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Research Article

The Vietoris hyperspace $\mathcal{F}(X)$ and certain generalized metric properties

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Abstract

In this paper, we study several generalized metric properties of the space $\mathcal{F}(X)$ of finite subsets of a space X endowed with the Vietoris topology. In particular, we consider such properties (P) for which $\mathcal{F}(X)$ has (P) if and only if X has (P). Also, we obtain some results related to the images of metric spaces under some kinds of continuous mappings.

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1. Introduction and preliminaries

Recently, the generalized metric properties on hyperspaces with the Vietoris topology have been studied by many authors ([8], [10], [16], [17], [20], [21], [22]). They considered several generalized metric properties and studied the relation between a space X satisfying such a property and its hyperspaces with the Vietoris topology, such as the n-fold symmetric product $\mathcal{F}_n(X)$, the hyperspace $\mathcal{F}(X)$ of finite subsets of X satisfying the same property.

In this paper, we study the relation between a space X satisfying certain generalized metric properties and its hyperspace of finite subsets $\mathcal{F}(X)$ with the Vietoris topology satisfying the same properties. We prove that

(1) X is an *sn*-symmetric space if and only if so is $\mathcal{F}(X)$;
(2) X has a σ-strong network consisting of *cs**-covers (*cs*-covers) if and only if so does $\mathcal{F}(X)$;
(3) X has a σ-(P)-strong network consisting of *cs**-covers (*cs*-covers) if and only if so does $\mathcal{F}(X)$.

By these results, we obtain that

(1) X is a semi-metric space if and only if so is $\mathcal{F}(X)$;

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(2) X is an sn-metrizable space (resp., an sn-developable space, a strongly sn-developable space) if and only if so is $\mathcal{F}(X)$;
(3) X is a weak Cauchy sn-symmetric space (resp., Cauchy sn-symmetric space) if and only if so is $\mathcal{F}(X)$;
(4) X is a Cauchy sn-symmetric space with a σ-(P)-property cs^*-network (resp., cs-network, sn-network) if and only if so is $\mathcal{F}(X)$;
(5) X is a space with a point-regular cs^*-network (resp., cs-network, sn-network) if and only if so is $\mathcal{F}(X)$.

By (5), we get that X has a point-regular base if and only if so does $\mathcal{F}(X)$. Moreover, we show that

(1) If $\mathcal{F}(X)$ is a g-metrizable space (resp., g-developable space, strongly g-developable space, Cauchy symmetric space, weak Cauchy symmetric space), then so is X, but the reverse is not true;
(2) If $\mathcal{F}(X)$ is a Cauchy symmetric space with a σ-(P)-property cs^*-network (resp., cs-network, sn-network, weak base), then so is X, but the reverse is not true;
(3) If $\mathcal{F}(X)$ has a point-regular weak base, then so does X, but the reverse is not true.

On the other hand, we also get some results about the images of metric spaces on Vietoris hyperspaces.

Throughout this paper, all spaces are assumed to be T_1 and regular, $\mathbb{N}$ denotes the set of all positive integers. For a sequence $\{x_n\}_{n\in\mathbb{N}}$ converging to x, we say that $\{x_n\}_{n\in\mathbb{N}}$ is *eventually* in P if $\{x\}\cup\{x_n : n\geq m\}\subset P$ for some $m\in\mathbb{N}$, and $\{x_n\}_{n\in\mathbb{N}}$ is *frequently* in P if some subsequence of $\{x_n\}_{n\in\mathbb{N}}$ is eventually in P.

Given a space X, we define its *hyperspaces* as the following sets:

(1) $CL(X)=\{A\subset X : A \text{ is closed and nonempty}\}$;
(2) $\mathbb{K}(X)=\{A\in CL(X) : A \text{ is compact}\}$;
(3) $\mathcal{F}_n(X)=\{A\in CL(X) : A \text{ has at most } n \text{ points}\}$, where $n\in\mathbb{N}$;
(4) $\mathcal{F}(X)=\{A\in CL(X) : A \text{ is finite}\}$.

The set $CL(X)$ is topologized by the *Vietoris topology* defined as the topology generated by

$$\mathcal{B}=\{\langle U_1,\dots,U_k\rangle : U_1,\dots,U_k \text{ are open subsets of } X,\ k\in\mathbb{N}\},$$

where

$$\langle U_1,\dots,U_k\rangle=\Big\{A\in CL(X) : A\subset \textstyle\bigcup_{i\leq k}U_i,\ A\cap U_i\neq\emptyset \text{ for each } i\leq k\Big\}.$$

Note that, by definition, $\mathbb{K}(X)$, $\mathcal{F}_n(X)$ and $\mathcal{F}(X)$ are subspaces of $CL(X)$. Hence, they are topologized with the appropriate restriction of the Vietoris topology. Moreover,

(1) $CL(X)$ is called the *hyperspace of nonempty closed subsets of* X;
(2) $\mathbb{K}(X)$ is called the *hyperspace of nonempty compact subsets of* X;
(3) $\mathcal{F}_n(X)$ is called the *n-fold symmetric product of* X;
(4) $\mathcal{F}(X)$ is called the *hyperspace of finite subsets of* X.

On the other hand, it is obvious that $\mathcal{F}(X)=\bigcup_{n=1}^{\infty}\mathcal{F}_n(X)$ and $\mathcal{F}_n(X)\subset\mathcal{F}_{n+1}(X)$ for each $n\in\mathbb{N}$.

Remark 1.1 ([20])**.** Let X be a space and let $n\in\mathbb{N}$.

(1) $\mathcal{F}_n(X)$ is closed in $\mathcal{F}(X)$.
(2) $f_1 : X\twoheadrightarrow\mathcal{F}_1(X)$, $(x\mapsto\{x\})$, is a homeomorphism.
(3) Every $\mathcal{F}_m(X)$ is a closed subset of $\mathcal{F}_n(X)$ for each $m,n\in\mathbb{N}$, $m<n$.

Notation 1.2 ([17])**.** If $U_1,\dots,U_s$ are open subsets of a space X, then $\langle U_1,\dots,U_s\rangle_{\mathcal{F}(X)}$ denotes the intersection of the open set $\langle U_1,\dots,U_s\rangle$ of the Vietoris topology, with $\mathcal{F}(X)$.

Notation 1.3 ([22])**.** Let X be a space. If $\{x_1,\dots,x_r\}$ is a point of $\mathcal{F}(X)$ and $\{x_1,\dots,x_r\}\in\langle U_1,\dots,U_s\rangle_{\mathcal{F}(X)}$, then for each $j\leq r$, we let

$$U_{x_j} = \bigcap\{U \in \{U_1, \ldots, U_s\} : x_j \in U\}.$$

Observe that $\langle U_{x_1}, \ldots, U_{x_r}\rangle_{\mathcal{F}(X)} \subset \langle U_1, \ldots, U_s\rangle_{\mathcal{F}(X)}$.

Definition 1.4 ([3]). For a cover $\mathcal{P}$ of a space X, let (P) be one of the following properties: point-finite, compact-finite, locally finite, point-countable, compact-countable, and locally countable. We say that $\mathcal{P}$ has the *σ-(P)-property*, if $\mathcal{P}$ can be expressed as $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$, where each $\mathcal{P}_n$ has the (P)-property.

Definition 1.5. Let $\mathcal{P}$ be a family of subsets of a space X and $P \subset X$.

(1) P is a *sequential neighborhood* at x [1], if each sequence L converging to x is eventually in P .
(2) $\mathcal{P}$ is a *cs^*-cover* [1] (resp., *cs-cover* [25]), if every convergent sequence is frequently (resp., eventually) in some $P \in \mathcal{P}$.
(3) $\mathcal{P}$ is a *cs^*-network* [1] (resp., *cs-network* [13]), if whenever L is a sequence converging to $x \in U$ with U open in X, then L is frequently (resp., eventually) in $P \subset U$ for some $P \in \mathcal{P}$.
(4) X is a *$\aleph$-space* [13], if it has a σ-locally finite cs-network.
(5) $\mathcal{P}$ is *point-regular* [1], if for every $x \in U$ with U open in X, the set $\{P \in \mathcal{P} : x \in P \not\subset U\}$ is finite.

Definition 1.6 ([3]). Let $\{\mathcal{P}_n : n \in \mathbb{N}\}$ be a sequence of covers of a space X. Put $\mathcal{P} = \bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$.

(1) $\mathcal{P}$ is a *σ-strong network* for X, if $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for every $n \in \mathbb{N}$ and $\{\mathtt{St}(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at each $x \in X$.
(2) $\mathcal{P}$ is a *σ-(P)-strong network* for X, if it is a σ-strong network and each $\mathcal{P}_n$ has the (P)-property.
(3) $\mathcal{P}$ is a *σ-(P)-strong network consisting of cs^*-covers (cs-covers)* for X, if it is a σ-(P)-strong network and each $\mathcal{P}_n$ is a cs^*-cover (cs-cover).

Definition 1.7 ([2]). Let $\mathcal{P} = \bigcup\{\mathcal{P}_x : x \in X\}$ be a cover of a space X such that $\mathcal{P}_x$ is a network at x, and if $P_1, P_2 \in \mathcal{P}_x$, then $P \subset P_1 \cap P_2$ for some $P \in \mathcal{P}_x$.

(1) $\mathcal{P}$ is a *weak base*, if for $G \subset X$, G is open in X if and only if for every $x \in G$, there exists $P \in \mathcal{P}_x$ such that $P \subset G$; $\mathcal{P}_x$ is said to be a *weak neighborhood base* at x.
(2) $\mathcal{P}$ is an *sn-network*, if every element of $\mathcal{P}_x$ is a sequential neighborhood of x for every $x \in X$; $\mathcal{P}_x$ is said to be an *sn-network* at x.

Remark 1.8. (1) Bases $\Rightarrow$ weak bases $\Rightarrow$ sn-networks [12] $\Rightarrow$ cs-networks $\Rightarrow$ cs^*-networks.
(2) In a sequential space, weak bases $\Leftrightarrow$ sn-networks [12].

Following [7], a function $d : X \times X \to [0, \infty)$ such that for all $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$ and $d(x, y) = d(y, x)$, is called a d-function on X.

Definition 1.9. [7, Definition 2.6] Let d be a d-function on a space X. For each $x \in X$ and $n \in \mathbb{N}$, put $S_n(x) = \{y \in X : d(x; y) < 1/n\}$. Then, X is *semi-metric* [24] (resp., *symmetric*, *sn-symmetric*), if $\{S_n(x) : n \in \mathbb{N}\}$ is a neighborhood base (resp., a weak neighborhood base, an sn-network) at x for all $x \in X$.

Definition 1.10. Let d be a d-function on a space X. Then:

(1) X is *Cauchy symmetric* [23] (resp., *Cauchy sn-symmetric* [2]), if (X, d) is a symmetric space (resp., an sn-symmetric space) and every convergent sequence is d-Cauchy.
(2) X is *weak Cauchy symmetric* (resp., *weak Cauchy sn-symmetric*) [5], if (X, d) is a symmetric space (resp., an sn-symmetric space) and every convergent sequence has a d-Cauchy subsequence.

Remark 1.11 ([3], [5], [7]). (1) symmetric spaces $\Longleftrightarrow$ sequential and sn-symmetric spaces.

(2) Cauchy symmetric spaces $\Longleftrightarrow$ sequential and Cauchy sn-symmetric spaces.
(3) weak Cauchy symmetric spaces $\Longleftrightarrow$ sequential and weak Cauchy sn-symmetric spaces.
(4) semi-metric spaces $\Longleftrightarrow$ first-countable and sn-symmetric spaces.

Definition 1.12 ([14]). Let

$$X = \{\infty\} \cup \{x_n : n \in \mathbb{N}\} \cup \{x_{nm} : n, m \in \mathbb{N}\},$$

where every x_n, x_{nm} and ∞ are different from each other. The set X endowed with the following topology is called the *Arens space* and denoted briefly as S_2: each x_{nm} is isolated; a basic neighborhood of x_n has the form $\{x_n\} \cup \{x_{nm} : m > k\}$ for some $k \in \mathbb{N}$; a basic neighborhood of ∞ has the form $\{\infty\} \cup \left(\bigcup\{V_n : n \geq k\}\right)$ for some $k \in \mathbb{N}$, where each V_n is a neighborhood of x_n.

Let us restrict the prefixes $\alpha(P_1)$ and $\alpha(P_2)$ to the following

(1) $\alpha(P_1)$ is compact if (P_1) is point-finite, $\alpha(P_1)$ is *mssc* if (P_1) is locally finite, and $\alpha(P_1)$ is *msk* if (P_1) is compact-finite.
(2) $\alpha(P_2)$ is *s* if (P_2) is point-countable, $\alpha(P_2)$ is *cs* if (P_2) is compact-countable, and $\alpha(P_2)$ is *msss* if (P_2) is locally countable.

For some undefined or related concepts, we refer the reader to [2], [3] and [13].

2. Main results

Let X be a space. We say that a sequence $\{A_n\}_{n\in\mathbb{N}}$ consisting of subsets of X converges to a subset $A \subset X$ if for each open set U in X with $A \subset U$, there exists $k \in \mathbb{N}$ such that $A_n \subset U$ for each $n > k$.

Lemma 2.1. *Let X be a space and $\{F_m\}_{m\in\mathbb{N}}$ be a sequence of points of $\mathcal{F}(X)$. If $\{F_m\}_{m\in\mathbb{N}}$ converges to a point $F = \{x_1, \ldots, x_r\}$ in $\mathcal{F}(X)$ and $\{U_1, \ldots, U_r\}$ is a family of pairwise disjoint open subsets of X such that $x_j \in U_j$ for each $j \leq r$, then $\{F_m \cap U_j\}_{m\in\mathbb{N}}$ converges to $\{x_j\}$ in X for each $j \leq r$.*

Proof. Fix $j \in \{1, \ldots, r\}$ and let V_j be any open neighborhood of x_j in X. Put $O_j = V_j \cap U_j$, then O_j is an open neighborhood of x_j in X. This implies that

$$F \in \langle U_1, \ldots, U_{j-1}, O_j, U_{j+1}, \ldots, U_r\rangle_{\mathcal{F}(X)}.$$

Since $\{F_m\}_{m\in\mathbb{N}}$ converges to the point $F = \{x_1, \ldots, x_r\}$ in $\mathcal{F}(X)$, there exists $k \in \mathbb{N}$ such that

$$\{F_m : m > k\} \subset \langle U_1, \ldots, U_{j-1}, O_j, U_{j+1}, \ldots, U_r\rangle_{\mathcal{F}(X)} \cap \langle U_1, \ldots, U_r\rangle_{\mathcal{F}(X)}.$$

Because $\{U_1, \ldots, U_r\}$ is a family of pairwise disjoint open subsets of X and $O_j \subset U_j$, we have that

$$F_m \cap U_j \subset O_j \subset V_j \text{ for each } m > k.$$

Therefore, $\{F_m \cap U_j\}_{m\in\mathbb{N}}$ converges to $\{x_j\}$ in X. $\square$

For each $n \in \mathbb{N}$, let $\mathcal{P}_n$ be a family of subsets of a space X. Put

$$\mathfrak{P}_n = \{\langle P_1^{(n)}, \ldots, P_s^{(n)}\rangle_{\mathcal{F}(X)} : P_1^{(n)}, \ldots, P_s^{(n)} \in \mathcal{P}_n, s \in \mathbb{N}\},$$

where $\langle P_1^{(n)}, \ldots, P_s^{(n)}\rangle_{\mathcal{F}(X)} = \langle P_1^{(n)}, \ldots, P_s^{(n)}\rangle \cap \mathcal{F}(X)$. Then, $\mathfrak{P}_n$ is a family of subsets of $\mathcal{F}(X)$ for each $n \in \mathbb{N}$.

If $\mathcal{A}$ is a family of subsets of a space Y and $B \subset Y$, then the star of B with respect to $\mathcal{A}$ is the set

$$\mathtt{St}(B, \mathcal{A}) := \cup\{A \in \mathcal{A} : A \cap B \neq \emptyset\}.$$

For $y \in Y$, we use the notation $\mathtt{St}(y, \mathcal{A})$ instead of $\mathtt{St}(\{y\}, \mathcal{A})$.

Lemma 2.2. *For each $n \in \mathbb{N}$,*

$$\mathtt{St}(\{x_1, \dots, x_s\}, \mathfrak{P}_n) = \langle \mathtt{St}(x_1, \mathcal{P}_n), \mathtt{St}(x_2, \mathcal{P}_n), \dots, \mathtt{St}(x_s, \mathcal{P}_n)\rangle_{\mathcal{F}(X)}.$$

Proof. Let $A \in \mathtt{St}(\{x_1, \dots, x_s\}, \mathfrak{P}_n)$. Then, there exist $P_1, \dots, P_l \in \mathcal{P}_n$ such that

$$A, \{x_1, \dots, x_s\} \in \langle P_1, \dots, P_l\rangle_{\mathcal{F}(X)}.$$

By [15, Lemma 2.3.1], we have that

$$A \in \langle P_1, \dots, P_l\rangle_{\mathcal{F}(X)} \subset \langle \mathtt{St}(x_1, \mathcal{P}_n), \dots, \mathtt{St}(x_s, \mathcal{P}_n)\rangle_{\mathcal{F}(X)}.$$

Therefore,

$$\mathtt{St}(\{x_1, \dots, x_s\}, \mathfrak{P}_n) \subset \langle \mathtt{St}(x_1, \mathcal{P}_n), \mathtt{St}(x_2, \mathcal{P}_n), \dots, \mathtt{St}(x_s, \mathcal{P}_n)\rangle_{\mathcal{F}(X)}. \tag{2.1}$$

Next, take any $A = \{y_1, \dots, y_k\} \in \langle \mathtt{St}(x_1, \mathcal{P}_n), \mathtt{St}(x_2, \mathcal{P}_n), \dots, \mathtt{St}(x_s, \mathcal{P}_n)\rangle_{\mathcal{F}(X)}$. Then, for each $i \le k$, since

$$A = \{y_1, \dots, y_k\} \subset \bigcup_{i \le s} \mathtt{St}(x_i, \mathcal{P}_n),$$

there exists $j \le s$ such that $y_i \in \mathtt{St}(x_j, \mathcal{P}_n)$. Hence, there exists $P_{y_i} \in \mathcal{P}_n$ such that $\{y_i, x_j\} \subset P_{y_i}$. On the other hand, for each $j \le s$, since $A \cap \mathtt{St}(x_j, \mathcal{P}_n) \ne \emptyset$, there exist $i \le k$ and $Q_{x_j} \in \mathcal{P}_n$ such that $\{x_j, y_i\} \subset Q_{x_j}$. If we put

$$\{P_{y_i} : i \le k\} \cup \{Q_{x_j} : j \le s\} = \{G_1, \dots, G_r\},$$

then

$$A \in \langle G_1, \dots, G_r\rangle_{\mathcal{F}(X)} \subset \mathtt{St}(\{x_1, \dots, x_s\}, \mathfrak{P}_n).$$

This shows that

$$\langle \mathtt{St}(x_1, \mathcal{P}_n), \mathtt{St}(x_2, \mathcal{P}_n), \dots, \mathtt{St}(x_s, \mathcal{P}_n)\rangle_{\mathcal{F}(X)} \subset \mathtt{St}(\{x_1, \dots, x_s\}, \mathfrak{P}_n). \tag{2.2}$$

By (2.1), (2.2), the lemma is proved. □

Lemma 2.3. (1) *If $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a σ-strong network for X, then $\bigcup\{\mathfrak{P}_n : n \in \mathbb{N}\}$ is a σ-strong network for $\mathcal{F}(X)$.*

(2) *For each $n \in \mathbb{N}$, if $\mathtt{St}(x, \mathcal{P}_n)$ is a sequential neighborhood of x for all $x \in X$, then $\mathtt{St}(F, \mathfrak{P}_n)$ is a sequential neighborhood of F for all $F \in \mathcal{F}(X)$.*

(3) *If $\mathcal{P}_n$ is a cs*-cover (resp., cs-cover) for X, then $\mathfrak{P}_n$ is a cs*-cover (resp., cs-cover) for $\mathcal{F}(X)$.*

(4) *If $\mathcal{P}_n$ has the (P)-property, then $\mathfrak{P}_n$ has the (P)-property.*

Proof. Assume that $F = \{x_1, \dots, x_r\} \in \mathcal{F}(X)$ and $\mathcal{U}$ is an open neighborhood of F in $\mathcal{F}(X)$. Then, there exist open subsets $U_1, \dots, U_s$ of X such that

$$F \in \langle U_1, \dots, U_s\rangle_{\mathcal{F}(X)} \subset \mathcal{U}.$$

Because X is Hausdorff, it follows from Notation 1.3 that we can find pairwise disjoint open subsets $O_1, \dots, O_r$ of X such that $x_j \in O_j$ for each $j \le r$ and

$$F \in \langle O_1, \dots, O_r\rangle_{\mathcal{F}(X)} \subset \langle U_1, \dots, U_s\rangle_{\mathcal{F}(X)} \subset \mathcal{U}.$$

Let $\{F_m\}_{m\in\mathbb{N}}$ be a sequence converging to F in $\mathcal{F}(X)$. By Lemma 2.1, for each $j \le r$, the sequence $\{F_m \cap U_j\}_{m\in\mathbb{N}}$ converges to $\{x_j\}$ in X.

(1) For each $n \in \mathbb{N}$, since $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$, it is obvious that $\mathfrak{P}_{n+1}$ refines $\mathfrak{P}_n$. On the other hand, for each $j \le r$, since $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a σ-strong network for X, $\{\mathtt{St}(x_j, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at x_j in X. Thus, there exists $m_j \in \mathbb{N}$ such that

$$x_j \in \mathtt{St}(x_j, \mathcal{P}_n) \subset U_j \text{ whenever } n \ge m_j.$$

Let $m = \max\{m_j : j \le r\}$. Then

$$F \in \langle \mathtt{St}(x_1, \mathcal{P}_n), \dots, \mathtt{St}(x_r, \mathcal{P}_n)\rangle_{\mathcal{F}(X)} \subset \langle U_1, \dots, U_r\rangle_{\mathcal{F}(X)}$$

whenever $n \ge m$. It follows from Lemma 2.2 that $F \in \mathtt{St}(F, \mathfrak{P}_n) \subset \mathcal{U}$ for every $n \ge m$. Therefore, $\{\mathtt{St}(F, \mathfrak{P}_n) : n \in \mathbb{N}\}$ is a network at F in $\mathcal{F}(X)$. This implies that $\bigcup\{\mathfrak{P}_n : n \in \mathbb{N}\}$ is a σ-strong network for $\mathcal{F}(X)$.

(2) For each $n \in \mathbb{N}$ and $j \le r$, since $\mathtt{St}(x_j, \mathcal{P}_n)$ is a sequential neighborhood of x_j, there exists $k_j \in \mathbb{N}$ such that

$$\{x_j\} \cup \Big(\bigcup\{F_m \cap U_j : m \ge k_j\}\Big) \subset \mathtt{St}(x_j, \mathcal{P}_n).$$

If we put $k = \max\{k_j : j \le r\}$, then it follows from Lemma 2.2 that

$$\{F\} \cup \{F_m : m > k\} \subset \langle \mathtt{St}(x_1, \mathcal{P}_n), \mathtt{St}(x_2, \mathcal{P}_n), \dots, \mathtt{St}(x_r, \mathcal{P}_n)\rangle_{\mathcal{F}(X)} = \mathtt{St}(F, \mathfrak{P}_n).$$

This shows that $\mathtt{St}(F, \mathfrak{P}_n)$ is a sequential neighborhood of F.

(3) If $\mathcal{P}_n$ is a cs^*-cover for X, by induction on r, then there exist $P_j^{(n)} \in \mathcal{P}_n$ and a subsequence $\{m_k\}_{k\in\mathbb{N}}$ of $\mathbb{N}$ such that

$$\{x_j\} \cup \Big(\bigcup\{F_{m_k} \cap U_j : k \in \mathbb{N}\}\Big) \subset P_j^{(n)}.$$

This implies that $\langle P_1^{(n)}, \dots, P_r^{(n)}\rangle_{\mathcal{F}(X)} \in \mathfrak{P}_n$ and

$$\{F\} \cup \{F_{m_k} : k \in \mathbb{N}\} \subset \langle P_1^{(n)}, \dots, P_r^{(n)}\rangle_{\mathcal{F}(X)}.$$

Hence, $\mathfrak{P}_n$ is a cs^*-cover for $\mathcal{F}(X)$.

If $\mathcal{P}_n$ is a cs-cover for X, then there exist $P_j^{(n)} \in \mathcal{P}_n$ and $k_j \in \mathbb{N}$ such that

$$\{x_j\} \cup \Big(\bigcup\{F_m \cap U_j : m \ge k_j\}\Big) \subset P_j^{(n)}.$$

Put $k = \max\{k_j : j \le r\}$. Then, $\langle P_1^{(n)}, \dots, P_r^{(n)}\rangle_{\mathcal{F}(X)} \in \mathfrak{P}_n$ and

$$\{F\} \cup \{F_m : m > k\} \subset \langle P_1^{(n)}, \dots, P_r^{(n)}\rangle_{\mathcal{F}(X)}.$$

Therefore, $\mathfrak{P}_n$ is a cs-cover for $\mathcal{F}(X)$.

(4) Because each $\mathcal{P}_n$ has the (P)-property, similar to the proof of [21, Lemma 2.2], we claim that $\mathfrak{P}_n$ has the (P)-property. □

Theorem 2.4. *A space X is an sn-symmetric space if and only if so is $\mathcal{F}(X)$.*

Proof. *Necessity.* Let X be an sn-symmetric space. By [3, Theorem 2.3], X has a σ-strong network $\bigcup\{\mathcal{P}_n : n \in \mathbb{N}\}$ such that $\{\mathtt{St}(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is an sn-network at x for all $x \in X$. It follows from Lemma 2.3(1) that $\bigcup\{\mathfrak{P}_n : n \in \mathbb{N}\}$ is a σ-strong network for $\mathcal{F}(X)$, where

$$\mathfrak{P}_n = \{\langle P_1^{(n)}, \dots, P_s^{(n)}\rangle_{\mathcal{F}(X)} : P_1^{(n)}, \dots, P_s^{(n)} \in \mathcal{P}_n, s \in \mathbb{N}\}.$$

Now, we will prove that $\{\mathtt{St}(F, \mathfrak{P}_n) : n \in \mathbb{N}\}$ is an sn-network at F for all $F \in \mathcal{F}(X)$. Indeed, take any $F = \{x_1, \dots, x_r\} \in \mathcal{F}(X)$. Then:

(1) Since $\bigcup\{\mathfrak{P}_n : n \in \mathbb{N}\}$ is a σ-strong network for $\mathcal{F}(X)$, $\{\mathtt{St}(F, \mathfrak{P}_n) : n \in \mathbb{N}\}$ is a network at F.

(2) Let $\mathtt{St}(F, \mathfrak{P}_{n_1}), \mathtt{St}(F, \mathfrak{P}_{n_2}) \in \{\mathtt{St}(F, \mathfrak{P}_n) : n \in \mathbb{N}\}$. Since $\mathfrak{P}_{n+1}$ refines $\mathfrak{P}_n$ for all $n \in \mathbb{N}$, if we put $m = \max\{n_1, n_2\}$, then

$$\mathtt{St}(F, \mathfrak{P}_m) = \mathtt{St}(F, \mathfrak{P}_{n_1}) \cap \mathtt{St}(F, \mathfrak{P}_{n_2}).$$

(3) Since $\{\mathtt{St}(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is an sn-network at x for all $x \in X$, $\mathtt{St}(x, \mathcal{P}_n)$ is a sequential neighborhood of x for all $x \in X$ and $n \in \mathbb{N}$. By Lemma 2.3(2), $\mathtt{St}(F, \mathfrak{P}_n)$ is a sequential neighborhood of F for each $n \in \mathbb{N}$,.

By [3, Theorem 2.3], $\mathcal{F}(X)$ is an sn-symmetric space.

Sufficiency. Let $\mathcal{F}(X)$ be an sn-symmetric space. Since every subspace of an sn-symmetric space is an sn-symmetric space, X is an sn-symmetric space by Remark 1.1. □

Corollary 2.5. *A space X is a semi-metric space if and only if so is $\mathcal{F}(X)$.*

Proof. By [15, Theorem 4.5.3] and Remark 1.1, we have that X is a first-countable space if and only if so is $\mathcal{F}(X)$. Therefore, X is a semi-metric space if and only if so is $\mathcal{F}(X)$ by Theorem 2.4 and Remark 1.11(4). □

Theorem 2.6. *Let X be a space. Then:*

(1) *X has a σ-strong network consisting of cs^*-covers (cs-covers) if and only if so does $\mathcal{F}(X)$;*
(2) *X has a σ-(P)-strong network consisting of cs^*-covers (cs-covers) if and only if so does $\mathcal{F}(X)$.*

Proof. *Necessity.* By Lemma 2.3.

Sufficiency. Assume that $\bigcup\{\mathfrak{P}_n : n \in \mathbb{N}\}$ is a σ-strong network consisting of cs^*-covers (cs-covers) (resp., σ-(P)-strong network consisting of cs^*-covers (cs-covers)) for $\mathcal{F}(X)$. For each $n \in \mathbb{N}$, we put

$$\mathfrak{G}_n = \{\mathcal{W} \cap \mathcal{F}_1(X) : \mathcal{W} \in \mathfrak{P}_n\}.$$

Then, $\bigcup\{\mathfrak{G}_n : n \in \mathbb{N}\}$ is a σ-strong network consisting of cs^*-covers (cs-covers) (resp., σ-(P)-strong network consisting of cs^*-covers (cs-covers)) for $\mathcal{F}_1(X)$. Thus, X has a σ-strong network consisting of cs^*-covers (cs-covers) (resp., σ-(P)-strong network consisting of cs^*-covers (cs-covers)) by Remark 1.1. □

By Theorem 2.6, [3, Theorems 2.5, 2.7, 2.9], [3, Corollaries 2.11, 2.13], [6, Theorem 1], [2, Theorems 2.3, 2.6, 2.9], [2, Corollaries 3.6, 3.8], [1, Theorem 2.3], [11, Theorem 1], we obtain the following corollaries.

Corollary 2.7. *Let X be a space. Then:*

(1) *X is an sn-metrizable space (resp., an sn-developable space, a strongly sn-developable space) if and only if so is $\mathcal{F}(X)$;*
(2) *X is a weak Cauchy sn-symmetric space (resp., Cauchy sn-symmetric space) if and only if so is $\mathcal{F}(X)$;*
(3) *X is a Cauchy sn-symmetric space with a σ-(P)-property cs^*-network (resp., cs-network, sn-network) if and only if so is $\mathcal{F}(X)$;*
(4) *X is a space with a point-regular cs^*-network (resp., cs-network, sn-network) if and only if so is $\mathcal{F}(X)$.*

Corollary 2.8. *Suppose a topological property γ satisfies the following:*

(1) *γ is a 1-sequence-covering and π-image of a metric space;*
(2) *γ is a sequence-covering and π-image of a metric space;*
(3) *γ is a compact-covering compact and mssc-image of a metric space;*
(4) *γ is a sequentially-quotient π and mssc-image of a metric space;*
(5) *γ is a 1-sequence-covering and mssc-image of a metric space;*
(6) *γ is a 1-sequence-covering compact and σ-image of a metric space;*
(7) *γ is a sequence-covering π and σ-image of a metric space;*
(8) *γ is a 1-sequence-covering and compact image of a metric space;*
(9) *γ is a sequence-covering and compact image of a metric space;*
(10) *γ is a pseudo-sequence-covering and compact image of a metric space;*
(11) *γ is a sequentially-quotient and π-image of a metric space;*
(12) *γ is a 1-sequence-covering compact, $\alpha(P_1)$-image of a metric space;*
(13) *γ is a sequence-covering π, $\alpha(P_1)$-image of a metric space;*
(14) *γ is a 1-sequence-covering π, $\alpha(P_2)$-image of a metric space;*

(15) γ *is a sequence-covering* π, $\alpha(P_2)$*-image of a metric space.*

Let X *be a space. Then,* X *has the property* γ *if and only if so does* $\mathcal{F}(X)$.

Remark 2.9. By Corollary 2.7(4), we obtain [10, Theorem 4.7] that X has a point-regular base if and only if so does $\mathcal{F}(X)$.

Proof. It follows from [4, Lemma 5.4.7] that a space has a point-regular base if and only if it is a first-countable space with a point-regular sn-network. On the other hand, by the proof of Corollary 2.5, we have that X is a first-countable space if and only if so is $\mathcal{F}(X)$. Therefore, X has a point-regular base if and only if so does $\mathcal{F}(X)$ by Corollary 2.7(4). $\square$

Since the property of sequential spaces is closed hereditary, by Remark 1.1 and Corollary 2.7, we obtain the following corollary.

Corollary 2.10. *Let* X *be a space. Then:*

(1) *If* $\mathcal{F}(X)$ *is a* g*-metrizable space (resp.,* g*-developable space, strongly* g*-developable space, Cauchy symmetric space, weak Cauchy symmetric space), then so is* X*;*
(2) *If* $\mathcal{F}(X)$ *is a Cauchy symmetric space with a* σ*-*(P)*-property* cs^**-network (resp.,* cs*-network,* sn*-network, weak base), then so is* X*;*
(3) *If* $\mathcal{F}(X)$ *has a point-regular weak base, then so does* X.

By Corollary 2.10 and [3, Corollaries 2.6, 2.8, 2.12, 2.14] and [2, Corollaries 2.7, 2.10], we get the following corollary.

Corollary 2.11. *Suppose a topological property* γ *satisfies the following:*

(1) γ *is a weak-open and* π*-image of a metric space;*
(2) γ *is a weak-open and mssc-image of a metric space;*
(3) γ *is a weak-open compact-covering compact and* σ*-image of a metric space;*
(4) γ *is a weak-open* π *and* σ*-image of a metric space;*
(5) γ *is a weak-open and compact image of a metric space;*
(6) γ *is a weak-open compact-covering compact,* $\alpha(P_1)$*-image of a metric space;*
(7) γ *is a weak-open* π, $\alpha(P_1)$*-image of a metric space;*
(8) γ *is a weak-open compact-covering* π, $\alpha(P_2)$*-image of a metric space;*
(9) γ *is a weak-open* π, $\alpha(P_2)$*-image of a metric space.*

Then, if X *is a space satisfying* $\mathcal{F}(X)$ *has the property* γ*, then so does* X.

Lemma 2.12. *If* X, Y *are Cauchy* sn*-symmetric spaces (resp., Cauchy symmetric spaces), then* $X \oplus Y$ *is a Cauchy* sn*-symmetric space (resp., Cauchy symmetric space).*

Proof. By [3, Theorem 2.5], the space X (resp., the space Y) has a σ-strong network consisting of cs-covers $\bigcup\{\mathcal{G}_n : n \in \mathbb{N}\}$ (resp., $\bigcup\{\mathcal{H}_n : n \in \mathbb{N}\}$). Then, $\mathcal{G}_{n+1} \cup \mathcal{H}_{n+1}$ refines $\mathcal{G}_n \cup \mathcal{H}_n$ for every $n \in \mathbb{N}$. Next, let $x \in X \oplus Y$ and V be an open neighborhood of x in $X \oplus Y$. Without loss of generality we can assume that $x \in X$. Since $V \cap X$ is an open neighborhood of x in X, there exists $n \in \mathbb{N}$ such that $\mathtt{St}(x, \mathcal{G}_n) \subset V \cap X$. On the other hand, since $x \notin H$ for every $H \in \mathcal{H}_n$, we claim that

$$\mathtt{St}(x, \mathcal{G}_n \cup \mathcal{H}_n) = \mathtt{St}(x, \mathcal{G}_n) \subset V \cap X \subset V.$$

Therefore, $\bigcup\{\mathcal{G}_n \cup \mathcal{H}_n : n \in \mathbb{N}\}$ is a σ-strong network for $X \oplus Y$.

Now, suppose that the sequence L converges to x in $X \oplus Y$. Without loss of generality we can assume that $L \subset X$. Thus, the sequence L converges to x in X. Because $\mathcal{G}_n$ is a cs-cover for X, there exists $G \in \mathcal{G}_n \subset \mathcal{G}_n \cup \mathcal{H}_n$ such that L is eventually in G. It shows that each $\mathcal{G}_n \cup \mathcal{H}_n$ is a cs-cover for $X \oplus Y$. It follows from [3, Theorem 2.5] that $X \oplus Y$ is a Cauchy sn-symmetric space. Moreover, if X, Y are sequential spaces, then $X \oplus Y$ is a sequential space. Hence, if X, Y are Cauchy symmetric spaces, then $X \oplus Y$ is a Cauchy symmetric space by Remark 1.11. $\square$

Lemma 2.13. *The Arens space S_2 is a Cauchy symmetric space.*

Proof. For each $n \in \mathbb{N}$, we put

$$\mathcal{P}_n = \Big\{\{x_{ij}\} : i, j \in \mathbb{N}\Big\} \cup \Big(\{\infty\} \cup \{x_i : i \geq n\}\Big) \cup \Big\{\{x_i\} \cup \{x_{im} : m \geq n\} : i \in \mathbb{N}\Big\}.$$

Then, $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for every $n \in \mathbb{N}$. Furthermore, we have:

(1) $\{\mathtt{St}(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at each $x \in S_2$.

Let $x \in S_2$ and V be an open neighborhood of x in S_2. If $x = x_{ij}$, then $\mathtt{St}(x, \mathcal{P}_n) = \{x\} \subset V$ for every $n > j$. If $x = x_m$, then we choose $n > m$ such that

$$\mathtt{St}(x, \mathcal{P}_n) = \{x_m\} \cup \{x_{mj} : j \geq n\} \subset V.$$

If $x = \infty$, then there exists $m \in \mathbb{N}$ such that $x_i \in V$ for every $i \geq m$. Take $n \geq m$, we have

$$\mathtt{St}(x, \mathcal{P}_n) = \{\infty\} \cup \{x_i : i \geq n\} \subset V.$$

Therefore, $\{\mathtt{St}(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at x.

(2) Each $\mathcal{P}_n$ is a *cs*-cover for S_2.

Let $x \in S_2$ and L be a sequence converging to x in S_2. If $x = x_{ij}$, then L is eventually in $P = \{x_{ij}\} \in \mathcal{P}_n$. If $x = x_i$, then L is eventually in $P = \{x_i\} \cup \{x_{ij} : j \geq n\} \in \mathcal{P}_n$. If $x = \infty$, then L is eventually in $P = \{\infty\} \cup \{x_i : i \geq m\} \in \mathcal{P}_n$. This shows that $\mathcal{P}_n$ is a *cs*-cover for S_2.

Then, S_2 is a Cauchy *sn*-symmetric space by Theorem 2.5 in [3] stating that a space X is Cauchy *sn*-symmetric if and only if X has a σ-strong network consisting of *cs*-covers. (Recall that for a sequence $\{\mathcal{P}_n : n \in \mathbb{N}\}$ of covers of a space X, $\cup\{\mathcal{P}_n : n \in \mathbb{N}\}$ is a σ-strong network for X [9] if $\mathcal{P}_{n+1}$ refines $\mathcal{P}_n$ for all $n \in \mathbb{N}$ and $\{\mathtt{St}(x, \mathcal{P}_n) : n \in \mathbb{N}\}$ is a network at x for each $x \in X$.) Since S_2 is a sequential space, S_2 is a Cauchy symmetric space by Remark 1.11. □

Remark 2.14. (1) In [2, Lemma 2.2] the authors described a (general) construction of the d-function which can work in the proof of the previous lemma.

(2) We gave a direct proof of the previous lemma. However, the result follows from the fact that S_2 is a 1-sequence-covering quotient and compact image of a metric space. Such spaces are Cauchy symmetric [18].

Example 2.15. There exists a Cauchy symmetric and $\aleph$-space X such that $\mathcal{F}_2(X)$ is not a k-space.

Proof. Let $Y = S_2 \times (\mathbb{P} \cup \{0\})$, where S_2 is the Arens space and $\mathbb{P}$ is the set of irrational numbers. Then, Y is not a k-space [13, Example 1.8.6]. Put $X = S_2 \oplus (\mathbb{P} \cup \{0\})$. Then, X is a $\aleph$-space because the space S_2 and $\mathbb{P} \cup \{0\}$ are $\aleph$-spaces. Moreover, since S_2 is Cauchy symmetric by Lemma 2.13, and $\mathbb{P} \cup \{0\}$ is Cauchy symmetric, we claim that X is Cauchy symmetric by Lemma 2.12. On the other hand, since Y is a closed subset of X^2 and the property of k-spaces is closed hereditary, we can conclude that the product X^2 is not a k-space. Therefore, $\mathcal{F}_2(X)$ is not a k-space by [20, Remark 4.2]. □

Remark 2.16. By Example 2.15, we claim that the inverse of Corollaries 2.10 and 2.11 is not true.

Proof. Let X be a Cauchy symmetric space and $\aleph$-space in Example 2.15. Observe that X is a weak Cauchy symmetric space. It follows from [2, Theorem 2.3], [3, Theorem 2.9], Remarks 1.8 and 1.11 that X is a Cauchy symmetric space with a σ-locally finite weak base. Furthermore, by [2, Corollary 2.7, Remark 2.8], X is a strong g-developable space and X is a weak-open compact-covering compact and *mssc*-image of a metric space. Thus, X is g-developable, g-metrizable, a weak-open π and σ-image of a metric space by

[3, Corollary 2.12]. It follows from [3, Corollary 2.14] that X has a point-regular weak base. By [3, Corollaries 2.6, 2.8, 2.12, 2.14] and [2, Corollaries 2.7, 2.10], we claim that X satisfies the properties γ in Corollary 2.11. On the other hand, if $\mathcal{F}_2(X)$ is a g-metrizable space or a g-developable space or a strongly g-developable space or a (weak) Cauchy symmetric space or a Cauchy symmetric space with a σ-(P)-property cs^*-network (resp., cs-network, sn-network, weak base) or a space with a point-regular weak base or a space satisfies one of the properties in Corollary 2.11, then $\mathcal{F}_2(X)$ is a k-space. This is a contradiction. Therefore, the converse of Corollaries 2.10 and 2.11 is not true. $\square$

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