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Research Article

The Source of Semi-Primeness of Γ-Rings

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Abstract: The notion of source of semi-primeness is firstly given by Aydın, Demir and Camcı in 2018 as the set of all elements a of R that satisfy $aRa = (0)$ for any associative ring R. They investigated some basic properties of this set and defined three types of rings which have not appeared in literature before. The theory of gamma ring has been introduced by Nobusawa in 1964 as a generalization of rings. In this work, we generalized the notion of source of semi-primeness for gamma rings and investigated its basic algebraic properties. We also defined $|S_M|$-strongly reduced Γ-ring, $|S_M|$-domain, $|S_M|$-division ring and examined the relationship between these structures. We determined all possible characteristic values of a $|S_M|$-domain and proved every finite $|S_M|$-domain Γ-ring M is a $|S_M|$-division Γ-ring.

Keywords: Γ-ring, source of semi-primeness, strong unity.

1. Introduction

The theory of gamma rings has been introduced by Nobusawa as a generalization of rings by defining triple products on two abelian groups [11]. His model was a pair (Γ, M), where M is a subgroup of $\mathrm{Hom}(A, B)$ and Γ is a subgroup of $\mathrm{Hom}(B, A)$ for additive abelian groups A and B and products $M \times \Gamma \times M$ and $\Gamma \times M \times \Gamma$, which are defined as ordinary composition of mappings. W. Barnes dropped the closedness of multiplications in Γ and then defined slightly generalized gamma rings [2]. After Barnes' definition a number of authors have done a lot of works and have obtained various generalizations analogous to the corresponding results in ring theory [3–6, 8, 9].

Prime and semiprime ideals of a Γ-ring M are beneficial to obtain the algebraic structure of M. The notion of a prime ideal was firstly defined by W. Barnes as an ideal P that satisfies $AB \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ for any ideals A and B of M [2]. Barnes also defined prime ideal and prime radical in this work. He obtained some equivalent conditions that of an ideal to be a prime ideal and introduced prime radical of a Γ-ring M by defining m-system in a manner

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analogous to that of McCoy [10]. Kyuno is also obtained some results on prime ideal, semiprime ideal and prime radical of a Γ-ring M [6].

The source of semi-primeness of a ring R which is denoted by S_R was firstly defined by Aydın et al. in 2018 as the set of all elements a of R satisfying $aRa = (0)$ [1]. They proved some of basic properties of the set S_R. Aydın et al. also defined other new notions which are $|S_R|$-strongly reduced ring, $|S_R|$-domain and $|S_R|$-field and obtained their relations with each other.

Our main interest is to define the source of semi-primeness $S_M(A)$ for any subset A of a Γ-ring M and to introduce some new notions such as $|S_M|$-strongly reduced ring, $|S_M|$-integral domain and $|S_M|$-field to understand the algebraic structure of the Γ-ring M.

2. Preliminaries

Let M and Γ be two additive Abelian groups. M is said to be a Γ-ring (in the sense of Barnes) if there exists ternary multiplication $M \times \Gamma \times M \to M$ satisfying below conditions for all $a, b, c \in M$, $\alpha, \beta \in \Gamma$:

(1) $(a + b)\,\alpha c = a\alpha c + b\alpha c,$
$a(\alpha + \beta)c = a\alpha c + a\beta c,$
$a\alpha\,(b + c) = a\alpha b + a\alpha c,$

(2) $(a\alpha b)\,\beta c = a\alpha\,(b\beta c).$

Let M be a Γ-ring. If there exist $\delta \in \Gamma$ and $e \in M$ such that $a\delta e = e\delta a = a$ for any $a \in M$, then a pair (δ, e) is called strong unity of the Γ-ring M [9]. A subset N of the Γ-ring M is said to be a subring if N is a subgroup of M and $n\alpha n' \in N$ for all $n, n' \in N$ and $\alpha \in \Gamma$. A subgroup U of M is called left ideal (resp. right ideal) if $M\Gamma U \subseteq U$ (resp. $U\Gamma M \subseteq U$). If U is both left and right ideal, then U is called an ideal of M. An ideal P of the Γ-ring M is said to be prime if $A\Gamma B \subseteq P$ implies $A \subseteq P$ or $B \subseteq P$ for any ideals A and B of M [2]. An ideal Q of M is said to be semi-prime if $A\Gamma A \subseteq P$ implies $A \subseteq P$ for any ideal A of M [6]. A Γ-ring M is said to be prime (resp. semi-prime) if the zero ideal is prime (resp. semi-prime) [9].

A nonzero element a in M is called zero divisor if there are nonzero elements $b, c \in M$ and $\beta, \gamma \in \Gamma$ such that $a\beta b = 0 = c\gamma a$. An element x of a Γ-ring M is called strongly nilpotent if there exists a positive integer n such that $(x\Gamma)^n x = (x\Gamma x\Gamma \ldots x\Gamma)x = (0)$ [8]. The smallest such n is called the index of x. A Γ-ring M with no nonzero strongly nilpotent elements is called a strongly reduced Γ-ring. A Γ-ring M is said to be a division Γ-ring if it has a strong unity (δ, e) and for each nonzero element a of M there exists b of M such that $a\delta b = b\delta a = e$. The prime radical of a Γ-ring M is the intersection of all prime ideals of M [9]. If there exists a positive integer n such that $nx = 0$ for all $x \in M$, then the smallest such positive integer is called the characteristic of M and denoted by char$M$. If there is no such positive integer, then M is said to be characteristic zero. Let M_1 be a Γ_1-ring and M_2 be a Γ_2-ring. An ordered pair (θ, φ) is called homomorphism if $\varphi : M_1 \longrightarrow M_2$ is a group homomorphism, $\theta : \Gamma_1 \longrightarrow \Gamma_2$ is a group homomorphism and $\varphi(a\alpha b) = \varphi(a)\theta(\alpha)\varphi(b)$ for all $a, b \in M$ and $\alpha \in \Gamma$ [9]. A subset A of a Γ-ring M is called semi-group ideal if $a\alpha m, m\alpha a \in A$ for all $a \in A$, $\alpha \in \Gamma$ and $m \in M$.

In this study, we introduced the notion of source of semi-primeness $S_M(A)$ as the set of all elements m of M that satisfy $m\Gamma A\Gamma m = (0)$ for any subset A of a Γ-ring M and prove some of its set theoretical properties. For instance, we show that $S_M(A)$ is a semi-group ideal of M and a condition is obtained for $S_M(A)$ to be an ideal of M. Also, the definitions of $|S_M|$-strongly reduced Γ-ring, $|S_M|$-domain and $|S_M|$-division Γ-ring are given and obtained some results about their relations. We determine all possible characteristic values of a $|S_M|$-domain and prove every finite $|S_M|$-domain Γ-ring M is a $|S_M|$-division Γ-ring.

3. Main Results

Definition 3.1 *Let A be a subset of a Γ-ring M. We define the source of semi-primeness of A as the set $S_M(A) = \{m \in M \mid m\Gamma A\Gamma m = (0)\}$. We write S_M instead of $S_M(M)$, when $A = M$.*

From the definition of source of semi-primeness it is clear that $S_A = S_M(A) \cap A$ and $S_M(B) \subseteq S_M(A)$ for any $A \subseteq B$. One can easily show that the source of semiprimeness of a Γ-ring M is equal to zero if and only if M is a semi-prime Γ-ring. Another observation about the source of semiprimeness of a Γ-ring M is that if $S_M = M$, then the Jordan product $(m,n)_{\alpha m'\beta} := m\alpha m'\beta n + n\alpha m'\beta m$ for any elements $m, m', n \in M$ with $\alpha, \beta \in \Gamma$ is equal to zero. Conversely, if the Jordan product for any elements $m, m', n \in M$ with $\alpha, \beta \in \Gamma$ is equal to zero, then S_M may not be equal to M. Indeed, if $M = \left\{\begin{bmatrix} 2\overline{a} & \overline{b} \end{bmatrix} \middle| \overline{a}, \overline{b} \in \mathbb{Z}_{18}\right\}$ and $\Gamma = \left\{\begin{bmatrix} 0 \\ 3\overline{x} \end{bmatrix} \middle| \overline{x} \in \mathbb{Z}_{18}\right\}$, then the equation $(m,n)_{\alpha m'\beta} = 0$ holds for all $m, m', n \in M$ and $\alpha, \beta \in \Gamma$. But, it can be shown that S_M is not equal to M. However, if one assume that the Γ-ring M being 2-torsion free, then converse of the proposition is true. It is also clear that every element in S_M is nilpotent of index at most 3.

We now give the other set-theoretical properties of the source of semi-primeness of a subset for a Γ-ring M.

Proposition 3.2 *Let M_1 and M_2 be two Γ-rings. If A and B are nonempty subsets of M_1 and M_2, respectively, then $S_{M_1 \times M_2}(A \times B) = S_{M_1}(A) \times S_{M_2}(B)$.*

Proof If M_1 and M_2 are two Γ-rings, then $M_1 \times M_2$ is a $\Gamma \times \Gamma$-ring with the ternary multiplication

$$(a,b)(\alpha,\beta)(c,d) = (a\alpha c, b\beta d).$$

Let $(a,b) \in S_{M_1 \times M_2}(A \times B)$. Then, $(a,b)(\alpha,\beta)(x,y)(\gamma,\theta)(a,b) = (0,0)$ for all $(x,y) \in A \times B$ and $(\alpha,\beta),(\gamma,\theta) \in \Gamma \times \Gamma$. Therefore, we get $a\alpha x\gamma a = 0$ and $b\beta y\theta b = 0$ for all $x \in A$, $y \in B$, $\alpha,\beta,\gamma,\theta \in \Gamma$, $a \in M_1$ and $b \in M_2$. Hence, $(a,b) \in S_{M_1}(A) \times S_{M_2}(B)$. Similarly, one can show that $S_{M_1}(A) \times S_{M_2}(B) \subseteq S_{M_1 \times M_2}(A \times B)$. Thus, the equality is obtained. □

Proposition 3.3 *Let M be a Γ-ring and A be an ideal of M. Then, the followings hold:*

(i) The source of semi-primeness of A is a semi-group ideal of M. In particular, it is a multiplicatively closed subset of M.

(ii) If $S_M(A)\Gamma S_M(A) = (0)$, then $S_M(A)$ is an ideal of M.

Proof (i) Let $m \in S_M(A)$, $\alpha \in \Gamma$ and $x \in M$. Then, $(x\alpha m)\Gamma A\Gamma(x\alpha m) = (0)$ since $m\Gamma A\Gamma m = (0)$. It follows that $x\alpha m \in S_M(A)$. Similarly, we have $m\alpha x \in S_M(A)$. Therefore, $S_M(A)$ is a semi-group ideal of M. The last part of the proposition is obvious.

(ii) Let $S_M(A)\Gamma S_M(A) = (0)$. It is enough to show that $S_M(A)$ is additively closed. Let $x,y \in S_M(A)$. Then,

$$(x+y)\Gamma A\Gamma(x+y) = x\Gamma A\Gamma x + x\Gamma A\Gamma y + y\Gamma A\Gamma x + y\Gamma A\Gamma y \subseteq x\Gamma A\Gamma y + y\Gamma A\Gamma x.$$

Since $S_M(A)$ is a semi-group ideal, we have $A\Gamma x \subseteq S_M(A)$ and $x\Gamma A \subseteq S_M(A)$. Therefore, $x\Gamma A\Gamma y + y\Gamma A\Gamma x = (0)$. Thus, $x+y \in S_M(A)$, that is, $S_M(A)$ is an ideal of M. □

Proposition 3.4 *If Q is a semi-prime ideal of a Γ-ring M, then $S_M \subseteq Q$. Moreover, S_M is contained in the prime radical of M.*

Proof Let $a \in S_M$. Since Q is semi-prime and $a\Gamma M\Gamma a = (0) \subseteq Q$, we have $a \in Q$. Therefore, $S_M \subseteq Q$. This also shows that S_M is contained in the prime radical of M. □

Theorem 3.5 *Let M_1 be a Γ_1-ring and M_2 be a Γ_2-ring. If the ordered pair (θ,φ) is a gamma ring homomorphism, then $\varphi(S_{M_1})$ is contained in $S_{\varphi(M_1)}$. Moreover, if φ is injective, then $\varphi(S_{M_1}) = S_{\varphi(M_1)}$.*

Proof Since (θ,φ) is a gamma ring homomorphism, we have $\varphi(M_1)$ is a $\theta(\Gamma_1)$-ring with ternary multiplication

$$\varphi(a)\theta(\alpha)\varphi(b) = \varphi(a\alpha b).$$

Therefore, the source of semi-primeness of $\varphi(M_1)$ is

$$\{\varphi(a) \in \varphi(M_1) \mid \varphi(a)\theta(\Gamma_1)\varphi(M_1)\theta(\Gamma_1)\varphi(a) = (0)\}.$$

Now, it is obvious that the set $\varphi(S_{M_1})$ is contained in $S_{\varphi(M_1)}$. Conversely, let φ be injective and $\varphi(a) \in S_{\varphi(M_1)}$. Then, we have $\varphi(a\Gamma_1 M_1 \Gamma_1 a) = \varphi(0)$. Hence, $a \in S_{M_1}$ since φ is injective. This shows that $S_{\varphi(M_1)} \subseteq \varphi(S_{M_1})$. □

Theorem 3.6 *Let M be a Γ-ring and $a \in S_M$. If $M\Gamma a \neq (0)$ and $a\Gamma M \neq (0)$, then a is a zero divisor. Consequently, an element of M which is a not a zero divisor is contained in $M - S_M$.*

Proof By hypothesis, there exist $b, c \in M$ and $\alpha, \gamma \in \Gamma$ such that $a\alpha b \neq 0 \neq c\gamma a$. Therefore, we get a is a zero divisor since $a\alpha b\delta a = 0 = a\varepsilon c\gamma a$, $a\alpha b \neq 0$ and $c\gamma a \neq 0$. Now assume that b is not a zero divisor of M. Hence, $b \in M - S_M$ since $b\Gamma M \neq (0) \neq M\Gamma b$. Otherwise, b would be a zero divisor. □

4. $|S_M|$-strongly Reduced Γ-ring, $|S_M|$-domain Γ-ring, $|S_M|$-division Γ-ring

Definition 4.1 *Let M be a Γ-ring and $M \neq S_M$.*

(1) *M is said to be a $|S_M|$-strongly reduced ring if there are no strongly nilpotent elements of $M - S_M$.*

(2) *M is said to be a $|S_M|$-domain if there are no left or right zero divisors of $M - S_M$. A $|S_M|$-domain M is called $|S_M|$-integral domain if M is commutative with strong unity.*

(3) *M is said to be a $|S_M|$-division ring if M has a strong unity and every element of $M - S_M$ is unit. A $|S_M|$-division ring M is called $|S_M|$-field if M is commutative.*

It is necessary to assume $M \neq S_M$ in the above definition. For instance, if M is the set of all 2×3 matrices of the form $\begin{bmatrix} \overline{a} & 0 & \overline{a} \\ 0 & \overline{b} & 0 \end{bmatrix}$ with $\overline{a}, \overline{b} \in 4\mathbb{Z}_{16}$ and Γ is the set of all 3×2 matrices of the form $\begin{bmatrix} \overline{x} & 0 \\ 0 & \overline{x} \\ \overline{x} & 0 \end{bmatrix}$ with $\overline{x} \in 4\mathbb{Z}_{16}$, then M is a Γ-ring with $S_M = M$.

From the Definition 4.1, it is clear that if M is a strongly reduced Γ-ring (Γ-domain or Γ-division ring), then M is a $|S_M|$-strongly reduced ring ($|S_M|$-domain or $|S_M|$-division ring). Also, one can show that every $|S_M|$-domain is a $|S_M|$-strongly reduced ring. Conversely, $|S_M|$-strongly reduced rings are not a $|S_M|$-domain in general. For example, if $M = \left\{ \begin{bmatrix} a & 0 & b \\ 0 & c & 0 \end{bmatrix} | a, b, c \in \mathbb{Z} \right\}$ and $\Gamma = \left\{ \begin{bmatrix} 0 & 0 \\ 0 & x \\ x & 0 \end{bmatrix} | x \in \mathbb{Z} \right\}$, then M is a $|S_M|$-strongly reduced Γ-ring but not a $|S_M|$-domain. Similarly, a $|S_M|$-division ring M may not be a $|S_M|$-domain. Let $M = \left\{ \begin{bmatrix} \overline{a} & \overline{a} \end{bmatrix} | \overline{a} \in \mathbb{Z}_p \right\}$ for any prime p

and $\Gamma = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} | x \in \mathbb{Z} \right\}$. Then, one can show that M is a $|S_M|$-division Γ-ring, but not a $|S_M|$-domain. Another observation on the Definition 4.1 is that if M_1 is a $|S_{M_1}|$-domain and M_2 is a $|S_{M_2}|$-domain, then the direct product $M_1 \times M_2$ is $|S_{M_1} \times S_{M_2}|$-strongly reduced ring. It is easy to show that the prime radical of a $|S_M|$-strongly reduced Γ-ring M contains every strongly nilpotent element. By the very nature of the gamma ring, every division gamma ring is not a gamma domain. Similarly, every $|S_M|$-division Γ-ring is not a $|S_M|$-domain. For example, the $\Gamma = \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix} | x \in \mathbb{Z} \right\}$-ring $M = \left\{ \begin{bmatrix} \overline{a} & \overline{a} \end{bmatrix} | \overline{a} \in \mathbb{Z}_p \right\}$ is a $|S_M|$-division Γ-ring that is not a $|S_M|$-domain for any prime p.

Proposition 4.2 *Let M be a Γ-ring with $M \neq S_M$ and $a \in M$. Then the followings are equivalent:*

(i) M is a $|S_M|$-strongly reduced ring.

(ii) If $a\Gamma a \subseteq S_M$, then $a \in S_M$.

(iii) If $(a\Gamma)^n a \subseteq S_M$ for any positive integer n, then $a \in S_M$.

Proof (i)$\Rightarrow$(ii) Let M be a $|S_M|$-strongly reduced ring and $a\Gamma a \subseteq S_M$. Therefore, we have $(a\Gamma)^4 a = (0)$ that is a is a strongly nilpotent element. Hence, $a \in S_M$ since M is a $|S_M|$-strongly reduced ring.

(ii)$\Rightarrow$(iii) Let $a \in M$ and n be the smallest positive integer such that $(a\Gamma)^n a \subseteq S_M$. There exists a positive integer k such that $n \leq 2k \leq n+1$. By Proposition 3.3, we have $(a\Gamma)^{2k+1} a \subseteq S_M$, that is, $(a\Gamma)^k a \subseteq S_M$. If $k = 1$, then $a \in S_M$ by (ii). Assume that $k > 1$. But, this contradicts with n to be the smallest positive integer since $k \leq n - k + 1 < n$. Hence, n cannot exceed 2.

(iii)$\Rightarrow$(i) Assume that $a \in M$ is a strongly nilpotent element. Then, there exists a positive integer n such that $(a\Gamma)^n a = (0)$. By hypothesis, we get $a \in S_M$ since $(a\Gamma)^n a \subseteq S_M$. Therefore, there is no strongly nilpotent element in $M - S_M$. So, M is a $|S_M|$-strongly reduced ring. □

Corollary 4.3 *If M is a $|S_M|$-strongly reduced Γ-ring, then* $S_M = \left\{ a \in M \,\middle|\, (a\Gamma)^2 a = (0) \right\}$.

Proof Let $T = \left\{ a \in M \,\middle|\, (a\Gamma)^2 a = (0) \right\}$ and $a \in S_M$. Then, clearly $a \in T$. Conversely, assume that $a \in T$. Then, we have $(a\Gamma)^2 a = (0)$, that is, a is a strongly nilpotent element. It follows that $a \in S_M$ since M is a $|S_M|$-strongly reduced Γ-ring. Consequently, we get $S_M = T$. □

Proposition 4.4 *Let M be a Γ-ring. If M is a $|S_M|$-domain, then $S_M(A) = S_M$ for any nonzero Γ-subring A of M. Besides, A is a $|S_A|$-domain.*

Proof From the definition of source of semi-primeness, it is clear that $S_M \subseteq S_M(A)$. Assume that there exists an element $m \in S_M(A)$ such that $m \notin S_M$. Then, we get $m\Gamma A = (0) = A\Gamma m$ since $m\Gamma A\Gamma m = (0)$ and M is a $|S_M|$-domain. This implies $A = (0)$, which is a contradiction. Hence, $S_M(A) = S_M$. Now, let $a \in A$ be a zero-divisor. Therefore, $a \in S_M$ since M is a $|S_M|$-domain. This implies $a \in S_M(A) \cap A = S_A$. It follows that A is a $|S_A|$-domain. □

We should note that $S_M(A) = S_A$ may not be provided even if M is a $|S_M|$-domain Γ-ring. For the $\Gamma = \left\{\begin{bmatrix} x & 0 \\ 0 & x \\ x & 0 \end{bmatrix} | x \in \mathbb{Z}\right\}$-ring $M = \left\{\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \end{bmatrix} | a, b, c \in \mathbb{Z}\right\}$, one can show that the M is a $|S_M|$-domain and $S_M(A) \neq S_A$ for the subset $A = \left\{\begin{bmatrix} a & 0 & b \\ 0 & 0 & 0 \end{bmatrix} | a, b \in \mathbb{Z}\right\}$ of M.

Proposition 4.4 is not true for a $|S_M|$-strongly reduced Γ-ring M in general. For example, let $M = \left\{\begin{bmatrix} a & 0 & c \\ 0 & b & 0 \end{bmatrix} | a, b, c \in \mathbb{Z}\right\}$ and $\Gamma = \left\{\begin{bmatrix} x & 0 \\ 0 & x \\ 0 & 0 \end{bmatrix} | x \in \mathbb{Z}\right\}$. Then, M is $|S_M|$-strongly reduced Γ-ring since there is no strongly nilpotent element in the set

$$M - S_M = \left\{\begin{bmatrix} a & 0 & c \\ 0 & b & 0 \end{bmatrix} | a, b, c \in \mathbb{Z},\ a \neq 0 \text{ or } b \neq 0\right\}.$$

For the Γ-subring $A = \left\{\begin{bmatrix} a & 0 & c \\ 0 & 0 & 0 \end{bmatrix} | a, c \in \mathbb{Z}\right\}$ of M, we have $S_M(A) = \left\{\begin{bmatrix} 0 & 0 & c \\ 0 & b & 0 \end{bmatrix} | b, c \in \mathbb{Z}\right\}$. Therefore, it is clear that $S_M(A) \neq S_M$.

Proposition 4.5 *If M is a $|S_M|$-strongly reduced Γ-ring and A is a non-zero Γ-subring of M, then A is a $|S_A|$-strongly reduced Γ-ring.*

Proof Let M be a $|S_M|$-strongly reduced Γ-ring and A be a nonzero Γ-subring of M. If $a \in A$ is a strongly nilpotent element, then $a \in S_M$ by hypothesis. This implies that $a \in S_A$ since $S_M \subseteq S_M(A)$. Hence, A is a $|S_A|$-strongly reduced Γ-ring. □

Lemma 4.6 *If M is a $|S_M|$-domain Γ-ring, then $M - S_M$ is a multiplicative set.*

Proof Let M be a $|S_M|$-domain Γ-ring. Assume that $a\alpha b$ is a zero-divisor for $a, b \in M - S_M$ and $\alpha \in \Gamma$. Then, there exist nonzero elements $c \in M - S_M$ and $\gamma \in \Gamma$ such that $(a\alpha b)\gamma c = 0$. Hence, a or b must be zero-divisors which contradicts with our hypothesis. This implies $a\alpha b$ is not a zero divisor, that is, $a\alpha b \in M - S_M$ by Theorem 3.6. Therefore, $M - S_M$ is a multiplicative set. □

Theorem 4.7 *Every finite $|S_M|$-domain Γ-ring M is a $|S_M|$-division ring.*

Proof Assume that M is a $|S_M|$-domain Γ-ring. Let $T = M - S_M = \{a_1, \ldots, a_n\}$ and a be any element of T. Since T is a multiplicative set by Lemma 4.6 and a is not a left (or right) zero divisor, we define injective maps on T such that $f(x) = a\gamma x$ and $g(x) = x\gamma a$ for all $x \in T$. Then, finite cardinality requires the maps to be surjective. Therefore, there exist $1 \leq i \leq n$ and $1 \leq j \leq n$ such that $a\gamma a_i = a = a_j\gamma a$. Since $a\gamma a_i\gamma a = a\gamma a = a\gamma a_j\gamma a$, we get $a_i = a_j$ and so $a\gamma a_i = a = a_i\gamma a$. By the same argument, we have an element $a_i' \in T$ such that $b\gamma a_i' = b = a_i'\gamma b$ for $b \in T$. Accordingly, one has

$$(a\gamma b)\,\gamma a_i' = a\gamma b = a_i\gamma\,(a\gamma b)$$

and since $a\gamma b \in T$, it follows that $a_i' = a_i$. Set $e = a_i$ and $\delta = \gamma$. Then, (δ, e) is a strong unity of the semigroup T and clearly $e\delta e = e$.

For an arbitrary element x of M, we either have $x \in S_M$ or $x \in T$. If $x \in T$, then we already have that $x\delta e = e\delta x = x$. Let $x \in S_M$. Assuming $e - e\delta x \in S_M$ implies that $e = 0$. But, it is a contradiction because $e \in T$. Thus, $e - e\delta x \in T$ and similarly we have $e - x\delta e \in T$. Then,

$$(e - e\delta x)\,\delta e = e - e\delta x \qquad \text{and} \qquad e\delta\,(e - x\delta e) = e - x\delta e$$

yields us that $e\delta x = x\delta e$. Therefore, we have $x\delta e = x = e\delta x$ since e is not a zero-divisor.

Consequently, (δ, e) is a strong unity of Γ-ring M. Moreover, considering the maps f and g, there exist $x, y \in T$ such that $a\delta x = e = y\delta a$. This shows that a is a unit in M. Hence, M is a $|S_M|$-division ring. □

Corollary 4.8 *If M is a finite $|S_M|$-integral domain, then it is $|S_M|$-field.*

Theorem 4.9 *Let M be a Γ-ring with strong unity (δ, e). If M is a $|S_M|$-domain, then the characteristic of M is either 0, or p for a prime p, or p^2 for a prime p.*

Proof Assume that $\operatorname{char} M = n > 1$ and p is a prime dividing n. Then, there exists an integer k such that $n = pk$. Hence, $0 = ne = (pe)\,\delta\,(ke)$. This implies that pe is a zero-divisor, that is, $pe \in S_M$. Therefore, we have $(pe)\,\delta m\delta\,(pe) = 0$ for all $m \in M$. It follows that $p^2 m = 0$ for all $m \in M$. Accordingly, we get $n = p$ or $n = p^2$ since $\operatorname{char} M = n$. □

Theorem 4.10 *Let M be a Γ-ring with strong unity (δ, e). If M is a $|S_M|$-strongly reduced ring, then the characteristic of M is a cube-free integer, that is, there is no prime p such that p^3 divides $\operatorname{char} M$.*

Proof Assume that $\operatorname{char} M = n > 1$ and p is a prime dividing n, say $n = p^t k$ for some $t \geq 1$ and

$1 \le k < n$ with $gcd(p,k) = 1$. Since

$$(pke)^t = p^t k^t e = k^{t-1}(ne) = 0 \Rightarrow pke \in S_M$$

$$\Rightarrow (pke)\delta m\delta(pke) = 0, \forall m \in M \ \Rightarrow p^2k^2m = 0, \forall m \in M$$

and char$M = n$, there exits $s \in \mathbb{Z}$ such that $p^t ks = p^2k^2$. If t were greater than or equal to 3, then we get $p|k$. But, this contradicts with $gcd(p,k) = 1$. Hence, n must be a cube-free integer. □

Declaration of Ethical Standards

The authors declare that the materials and methods used in their study do not require ethical committee and/or legal special permission.

Authors Contributions

Author [Okan Arslan]: Thought and designed the research/problem, contributed to research method or evaluation of data, collected the data, wrote the manuscript (%70).

Author [Nurcan Düzkaya]: Collected the data, contributed to completing the research and solving the problem (%30).

Conflicts of Interest

The authors declare no conflict of interest.

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