \sigma_{2l}(q)$.

If $\alpha_1 \equiv 1 \pmod{4}$ and $\alpha_2 = \alpha_3 = ... = \alpha_r = 0$, then

$$\sigma_{2l}(p_1^{\alpha_1}) = \sigma_{2l}(p_1^{4m+1}) = 1 + p_1^{2l} + p_1^{2(2l)} + ... + p_1^{(4m+1)(2l)},$$

$$\sigma_{2l}(q) = 1 + q^{2l}$$

and

$$p_1^{2l}(1 + p_1^{2l} + ... + p_1^{8ml}) = q^{2l}.$$

It is easy to verify that $p_1 \nmid q$ and $\sigma(p_1^{\alpha_1}) \neq \sigma(q)$.

If $\alpha_i \equiv 3 \pmod{4}$ and α_j is even $(i \neq j)$ for all $1 \leq j \leq r$, then

$$\sigma_{2l}(p_1^{\alpha_1} ... p_r^{\alpha_r}) = (1 + p_1^{2l} + ... + p_1^{\alpha_1(2l)})...(1 + p_r^{2l} + ... + p_r^{\alpha_r(2l)}) \equiv 0 \pmod{4}. \tag{2.8}$$

By (2.7) and (2.8), $\sigma_{2l}(\mathrm{n}) \neq \sigma_{2l}(q)$. From the above computations, the proof of this theorem is completed.

Remark 2.5. The results of Table 1 and Table 2 were realized by combining several computers and by using Mathematica 9.0 Software:

Table 1. The number of $\#(2k, p_1.p_2, q)$, $\#(2k, p_1.p_2.p_3, q)$ and $\#(2k, p_1.p_2.p_3.p_4, q)$

	$\#(2k, p_1.p_2, q)$	$\#(2k, p_1.p_2.p_3, q)$	$\#(2k, p_1.p_2.p_3.p_4, q)$
$2 \leq 2k \leq 100$	*73*	*1*	*0*
$102 \leq 2k \leq 1000$	*1808*	*57*	*0*
$1002 \leq 2k \leq 10000$	*63906*	*1261*	*53*
$10002 \leq 2k \leq 100000$	*2911232*	*18356*	*1571*

Table 1 gives us many different leaves model types derived from Theorem 1.1. In general, we have considered the solutions of the shifted divisor functions

$$\sigma(p_1 ... p_r) = \sigma(q) \tag{2.9}$$

with $q = p_1 p_2 ... p_r + 2k$ by using Mathematica 9.0 Software. This is a very big list. So, in this article, we only write the lower bound of *2k* (*LB(2k)*) in (2.9) as follows:

Table 2. The list of *LB(2k)*

n	*LB(2k)*	*n*	*LB(2k)*
$p_1.p_2$	*8*	$p_1 \cdots p_7$	*10015844*
$p_1.p_2.p_3$	*86*	$p_1 \cdots p_8$	*302391704*
$p_1 \cdots p_4$	*1322*	$p_1 \cdots p_9$	*7465944254*
$p_1 \cdots p_5$	*25178*	$p_1 \cdots p_{10}$	*249278458694*
$p_1 \cdots p_6$	*325352*	$p_1 \cdots p_{18}$	*838962489663670353881 2454*

Let $\sigma_{1,1}(n) := \sum_{\substack{d \mid n \\ d\ odd}} d^k$ in Table 3, we find $\mathrm{t}_k := \#\left\{ n \middle| \sigma_{1,1}(n) = \sigma_{1,1}(n+k),\ 1 \leq n \leq 2^{31} \right\}$ }. The computation by Mathematica 9.0 Software took about one month.

Table 3. A list of t_k $(1 \le k \le 42)$.

k	t_k	k	t_k	k	t_k
1	53	**15**	3	**29**	60
2	731	**16**	1877	**30**	4027
3	2	**17**	54	**31**	70
4	1394	**18**	2857	**32**	2973
5	3	**19**	77	**33**	2
6	1967	**20**	2340	**34**	999
7	32	**21**	3	**35**	4
8	1850	**22**	1050	**36**	5750
9	2	**23**	54	**37**	77
10	784	**24**	5684	**38**	1054
11	55	**25**	3	**39**	3
12	2767	**26**	1012	**40**	3422
13	60	**27**	2	**41**	69
14	251	**28**	2203	**42**	3563

3. Divisor leaves model (DLM) derived from the shifted divisor function

Leaves are the vital part of plants and aid the plants in a variety of ways, including producing food, oxygen through photosynthesis, etc. The basic formula in the procedural modeling has been proposed in this study and the properties of divisor functions have been used to model the area of the leaves and describes the growth process of various leaves [9].

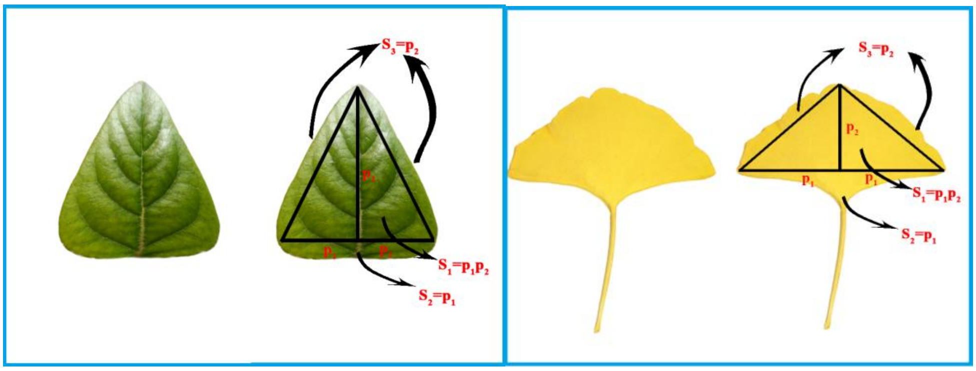

Figure 1. Elliptic DLM (Cotoneaster sp.) and Flabellate DLM (Ginkgo)

3.1. Elliptic and flabellate divisor leaves model

3.1.1. Main structure of divisor leaves model. The area of leaves (Elliptic or Flabellate) can be modeled using the divisor function (Figure 1). First, the leaf is separated into three areas as S_1, S_2, S_3. S_1 is equal to the area of an isosceles triangle. The height of isosceles triangle is equal to p_2. The base of isosceles triangle is equal to $2p_1$. Then, we have

$$S_1 := (base\ x\ height)/2 = p_1 p_2.$$

Archimedes's sum of geometric series was used to calculate the area enclosed by a parabola and a line [10]. The underlying method is defined as the separation of the many infinite areas of the triangle. We note that the area of each triangle B_1 is one eighth of the area of the triangle A_1 (See Figure 2).

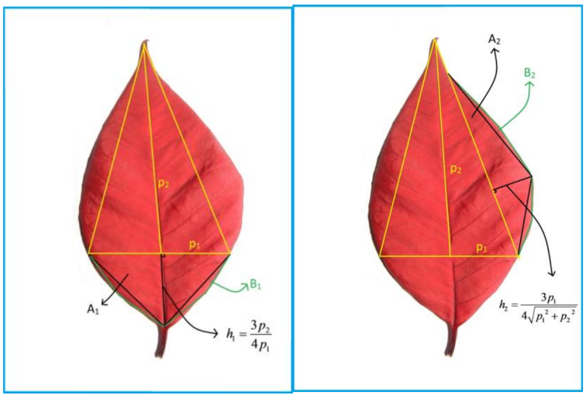

Figure 2. S_2, S_3 and check list of Elliptic DLM (Euphorbia pulcherrima)

Then the area of S_2 can be expressed by

$$S_2 = A_1 + 2\left(\frac{A_1}{8}\right) + 4\left(\frac{A_1}{8^2}\right) + 8\left(\frac{A_1}{8^3}\right) + \dots \quad with\ A_1 = \frac{1}{2}(2p_1h_1) = p_1h_1$$

and $S_2 = \frac{4}{3} p_1 h_1$. Similarly, we get

$$S_3 = A_2 + 2\left(\frac{A_2}{8}\right) + 4\left(\frac{A_2}{8^2}\right) + 8\left(\frac{A_2}{8^3}\right) + \dots = \frac{4}{3}\left(\sqrt{p_1^2 + p_2^2}\right) h_2 .$$

Put $h_1 = \frac{3p_2}{4p_1}$ and $h_2 = \frac{3p_1}{4\sqrt{p_1^2 + p_2^2}}$. We get

and

$$S := S_1 + S_2 + S_3 = p_1 p_2 + p_1 + p_2.$$

Let p_1, p_2, q be odd distinct primes and fixed k be positive integer. By Theorem 1.1, we consider the shifted divisor function $\sigma(p_1p_2) = \sigma(q)$ with $q = p_1p_2 + 2k$. For example, if we choose fix $k = 4$, $p_1 = 3$, $p_2 = 5$, then we get *(8, 3.5, 23)-DLM*

$$S_1 = 3.5,\ S_2 = 5,\ S_3 = 3\ \text{ and } S = 23$$

derived from

$$\sigma(3.5) = \sigma(23).$$

For $k = 4, 5, 6, 7, 8, 9, 10, 11$ we can give the following table.

Table 4. The First Eight Values of $(2k, p_1, p_2, n, q, \frac{p_1}{p_2}, h_1, h_2)$.

$2k$	p_1	p_2	n	q	p_2 / p_1	h_1	h_2
8	3	5	15	23	1,666667	1,250000	0,385872
10	3	7	21	31	2,333333	1,750000	0,295439
12	5	7	35	47	1,400000	1,050000	0,435929
14	3	11	33	47	3,666667	2,750000	0,197338
16	5	11	55	71	2,200000	1,650000	0,310352
18	5	13	65	83	2,600000	1,950000	0,269234
20	3	17	51	71	5,666667	4,250000	0,130339
22	5	17	85	107	3,400000	2,550000	0,211625

3.1.2. Shape of S_3 in divisor leaves model. We make leaves with ordered pairs $(2k, p_1, p_2, n, q)$ that meet the equation, $\sigma(n) = \sigma(n + 2k)$ when p_1, p_2, q are odd primes, $n = p_1p_2$ and $q = n + 2k$. We embody the leaves like Figure 3.

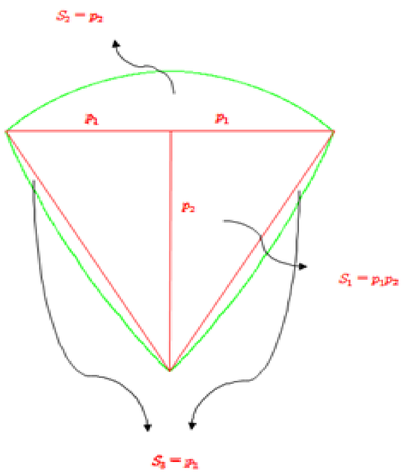

Figure 3. Area of Divisor Leaves Model.

First, the triangle (S_1) with red line at the center is an isosceles triangle and its height is p_2, base line is $2p_1$ and the area is $p_1.p_2$. We make S_2 and S_3 using Archimedes's idea. S_2's area is p_2. S_3 is made by using modified Archimedes's idea. The number of triangles in each step increases twice, but the area of triangles decreases *8* times. So common ratio becomes $\frac{1}{4}$.

In the process of making S_3, the common ratio is r. Let's call the triangle's area, base line, height A_n, K_n, H_n. Then, each of the A_n, K_n, H_n's relational equation is named as Table 5. And N_n means the total number of triangles that are newly made at nth step of geometric series.

Table 5. A_n, K_n, H_n and N_n .

	$n=1$	Recurrence Relation $(n \geq 2)$
A_n	$A_1 = \frac{1-r}{2} p_1$	$A_n = \frac{r}{2} A_{n-1}$
K_n	$K_1 = \sqrt{p_1^2 + p_2^2}$	$K_n = \sqrt{\left(\frac{1}{2} K_{n-1}\right)^2 + \left(H_{n-1}\right)^2}$
H_n	$H_1 = \frac{2A_1}{\sqrt{p_1^2 + p_2^2}}$	$H_n = \frac{rA_{n-1}}{K_n}$
N_n	1	$N_n = 2^{n-1}$

The common ratio is $\frac{1}{4}$ that $\frac{1}{8}x2$ (decrease ratio of triangle's area X increase ratio of the number of triangles). When we calculate S_3, we use common ratio r that is $\frac{r}{2}\left(=\frac{A_{n+1}}{A_n}\right) x\ 2\left(=\frac{N_{n+1}}{N_n}\right)$. The Figure 4 is S_3's shape according to r. We use Python 2.7.9 (Turtle Module) to draw illustrations in Figure 4 and Figure 5.

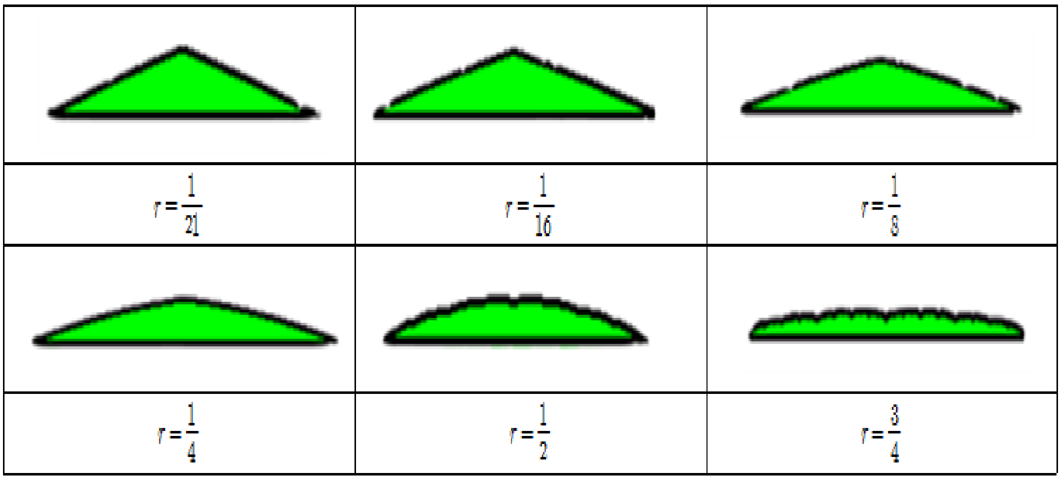

Figure 4. r connected to S_3 ($p_1 = 3, p_2 = 5, n = 10$).

Here, n means that we draw S_3 with nth step of geometric series in Python 2.7.9 (Turtle Module). The Figure 5 is the shape of leaves according to r.

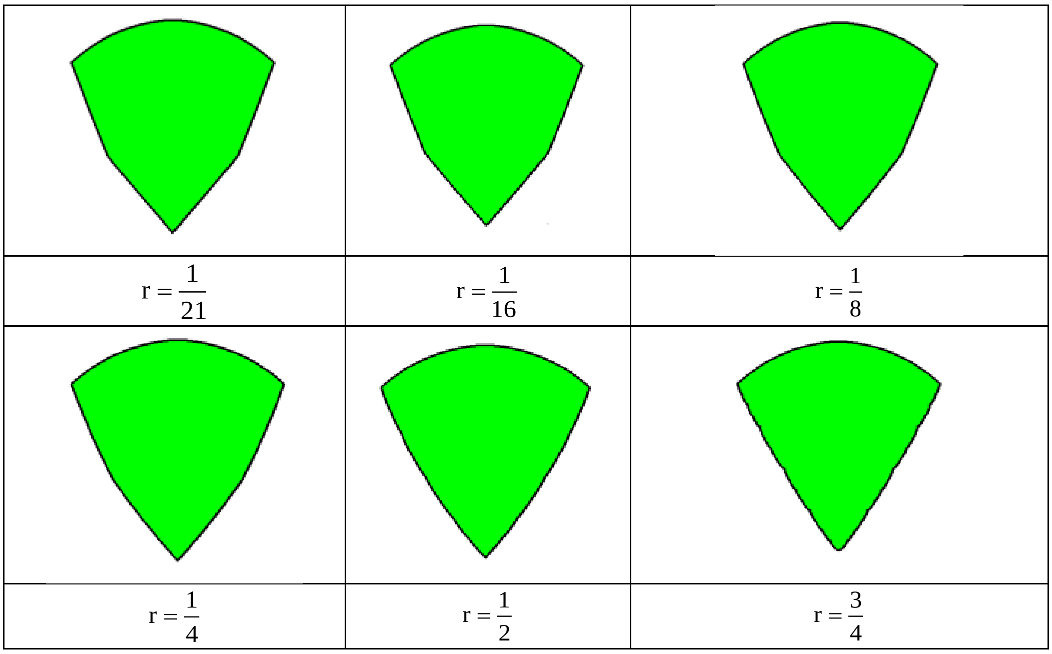

Figure 5. Shape of Leaves ($p_1 = 3, p_2 = 5, n = 10$)

3.2. Five-lobed divisor leaves model. A similiar new model can be created for five-lobed leaves in Figure 6. This can be done similary to the other leaves model. Also, the area of five-lobed leaves can be calculated using the divisor function. First, the five-lobed leaves are separated into three areas as S_1, S_2, S_3, that is, $S := S_1 + S_2 + S_3$.

S_1 is equal to the area of an isosceles triangle. The height of isosceles triangle is equal to p_2. The base of isosceles triangle is equal to $2p_1$. Then $S_1 = p_1 p_2$.

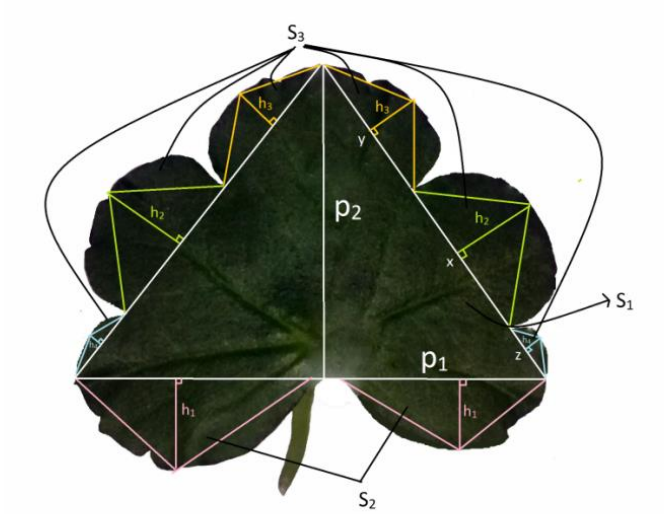

Figure 6. S_2, S_3 and check list of Five-Lobed DLM (Pelargonium sp.)

The area of S_2 can be expressed by

$$S_2 = 2\left[A_1 + 2\left(\frac{A_1}{8}\right) + 4\left(\frac{A_1}{8^2}\right) + 8\left(\frac{A_1}{8^3}\right) + \ldots \right]$$

with $A_1 = \frac{1}{2}(p_1 h_1)$. Put $h_1 = \frac{3p_2}{4p_1}$ and we get $S_2 := p_2$.

Then the area of S_3 can be represented as

$$S_3 := 2\left\{ 2\left[A_2 + 2\left(\frac{A_2}{8}\right) + 4\left(\frac{A_2}{8^2}\right) + 8\left(\frac{A_2}{8^3}\right) + \ldots \right] \right.$$

$$+2\left[A_3 + 2\left(\frac{A_3}{8}\right) + 4\left(\frac{A_3}{8^2}\right) + 8\left(\frac{A_3}{8^3}\right) + \ldots \right]$$

$$\left. +2\left[A_4 + 2\left(\frac{A_4}{8}\right) + 4\left(\frac{A_4}{8^2}\right) + 8\left(\frac{A_4}{8^3}\right) + \ldots \right] \right\}$$

$$= \frac{16}{3}\left[A_2 + A_3 + A_4 \right]$$

with $x + y + z = \sqrt{p_1^{\,2} + p_2^{\,2}}$, $A_2 = \frac{1}{2}(xh_2)$, $A_3 = \frac{1}{2}(yh_3)$ and $A_4 = \frac{1}{2}(zh_4)$.

Put $h_2 = \frac{p_1}{8x}$, $h_3 = \frac{p_1}{8y}$ and $h_4 = \frac{p_1}{8z}$. We obtain $S_3 := p_1$.

Finally, we have $S := p_1 + p_2 + p_1 p_2$.

3.3. Four-growing step. A leaf model in this study is a structure for determining the growth pattern of a leaf based on the shifted divisor functions. The mathematical meaning of divisor functions are analyzed, and the advantages when divisor functions are applied to a leaf model are discovered. The propagation rules must be rule-based modeling and not simple and intuitive, and this method also required the assignment of complex parameters.

With the general perspective of leaf's growth, the early phase of young leaf seems to grow slowly. At the second phase, the growing speed of leaf becomes fast and at the apotheosis of leaf's growth, we can observe that leaf grows very large. Finally, when the leaf almost has grown, we can find that the leaf's growing speed becomes slower. We suggest more natural leaf's growing model by applying four phases that have four different growing speeds in the same period (Figure 7).

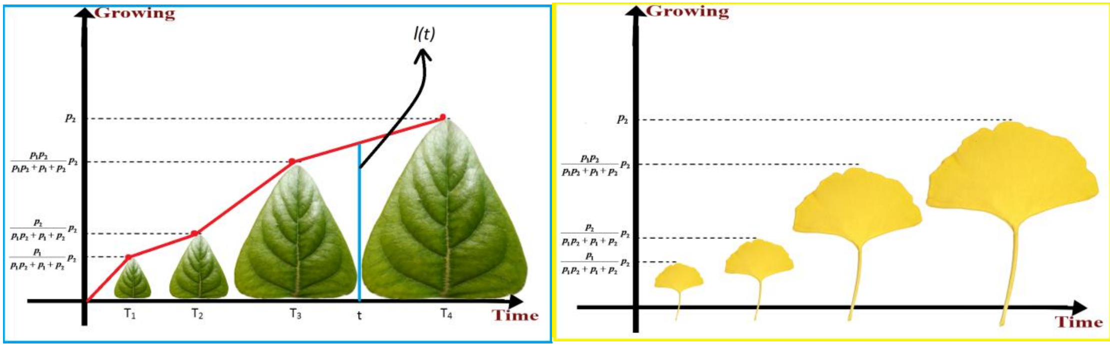

Figure 7. Growing step of DLM

Let t be time and I be fixed period, $I_0 = [0, T_1]$ and $I_i = [T_i, T_{i+1}]$ with time T_i $(i = 1, 2, 3)$. Assume $l(t)$ is the height of leaf at time t. Then the height of $(2k, p_1 p_2, q)$-DLM is

$$l(t) = \begin{cases} \dfrac{p_1 p_2}{(p_1 p_2 + p_1 + p_2) T_1} t & \text{, if } t \in I_0 \\ \dfrac{(p_2 - p_1) p_2}{(p_1 p_2 + p_1 + p_2)(T_2 - T_1)} t + \dfrac{(p_1 T_2 - p_2 T_1) p_2}{(p_1 p_2 + p_1 + p_2)(T_2 - T_1)} & \text{, if } t \in I_1 \\ \dfrac{(p_2 - 1) p_1 p_2}{(p_1 p_2 + p_1 + p_2)(T_3 - T_2)} t + \dfrac{(T_3 - p_2 T_2) p_1 p_2}{(p_1 p_2 + p_1 + p_2)(T_3 - T_2)} & \text{, if } t \in I_2 \\ \dfrac{(p_1 p_2 (1 - p_2)) + p_1 + p_2)}{(p_1 p_2 + p_1 + p_2)(T_4 - T_3)} t + \dfrac{T_4 (p_2 (2 p_1 - 1) - p_1) - T_3 (p_2 (p_1 p_2 + p_1 + p_2))}{(p_1 p_2 + p_1 + p_2)(T_4 - T_3)} & \text{, if } t \in I_3 \end{cases}$$

This is the formula that provides leaf size when the real-time model of growing leaves is embodied by animation. By this observation, we propose 4-growing steps derived from $(2k, p_1 p_2, q)$-DLM.

Remark 3.3.1. In the leaves, sunflower, pine cones, palm, ... , we can see the golden ratio a lot $\left(\frac{1+\sqrt{5}}{2} \approx 1{,}618033 \right)$ and Fibonacci numbers. In the following table, for p_1, p_2 are primes, p_1 / p_2 ratio, the golden ratio is very close to the values given. In Table 6, using Mathematica 9.0 Software (*0 < 2k < 199900*), we find almost the golden ratio leaves of $(2k, p_1 p_2, q)$-DLM as follows.

Table 6. Golden ratio $(2k, p_1 p_2, q)$-*DLM*.

$2k$	p_1	p_2	q	p_1 / p_2
13962	*8629*	*5333*	*46032419*	*1,618039*
41200	*25463*	*15737*	*400752431*	*1,618034*
60144	*37171*	*22973*	*853989527*	*1,618030*
63490	*39239*	*24251*	*951648479*	*1,618036*
68616	*42407*	*26209*	*1111513679*	*1,618032*
68878	*42569*	*26309*	*1120016699*	*1,618039*
75868	*46889*	*28979*	*1358872199*	*1,618034*
83052	*51329*	*31723*	*1628392919*	*1,618037*
85314	*52727*	*32587*	*1718300063*	*1,618038*
87780	*54251*	*33529*	*1819069559*	*1,618032*
92304	*57047*	*35257*	*2011398383*	*1,618033*
98488	*60869*	*37619*	*2289929399*	*1,618039*
113730	*70289*	*43441*	*3053538179*	*1,618034*
114636	*70849*	*43787*	*3102379799*	*1,618037*
117390	*72551*	*44839*	*3253231679*	*1,618033*
125904	*77813*	*48091*	*3742230887*	*1,618037*
130030	*80363*	*49667*	*3991519151*	*1,618036*
134664	*83227*	*51437*	*4281081863*	*1,618038*
137790	*85159*	*52631*	*4482141119*	*1,618039*

Using Mathematica 9.0 Software, we deduce that (*842538, 321821.520717, 74783651*) represents the closest golden ratio leaves of $(2k, p_1 p_2, q)$-*DLM* of (*0 < 2k < 1000000*). Let p_1 and p_2 be prime integers smaller than *100000*. Then the smallest of $\frac{p_1}{p_2}$ is *1,000020056358367* with *(199440, 99721.99719, 9944277839)-DLM* and the biggest of $\frac{p_1}{p_2}$ is *33323,666666666664* with *(99974, 99971.3, 399887)-DLM*. Finally, we compared the real leaf size in Balikesir University in Turkey with $(2k, p_1 p_2, q)$-*DLM* (Figure 8).

Figure 8. Examples leaves of $(2k, p_1 p_2, q)$-DLM in Balikesir.

In this study, a procedural modeling method based on odd divisor functions generated various leaves constructed in a real-time virtual ecosystem. We establish a relationship between the area of leaves and prime numbers using odd divisor functions. We have also established a model associated with the growing patterns of the leaves. In future, other ways to model different leaves may become possible through research on methods using the divisor functions.

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