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Fixed Points of Multivalued Mappings Useful in the Theory of Differential and Random Differential Inclusions

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Abstract

Fixed point theory is very useful in nonlinear analysis, diferential equations, differential and random differential inclusions. It is well known that different types of fixed points implies the existence of specific solutions of the respective problem concerning differential equations or inclusions. There are several classifications of fixed points for single valued mappings. Recall that in 1949 M.K. Fort [19] introduced the notion of essential fixed points. In 1965 F.E. Browder [12], [13] introduced the notions of ejective and repulsive fixed points. In 1965 A.N. Sharkovsky [31] provided another classification of fixed points but only for continous mappings of subsets of the Euclidean space $\mathbb{R}^n$. For more information see also: [15], [18]–[22], [3], [25], [27], [31]. Note that for multivalued mappings these problems were considered only in a few papers (see: [2]–[8], [14], [23], [24], [32]) – always for admissible multivalued mappings of absolute neighbourhood retracts (ANR-s). In this paper ejective, repulsive and essential fixed points for admissible multivalued mappings of absolute neighbourhood multi retracts (ANMR-s) are studied. Let as remark that the class of MANR-s is much larger as the class of ANR-s (see: [32]). In order to study the above notions we generalize the fixed point index from the case of ANR-s onto the case of ANMR-s. Next using the above fixed point index we are able to prove several new results concerning repulsive ejective and essential fixed points of admissible multivalued mappings. Moreover, the random case is mentioned. For possible applications to differential and random differential inclusions see: [1], [2], [8]–[11], [16], [25], [26].

Keywords: fixed point index essential ejective and repulsive fixed points multivalued mappings compact absorbing contractions absolute neighbourhood multi retracts differential inclusions random differential inclusions.

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1. Introduction

The aim of this paper is to develop of topological techniques for the solvalibity of differential inclusions and implicit differential inclusions. We mean also initial value problems and boundary problems both for ordinary and patrial differential inclusions. The main tool for the above investigsation is the appropriate topological fixed point theory for multivalued mappings, i.e. the fixed point index, the Lefschetz fixed point theorem in particular the Schauder fixed point theorem for a large class of multivalued mappings so called compact absorbing contractions mapped ANR-s or ANMR-s into itself (see: [23], [5], [16], [25], [32]). Important new result is also look into essential, repulsive and ejective fixed points of such mappings.

Our paper is organized as follows. After recalling some auxiliary defitions, the class of admissible multi valued mappings is presened. Next we consider the main class of mappings so called compact absorbing contractions (CAC-mappings) and its properties (for more details see also [4], [5], [23]). Most important part of this paper stand Section 4, where the fixed point index theory for CAC-mappings of ANRS is generalized to the case of CAC-maps of ANMR-s i (comp. also [4], [23], [32]). Then in Sections 5 and 6 are considered repulsive, ejective and essential fixed points of CAC-mappings. Using fixed point index defined in Section 4 some new existence theorems are proved (comp. [2]–[4], [6], [8], [11], [21]). In last section random CAC-mappings are considered. For such mappings the Lefschetz- type fixed point theorem is proved. Moreover, some open problems are formulated.

2. Some Auxiliary Definitions

In the entire text, all topological spaces are metric and all single-valued mappings are continuous. Let X be a metric space and let x be a point of X. By $U(x)$ we shall denote the family of all open neighbourhoods of x in X.

Let Top_2 be the category of pairs of topological spaces and continuous mappings of such pairs. By a pair (X, A) in Top_2, we understand a space X and its subset A; a pair $(X, \emptyset)$ will be denoted for short by X. By a map $f\colon (X, A) \to (Y, B)$, we shall understand a continuous map from X to Y such that $f(A) \subset B$.

We shall use the following notations: if $f\colon (X, A) \to (Y, B)$ is a map of pairs, then by $f_X\colon X \to Y$ and $f_A\colon A \to B$, we shall understand the respective induced mappings. Let us also denote by Vect_G the category of graded vector spaces over the field of rational numbers $\mathbb{Q}$ and linear maps of degree zero between such spaces. By $H\colon \mathrm{Top}_2 \to \mathrm{Vect}_G$, we shall denote the Čech homology functor with compact carriers and coefficients in $\mathbb{Q}$.

Thus, for any pair (X, A), we have $H(X, A) = \{H_q(X, A)\}_{q\geq 0}$, a graded vector space in Vect_G and, for any map $f\colon (X, A) \to (Y, B)$, we have the induced linear map $f_* = \{f_{*q}\}\colon H(X, A) \to H(Y, B)$, where $f_{*q}\colon H_q(X, A) \to H_q(Y, B)$ is a linear map from the q-dimensional homology $H_q(X, A)$ of the pair (X, A) into the q-dimensional homology $H_q(Y, B)$ of the pair (Y, B).

For the properties of H, we recommend [13].

A non-empty space X is called *acyclic* provided:

(i) $H_q(X) = 0$, for every $q \geq 1$, and
(ii) $H_0(X) = Q$.

Definition 2.1. *A map $p\colon \Gamma \to X$ is called a Vietoris map if the following conditions are safisfied:*

(i) *p is onto and proper, i.e. $p^{-1}(K)$ is compact for every compact $K \subset X$,*

(ii) *for every $x \in X$, the set $p^{-1}(x)$ is acyclic.*

Theorem 2.2 (Vietoris, see e.g. [23])**.** *If $p\colon \Gamma \to X$ is a Vietoris map, then the induced linear map $p_*\colon H(\Gamma) \xrightarrow{\sim} H(X)$ is an isomorphism, i.e. for every $q \geq 0$ the linear map $p_{*q}\colon H_q(\Gamma) \xrightarrow{\sim} H_q(X)$ is a linear isomorphism.*

For further properties of Vietoris mappings, see e.g. [23].

The following notions will play a crucial role. At first, by $\varphi\colon X \multimap Y$, we shall denote a multivalued map, i.e. a map which assigns to every point $x \in X$ a compact nonempty set $\varphi(x) \subset Y$.

A multivalued map $\varphi\colon X \multimap Y$is called *admissible* (see [23]) provided there exists a diagram

$$X \xleftarrow{p} \Gamma \xrightarrow{q} Y$$

in which p is a Vietoris map, such that $\varphi(x) = q(p^{-1}(x))$. The pair (p, q) is called a *selected pair* of φ (write $(p, q) \subset \varphi$). In what follows, we shall use the following notation:

$$\Gamma \overset{p}{\Longrightarrow} X$$

for Vietoris mappings.

Note that the superposition $\psi \circ \varphi\colon X \multimap Z$ of two admissible maps $\varphi\colon X \multimap Y$ and $\psi\colon Y \multimap Z$ is again an admissible map. It is easy to see that any admissible map is usc (upper semi continuous).

For a map $\varphi\colon X \multimap X$, we shall consider the set $\mathrm{Fix}(\varphi)$ of fixed points φ, i.e.,

$$\mathrm{Fix}(\varphi) := \{x \in X \mid x \in \varphi(x)\}.$$

More information about admissible mappings will be presented in the next section.

Recall that the space X is an *absolute neighbourhood retract* ($X \in \mathrm{ANR}$), provided there exists an open set U in a normed space E and two maps:

$$r\colon U \to X \quad \text{and} \quad s\colon X \to U$$

such that $r \circ s = \mathrm{id}_X$.

We shall also use the notion of a multiretraction.

Definition 2.3 ([32], [5])**.** *A map $r\colon Y \to X$ is said to be a multiretraction if there exists an admissible map $\varphi\colon X \multimap Y$ such that $r \circ \varphi = \mathrm{id}_X$.*

Definition 2.4 ([32])**.** *A space X is called an absolute neighbourhood multiretract ($X \in \mathrm{ANMR}$) if there exists an open set U of a normed space E and a multiretraction $r\colon U \to X$; if U is an arbitrary convex set then X is an absolute multi retract ($X \in \mathrm{AMR}$).*

Evidently, we have:

$$\mathrm{ANR} \subset \mathrm{ANMR},$$

i.e. that the class of ANMR-spaces is obviously larger than the one of ANR- spaces (see [5] and [23]).

For some nontrivial examples and more details concerning ANMR-spaces, we recommend [32]. Note that any open subset of ANMR (ANR) is ANMR (ANR) too.

3. Compact Absorbing Contraction Mappings

Let $\varphi\colon X \multimap Y$ be an admissible mapping and $(p, q) \subset \varphi$ be a selected pair of φ.

Using the Vietoris Theorem 2.2, we are able to define the induced by (p, q) linear map by putting:

$$q_* \circ p_*^{-1}\colon H_*(X) \to H_*(Y).$$

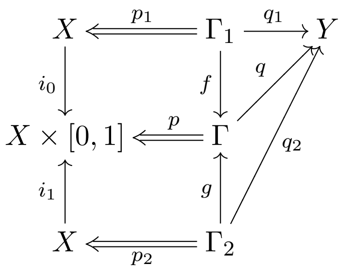
We let: $\varphi_* = \{q_* \circ p_*^{-1} \mid (p, q) \subset \varphi\}$.

Now, let us consider two admissible mappings $\varphi, \psi\colon X \multimap Y$. We shall say that φ is *homotopic* to ψ (written: $\varphi \sim \psi$), provided there exists an admissible mapping $\chi\colon X \times [0, 1] \multimap Y$ such that $\chi(x, 0) = \varphi(x)$ and $\chi(x, 1) = \psi(x)$, for every $x \in X$.

We have the following proposition (for its proof, see [23]):

Proposition 3.1. *If $\varphi \sim \psi$, then $\varphi_* \cap \psi_* \neq \emptyset$.*

Let $(p_1, q_1) \subset \varphi$ and $(p_2, q_2) \subset \psi$. We shall say that the above selected pairs are *homotopic* (written $(p_1, q_1) \sim (p_2, q_2)$), provided there exists the following commutative diagram:



where $i_0(x) = (x, 0)$, $i_1(x) = (x, 1)$, Γ is a given space and f, g are also given.

Evidently, we have:

Proposition 3.2. *If $(p_1, q_1) \sim (p_2, q_2)$, then $q_{1*} \circ p_{1*}^{-1} = q_{2*} p_{2*}^{-1}$.*

We say that an admissible map $\varphi \colon X \multimap X$ is a Lefschetz map provided, for every selected pair $(p, q) \subset \varphi$, the generalized Lefschetz number $\Lambda(p, q) = \Lambda(q_* \circ p_*^{-1})$ is well defined (for details, see [23]).

For a Lefschetz map $\varphi \colon X \multimap X$, we define the *Lefschetz set* $\Lambda(\varphi)$ of φ by putting:

$$\Lambda(\varphi) = \{\Lambda(p, q) \mid (p, q) \subset \varphi\}.$$

We have (see [23]):

(a) If $\varphi \sim \psi$, then $\Lambda(\varphi) \cap \Lambda(\psi) \neq \emptyset$.

(b) If $(p_1, q_1) \sim (p_2, q_2)$, then $\Lambda((p_1, q_1)) = \Lambda((p_2, q_2))$.

Definition 3.3 ([5], [23]). *An admissible map $\varphi \colon X \multimap X$ is called a compact absorbing contraction ($\varphi \in \mathrm{CAC}(X)$) if there exists an open set $U \subset X$ such that:*

(i) *$\varphi(U) \subset U$,*

(ii) *the closure $\overline{\varphi(U)}$ of $\varphi(U)$ is contained in a compact subset of U,*

(iii) *for every $x \in X$, there exists a natural number n_x such that $\varphi^{n_x}(x) \subset U$.*

We say that $\varphi \colon X \multimap X$ is a *locally compact map* provided, for every $x \in X$, there exists $V \in U(x)$ such that $\varphi|_V \colon V \multimap X$ is a compact map, i.e. $\overline{\varphi|_V(V)}$ is compact.

We let:

$\mathrm{K}(X) = \{\varphi \colon X \multimap X \mid \varphi$ is admissible and compact$\}$.

$\mathrm{EC}(X) = \{\varphi \colon X \multimap X \mid \varphi$ is admissible locally compact and there exists a natural number n such that the n-th iteration $\varphi^n \colon X \multimap X$ of φ is a compact map$\}$.

$\mathrm{ASC}(X) = \Big\{\varphi \colon X \multimap X \;\Big|\; \varphi$ is admissible locally compact, the orbit $O(x) = \bigcup_{n=1}^{\infty} \varphi^n(x)$ is, for every $x \in X$, relatively compact and the core $C(\varphi) = \bigcap_{n=1}^{\infty} \varphi^n(x)$ is nonempty and relatively compact$\Big\}$.

$\mathrm{CA}(X) = \{\varphi \colon X \multimap X \mid \varphi$ is admissible locally compact and has a compact attractor, i.e., then exists a compact set $A \subset X$ such that, for every open set $W \subset X$ containing A and for every point $x \in X$, there is n_x such that $\varphi^{n_x}(x) \subset W\}$.

The following hierarchy holds ([23]):

$$\mathrm{K}(X) \subset \mathrm{EC}(X) \subset \mathrm{ASC}(X) \subset \mathrm{CA}(X) \subset \mathrm{CAC}(X). \tag{1}$$

Moreover, each of the above inclusions is proper.

Let $\varphi \in \mathrm{CAC}(X)$ and let U be chosen according to Definition 3.3. Then

$$\begin{aligned}&\varphi_U\colon U \multimap U, \text{ defined by the formula } \varphi_U(x) = \varphi(x),\\ &\text{for every } x \in U, \text{ is a compact admissible map.}\end{aligned} \tag{2}$$

Recall that if $\psi\colon Y \multimap Y$ is a compact admissible map and $Y \in \mathrm{ANMR}$, then ψ is a Lefschetz map and $\Lambda(\psi) \neq \{\emptyset\}$ implies that ψ has a fixed point (see [23]).

We prove the following theorem.

Theorem 3.4. *Let $\varphi \in \mathrm{CAC}(X)$, where $X \in \mathrm{ANMR}$. Assume further that U is chosen according to Definition 3.3 and $\varphi_U\colon U \multimap U$ be a map defined in (2). Then φ is a Lefschetz map and*

$$\Lambda(\varphi) \subset \Lambda(\varphi_U).$$

Proof. Let (p, q) be a selected pair of φ, i.e., we have a diagram:

$$X \overset{p}{\Longleftarrow} \Gamma \overset{q}{\longrightarrow} Y$$

such that $\varphi(x) = q(p^{-1}(x))$, for every $x \in X$. Consider still the following diagram:

$$U \overset{p_1}{\Longleftarrow} p^{-1}(U) \overset{q_1}{\longrightarrow} U$$

in which p_1 and q_1 are respective contractions of p and q.

We have also the following diagram:

$$(X, U) \overset{\overline{p}}{\Longleftarrow} (\Gamma, p^{-1}(U)) \overset{\overline{q}}{\longrightarrow} (X, U)$$

in which $\overline{p}(y) = p(y)$ and $\overline{q}(y) = q(y)$, for every $y \in \Gamma$.

Now, we shall use the following formula proved in [23]. If two Lefschetz numbers from the following three numbers $\Lambda(\overline{p}, \overline{q})$, $\Lambda(p, q)$ and $\Lambda(p_1, q_1)$ are well defined, then the third one is well defined too, and we have:

$$\Lambda(p, q) = \Lambda(\overline{p}, \overline{q}) + \Lambda(p_1, q_1).$$

Since an open subset of an ANMR-space is an ANMR-space, too, we infer from above that $\Lambda(p_1, q_1)$ is well defined.

Now, since we consider the homology with compact carriers from (b), it follows that $\Lambda(\overline{p}, \overline{q}) = 0$. Consequently, we get that $\Lambda(p, q)$ is well defined, and

$$\Lambda(p, q) = \Lambda(p_1, q_1).$$

The proof is completed. □

Corollary 3.5. *If $\varphi \in \mathrm{CAC}(X)$ and $X \in \mathrm{ANMR}$, then φ is a Lefschetz map and $\Lambda(\varphi) \neq \{\emptyset\}$ implies that φ has a fixed point.*

4. The Fixed Point Index

Firstly, let us assume that $\varphi\colon X \multimap X$ is a compact admissible map, where $X \in \mathrm{ANR}$.

Let $(p, q) \subset \varphi$ and $V \subset X$ be an open set such that $\{x \in V \mid x \in \varphi(x)\}$ is compact. Then the fixed point index $\mathrm{ind}((p, q), V)$ of the pair (p, q) with respect to V is well defined (see [23] and also [32]). Note that $\mathrm{ind}((p, q), V)$ is an integer.

We define the fixed point index of φ as the following set:

$$\mathrm{Ind}(\varphi, V) = \{\mathrm{ind}((p, q), V) \mid (p, q) \subset \varphi\}. \tag{3}$$

Below, we shall list the important properties of the fixed index which we shall need in the next section.

(a) (Existence). If $\mathrm{ind}((p,q),V) \neq 0$ ($\mathrm{Ind}((\varphi,V) \neq \{0\}$), then

$$\mathrm{Fix}(p,q) \cap V \neq \emptyset.$$

(b) (Excision). If $\mathrm{Fix}(\varphi) \cap W \subset V \subset W$ is compact, then

$$\mathrm{ind}((p,q),V) = \mathrm{ind}((p,q),W) \qquad (\mathrm{Ind}(\varphi,V) = \mathrm{Ind}(\varphi,W)).$$

(c) (Additivity). If V_1, V_2 are open subsets of X such that $V_1 \cap V_2 = \emptyset$ and $\mathrm{Fix}(\varphi) \cap V_1$, $\mathrm{Fix}(\varphi) \cap V_2$ are compact sets, then

$$\mathrm{ind}((p,q),V_1 \cup V_2) = \mathrm{ind}((p,q),V_1) + \mathrm{ind}((p,q),V_2).$$

(d) If $(p_1,q_1) \sim (p_2,q_2)$ ($\varphi \sim \psi$), then

$$\mathrm{ind}((p_1,q_1),V) = \mathrm{ind}((p_2,q_2),V) \qquad (\mathrm{Ind}(\varphi,V) \cap \mathrm{Ind}(\psi,V) \neq \emptyset),$$

where $(p_1,q_1) \subset \varphi$ and $(p_2,q_2) \subset \psi$.

(e) (Normalization). If $V = X$, then

$$\mathrm{ind}((p,q),V) = \Lambda((p,q)) \quad \text{and} \quad \Lambda(\varphi) = \mathrm{Ind}(\varphi,V).$$

Now, we shall consider the noncompact case. Assume that $\varphi\colon X \multimap X$ is an admissible compact absorbing contraction and $X \in \mathrm{ANR}$. Assume, furthermore, that V is an open set such that $\{x \in V \mid x \in \varphi(x)\}$ is compact. According to the Definition 3.3, we select an open set U satisfying all assumptions of (i)–(iii). Evidently, $\mathrm{Fix}(\varphi) \subset U$. Moreover, we have that $\varphi_U\colon U \multimap U$ is a compact admissible map, where $\varphi_U(x) = \varphi(x)$, for every $x \in U$. Let $(p,q) \subset \varphi$. Then $(p_U,q_U) \subset \varphi_U$, where $p_U\colon p^{-1}(U) \Rightarrow U$ and $q_U\colon p^{-1}(U) \to U$ are defined as follows: $p_U(y) = p(y)$ and $q_U(y) = q(y)$, for every $y \in p^{-1}(U)$.

We let:

$$\mathrm{ind}((p,q),V) = \mathrm{ind}((p_U,q_U),V \cap U) \tag{4}$$

and

$$\mathrm{Ind}(\varphi,V) = \{\mathrm{ind}((p,q),V) \mid (p,q) \subset \varphi\}. \tag{5}$$

By means of (c), we deduce that the definitions (4) and (5) do not depend on the choice of U. Thus, all properties (a)–(e) are satisfied.

For more details, we recommend [5], [7], [14], [23].

Now we shall generalize the above presented fixed point index to the case of CAC-mappings of ANMR-space X into itself.

Let $X \in \mathrm{ANMR}$. In what follows we fix an open subset of a normed space X and multiretractions

$$r\colon U \to X \quad \text{and} \quad \rho\colon X \multimap U$$

given according to the Definition 2.4. Then we have $\rho \circ r = \mathrm{id}_X$.

Now assume that $\varphi\colon X \multimap X$ is compact admissible map. Then we associate with φ the map $\widetilde{\varphi}\colon U \multimap U$ defined by the following formula

$$\widetilde{\varphi} = \rho \circ \varphi \circ r. \tag{6}$$

Since φ is compact admissible map hence $\widetilde{\varphi}$ is also compact and admissible map. Consequently $\mathrm{Fix}(\varphi)$ and $\mathrm{Fix}(\widetilde{\varphi})$ are compact (possible empty) sets. We will proof the following:

Lemma 4.1. $\mathrm{Fix}(\widetilde{\varphi}) = r^{-1}(\mathrm{Fix}(\varphi))$.

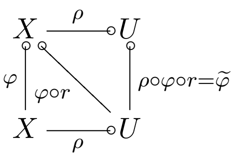
Proof. For the proof we shall show that $u \in \mathrm{Fix}(\widetilde{\varphi}))$ if and only if $r(u) \in \mathrm{Fix}(\varphi)$. If $u \in \mathrm{Fix}(\widetilde{\varphi})$, then we obtain $u \in \rho(\varphi(r(u)))$ and hence we have

$$r(u) \in r(\rho(\varphi(r(u)))) = \varphi(r(u)).$$

If $r(u) \in \mathrm{Fix}(\varphi)$ then we have $r(u) \in \varphi(r(u))$ and hence we obtain $\varphi(r(u)) \in \rho(\varphi(r(u)))$. It means that $u \in \rho(\varphi(r(u)))$ and the proof is completed. □

Observe that if A is a compact subset of $\mathrm{Fix}(\varphi)$, then $r^{-1}(A)$ is a compact subset of $\mathrm{Fix}(\widetilde{\varphi})$.

We have the following diagram of admissible mappings in which the vertical mappings are compact:



It is well known that φ and $\widetilde{\varphi}$ are Lefschetz maps and $\Lambda(\varphi) = \Lambda(\widetilde{\varphi})$ (see [4], [23], [32]).

We need the following lemma:

Lemma 4.2 ([23], see also [32]). *Let us consider the following diagram of admissible mappings:*

$$X \xrightarrow{\varphi_1} X_1 \xrightarrow{\varphi_2} X_2 \xrightarrow{\varphi_2} X_3$$

If $(p_1, q_1) \subset \varphi_1$, $(p, q) \subset \varphi$ and $(p_1, q_2) \subset \varphi_2$ and $\Psi = \varphi_2 \circ \varphi \circ \varphi_1$ the there exists a selected pair $(\widetilde{p}, \widetilde{q}) \subset \Psi$ such that

$$\widetilde{q}_* \circ \widetilde{l}_*^{-1} = q_{2*} \circ p_{2*}^{-1} \circ q_* \circ p_*^{-1} \circ q_{1*} \circ q_{1*}^{-1}.$$

Moreover, if for example $\varphi_1 = r$ is a continuous single valued map then $q_{1*} \circ p_{1*}^{-1} = r_*$.

Now we return to the notations used in the above diagram. Let $(p, q) \subset \varphi$, $(p_1, q_1) \subset r$ and $(p_1, q_2) \subset \rho$. Using the above lemma we get the selection pair $(\widetilde{p}, \widetilde{q}) \subset \widetilde{\varphi}$ such that:

$$\widetilde{q}_* \circ \widetilde{p}_*^{-1} = q_{2*}^{-1} \circ p_{2*}^{-1} \circ q_* \circ p_*^{-1} \circ r_*. \tag{7}$$

In what follows $(\widetilde{p}, \widetilde{q})$ we shall call the associated the pair with (p, q).

Observe that $(\widetilde{p}, \widetilde{q})$ depends also on the choice of $(p_2, q_2) \subset \rho$. In what follows for the simplicity we shall fix a selected pair (p_2, q_2) of ρ.

Remark 4.3. *The commutativity of the above diagram and* (7) *implies that the Lefschetz number $\Lambda(p, q)$ and $\Lambda(\widetilde{p}, \widetilde{q})$ are equal. We deduce it by applying homology function* (*comp.* [23], [4] *and* [24]).

Now assume that $X \in$ ANMR and $\varphi \colon X \multimap X$ is a compact admissible map. Assume that V is an open subset of X such that $\mathrm{Fix}(\varphi) \cap V$ is compact. Then, for every (p, q) we define the fixed point index $\mathrm{ind}((p, q), V)$ by putting

$$\mathrm{ind}((p, q), V) = \mathrm{ind}((\widetilde{p}, \widetilde{q}), r^{-1}(V)) \tag{8}$$

and

$$\mathrm{Ind}(\varphi, V) = \{\mathrm{ind}((p, q), V), (p, q) \subset \varphi\}. \tag{9}$$

It is evident that properties (a)–(e) can be formulated in the case of ANMR-retracts.

Finally, let us assume that $\varphi \colon X \multimap X$ is a CAC-map. Assume further that $\mathrm{Fix}(\varphi) \cap V$ is compact set and $V \subset X$ is open. According to the Definition 2.4 there exists an open set $W \subset X$ satisfying all assumptions of Definition 2.4. Ewidently $\mathrm{Fix}(\varphi) \subset W$. Moreover, we have compact admissible map $\varphi_W \colon \multimap W$. Then, for every $(p, q) \subset \varphi$, we define $(p_W, q_W) \subset \varphi_W$ by the formula:

$$\begin{aligned} &p_W \colon p^{-1}(W) \to W \quad \text{and} \quad q_W \colon p^{-1}(W) \to W,\\ &p_W(y) = p(y) \qquad\quad\ \text{and} \quad q_W(y) = q(y), \end{aligned}$$

for every $y \in p^{-1}(W)$.

We let

$$\mathrm{Ind}((p,q),W) = \mathrm{ind}((p_W, q_W), U \cap W) \tag{10}$$

and

$$\mathrm{ind}(\varphi, K) = \mathrm{ind}(\varphi_U, U \cap W). \tag{11}$$

Not that all properties (a)–(e) can be formulated for compact absorbing contraction mappings of ANMR-s spaces. We left it to the reader.

We recomend to compare the following: [4], [5], [1], [23], [21], [25], [32].

5. Ejective Fixed Points

In this section we shall assume that $X \in \mathrm{ANMR}$ and $\varphi\colon X \multimap X$ is a CAC-map.

Definition 5.1 ([8], [1])**.** *Let $\varphi\colon X \multimap X$ be a given map and let $x_0 \in \mathrm{Fix}(\varphi)$.*

(i) We shall say that x_0 is an ejective fixed point *provided then exists an open neighbourhood V of x_0 in X such that, for any $x \in \overline{V} \setminus \{x_0\}$, there is an integer $n \geq 1$ such that $\varphi^n(x) \subset X \setminus \overline{V}$.*

(ii) A fixed point $x_0 \in \mathrm{Fix}(\varphi)$ is called a repulsive fixed point *provided there exists am open neighbourhood V of x_0 in X such that for any open neighbourhood W of x_0 in X there is an integer $n(W) \geq 1$ such that $\varphi^n(X \setminus W) \subset X \setminus \overline{V}$ for all $n \geq n(W)$.*

We let

$$\mathrm{Fix}_e(\varphi) = \{x \in \mathrm{Fix}(\varphi);\, x \text{ is ejective}\}, \qquad \mathrm{Fix}_r(\varphi) = \{x \in \mathrm{Fix}(\varphi);\, x \text{ is repulsive}\}.$$

As an immediate consequence of the above definitions we have:

$$\mathrm{Fix}_r(\varphi) \subset \mathrm{Fix}_e(\varphi).$$

Remark 5.2. *(i) The following example shows that $\mathrm{Fix}_r(\varphi) \neq \mathrm{Fix}_e(\varphi)$. Let $f\colon [0,1] \to [0,1]$ be defined as $f(x) = 2(-x^2 + x)$. Then $x_0 = 0$ is an ejective fixed point but not repulsive.*

(ii) Observe that any ejective fixed point is isolated in $\mathrm{Fix}(\varphi)$. Therefore, if $\#\mathrm{Fix}(\varphi) < \infty$ then $\mathrm{Fix}_e(\varphi)$ is open and compact in $\mathrm{Fix}(\varphi)$.

Since $\varphi\colon X \multimap X$ is a CAC-mapping, according to the Definition 3.3, we have an open subset $U \subset X$ and a compact admissible map $\varphi_U\colon U \multimap U$ defined by the formula: $\varphi_U(x) = \varphi(x)$, for every $x \in U$ such that:

$$\begin{cases} \mathrm{Fix}(\varphi) = \mathrm{Fix}(\varphi_U), \\ \mathrm{Fix}_e(\varphi) = \mathrm{Fix}_e(\varphi_U), \\ \mathrm{Fix}_r(\varphi) = \mathrm{Fix}_r(\varphi_U). \end{cases} \tag{12}$$

Consequently, from properties of the fixed point index we can deduce the same results for compact admissible mappings on ANMR-s and for CAC-mappings.

Now let shall formulate existence results concerning ejective and repulsive fixed points of CAC-mappings.

Theorem 5.3. *Let $X \in \mathrm{ANMR}$ and $\varphi\colon X \multimap X$ be a CAC-map. Assume further that $x_0 \in X$ is a repulsive fixed point of φ with respect to V. If there exists an open neighbourhood W of x_0 such that:*

(i) $\overline{V} \subset W$,

(ii) the inclusion map $i\colon X \setminus W \to X$ induces the isomorphism $i_*\colon H_*(X \setminus W) \xrightarrow{\sim} H_*(X)$,

then $\mathrm{ind}(\varphi, V) = \{0\}$.

Corollary 5.4. *If we assume additionaly that $\mathrm{Fix}(\varphi)$ is a finite set and $\lambda(\varphi) \neq \{0\}$, the there exists a non-repulsive fixed point of φ.*

For some possible applications of ejective fixed points see [4], [3], [5] and references there in. Topological study of ejective fixed points is presented in [13], [14], [17], [18], [21], [22], [20], [30], [31].

6. Essential Fixed Points of CAC-Maps

Unttil end of this section we shall assume that $X \in$ ANMR and $\varphi\colon X \multimap X$ is a CAC-mapping.

Definition 6.1. *Let $x_0 \in \mathrm{Fix}(\varphi)$. We shall say that is an* essential fixed point of φ *if, for every open $U \subset X$, there exists an open subset $V \subset U$ such that:*

(i) $x_0 \in V$,

(ii) $\partial V \cap \mathrm{Fix}(\varphi) = \emptyset$,

(iii) $\mathrm{Ind}(\varphi, V) \neq \{0\}$.

We let $\mathrm{Ess}(\varphi) = \{x \in \mathrm{Fix}(\varphi);\ x \text{ is essential}\}$.

Theorem 6.2. *Let $\varphi\colon X \multimap X$ be a* CAC*-map. Assume further that* $\mathrm{Fix}(\varphi) \neq \emptyset$ *and the topological dimension* $\dim \mathrm{Fix}(\varphi)$ *is equal* 0*. Then* $\mathrm{Ess}(\varphi) \neq \emptyset$.

Proof. In the case when φ is compact and $X \in$ ANR the above theorem was proved in [8] (see also [11]). Using exactly the same arguments we obtain this theorem for component φ on $X \in$ ANMR. Assume that $\varphi \in$ CAC. According to the Definition 3.3 there exists an open set $V \subset X$ such that the map $\widetilde{\varphi}\colon V \multimap V$, $\widetilde{\varphi}(x) = \varphi(x)$ for every x is a compact admissible map and $\mathrm{Fix}(\varphi) = \mathrm{Fix}(\widetilde{\varphi})$. Since $\mathrm{Ess}(\widetilde{\varphi}) \neq \emptyset$ the proof is completed. □

Now we are going to present two theorems which are very important in the theory of implicit differential inclusions and equations (comp. [8], [11]).

In what follows we shall assume that $\varphi\colon A \times X \multimap X$ is an usc map such that, for every $a \in A$, the map $\varphi(a, \cdot)\colon X \multimap X$, $\varphi(a, \cdot)(x) = \varphi(a, x)$ is a CAC-map and A is a metric locally arc connected space. Assume further that $\Lambda(\varphi(a, \cdot)) \neq \{0\}$ for every $a \in A$. So we can define the following map

$$\widetilde{\varphi}\colon A \multimap X, \qquad \widetilde{\varphi}(a) = \mathrm{Fix}(\varphi(a, \cdot))$$

and, if we assume that $\dim \mathrm{Fix}(\varphi, a)) \neq \emptyset$ for every $a \in A$, then in view of Theorem 6.2 we can define the map

$$\widehat{\varphi}\colon A \multimap X, \qquad \widehat{\varphi}(a) = \mathrm{Ess}(\varphi(a, \cdot))$$

for every $a \in A$.

Theorem 6.3. *We have:*

(i) $\widetilde{\varphi}\colon A \multimap X$ *is usc with compact values,*

(ii) $\widehat{\varphi}\colon A \multimap X$ *is a lsc map.*

Evidently Theorems 6.2 and 6.3 are automatically true for $X \in$ AMR.

For more applications of essential fixed points see [5]–[11], [24]–[26], [29].

7. Random Case

A systematic study of random operators was iniciated in 1950 by Czech mathematicians. We can do it for random CAC-mappings (RCAC-mappings). Random admissible operators were studied in [5], [2], [23], [24].

By a measure space we shall mean the pair (Ω, Σ) where wthe set Ω is equipped in σ-algebra Σ of subsets. We shall use $B(X)$ to denote the Borel σ-algebra on X. The symbol $\Sigma \otimes B(X)$ denotes the smallest σ-algebra on $\Omega \times X$ which contains the sets $A \times B$ where $A \in \Sigma$ and $B \in B(X)$.

Definition 7.1. *Let* (Ω, Σ) *be a measurable space and* X *a metric space. A map* $\varphi\colon \Omega \multimap Y$ *is called* measurable *if*

$$\varphi^{-1}(B) = \{\omega \in \Omega;\, \varphi(\omega) \subset B\} \in \Sigma$$

for each open $B \subset X$.

Definition 7.2. *A map* $\varphi\colon \Omega \times X \multimap X$ *is called a random operator provided:*

(i) φ *is measurable (on whole space),*

(ii) the map $\varphi(\omega, \cdot)\colon X \multimap X$ *is usc with closed values for every* $\omega \in \Omega$.

Definition 7.3. *Let* $\varphi\colon \Omega \times X \multimap X$ *be a random map. A measurable mapping* $\xi\colon \Omega \to X$ *is called a* random fixed point of φ *provided:*

$$\xi(\omega) \in \varphi(\Omega, \xi(\omega)), \quad \textit{for each } \omega \in \Omega.$$

The following proposition is crucial in what follows.

Proposition 7.4. *Let* $\varphi\colon \Omega \times X \multimap X$ *be a random map and* X *be a separable space. If*

$$\mathrm{Fix}(\varphi(\omega, \cdot)) \neq \emptyset \quad \textit{for every } \omega \in \Omega,$$

then φ *has a random fixed point.*

For the proof see [5], [2], [23], [24].

As an immediate consequence of Proposition 7.4 and the Lefschetz fixed point theorem for CAC-mappings we obtain (see also [32]):

Corollary 7.5. *Let* $\varphi\colon \Omega \times X \multimap X$ *be a random operator where* $X \in$ ANR *is a separable space. Assume further that* $\varphi(\omega, \cdot)\colon X \multimap X$ *is a* CAC*-map such that* $\Lambda(\varphi(\omega, \cdot)) \neq \{0\}$ *for every* $\omega \in \Omega$. *Then* φ *has a random fixed point.*

Evidently Corollary 7.5 is true for separable $X \in$ ANR or AMR or AR.

A random map $\varphi\colon \Omega \times X \multimap X$ such that, for every $\omega \in \Omega$ the map $\varphi(\omega, \cdot)$ is a CAC-mapping, we shall call random CAC-mapping.

Open problems

How to define the notion of repulsive, ejective and essential random fixed points of random CAC-maps of ANMR (ANR) into itself or in particular, for random compact admissible operators of ANMR or ANR into itself.

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