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# PULLBACK CAT-1 RACKS

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Abstract. In this paper we introduce the notion of pullbak cat-1 racks, which is obtained by a certain kind of pullback diagram in the category of racks.

## 1. Introduction

A rack [4] is a set $R$ with a non-associative binary operation satisfying certain two rack conditions. The theory of racks is strongly connected to the theory of groups in the sense of conjugation. This relation leads to the adjoint functors between the category of racks and the category of groups [5, 6].

The earliest work on racks is due to Conway and Wraith [4], which inspired by the conjugacy operation in a group and focuses in the special case of racks, called quandles; but they also were aware of the generalization. In the history, racks are also called as "automorphic sets" [2], "crystals" [7] and "(left) distributive quasigroups" [9].

Loday [8] introduced the notion of strict 2-groups which can be seen as a reformulation of cat-1 groups. There exists an equivalence between the category of cat-1 groups and the category of crossed modules of groups [10]. Pullback cat-1 groups are introduced by Alp in [1]. In the same paper, he also discovered some functorial relations between pullback cat-1 groups and pullback crossed modules which is defined by Brown in [3].

In this paper we examine the cat-1 rack [5] case. On other words, we introduce the pullbak cat-1 racks, which are obtained by a certain kind of pullback diagrams in the category of racks.

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## 2. Preliminaries

In this section, we recall some notions from [5, 6].

### 2.1. Racks.

**Definition 2.1.** A (right) rack $R$ is a set equipped with a binary operation with the following conditions:

**i**) There exists a unique $c \in R$ such that:

$$c \lhd a = b$$

for all $a, b \in R$.

**ii**) For all $a, b, c \in R$, we have:

$$(a \lhd b) \lhd c = (a \lhd c) \lhd (b \lhd c)\,.$$

A pointed rack is a rack $R$ with an element $1 \in R$ such that:

$$1 \lhd a = 1 \qquad \text{and} \qquad a \lhd 1 = a$$

for all $a \in R$.

Let $R, S$ be two (pointed) racks. A rack homomorphism is a map $f \colon R \to S$ such that:

$$f\,(a \lhd b) = f\,(a) \lhd f\,(b) \qquad\quad (\text{and } f\,(1) = 1)$$

for all $a, b \in R$.

Therefore we have **Rack**, the category of racks.

Some well-known examples of racks are the followings:

**1)** Given a group $G$, we may define rack structure on it by:

$$g \lhd h = h^{-1}gh$$

for all $g, h \in G$. This rack is called the conjugation rack of $G$ which provides the functor between the category of groups and the category of racks as:

$$\mathbf{Conj} : \mathbf{Grp} \to \mathbf{Rack}$$

**2)** Another rack structure in a group $G$, called the core rack defined by:

$$g \lhd h = hg^{-1}h$$

for all $g, h \in G$; however this construction is not functorial.

**3** Let $P, R$ be two racks, then the set $P \times R = \{(p, r) \mid p \in P, r \in R\}$ defines a rack with:

$$(p, r) \lhd (p', r') = (p \lhd p', r \lhd r')\,.$$

**Definition 2.2.** Let $R, S$ be two racks. The map $\cdot \colon S \times R \to S$ is called a (right) rack action of $R$ on $S$ if it satisfies:

**i**) $(s \cdot r) \cdot r' = (s \cdot r') \cdot (r \lhd r')$

**ii**) $(s \lhd s') \cdot r = (s \cdot r) \lhd (s' \cdot r)$

for all $s, s' \in S$ and $r, r' \in R$.

**Definition 2.3.** If there exists a (right) rack action of $S$ on $R$, the hemi-semi-direct product $S \ltimes R \subset S \times R$ is the rack defined by the rack operation:

$$(s, r) \lhd (s', r') = (s \lhd s', r \cdot s')$$

for all $s, s' \in S$ and $r, r' \in R$.

**Definition 2.4.** A crossed module of racks is given by a rack homomorphism $\partial : R \to S$ together with an (right) rack action of $S$ on $R$ such that;

**i**) $\partial$ is equivariant, i.e.

$$\partial\,(r \cdot s) = \partial\,(r) \lhd s,$$

**ii**) Peiffer identity is satisfied, i.e.

$$r \cdot \partial\,(r') = r \lhd r'$$

for all $r, r' \in R$ and $s \in S$.

2.2. **Cat-1 Racks.**

**Definition 2.5.** A cat-1 rack $C = (e; s, t : R \to S)$, consists of a rack $R$, a subrack $S$ and the rack homomorphisms:

$$s, t : R \to S\,, \qquad e : S \to R$$

such that:

**i**) $se = id_S$ and $te = id_S$

**ii**) for all $x \in \ker(s)$ and $y \in \ker(t)$:

$$x \lhd y = x.$$

Let $C = (e; s, t : R \to S)$ and $C' = (e'; s', t' : R' \to S')$ be cat-1 racks. A cat-1 rack morphism $(\phi, \varphi) : C \to C'$ is a tuple which consists of the rack homomorphisms $\phi : R \to R'$ and $\varphi : S \to S'$ such that the following diagram commutes:

$$\begin{tikzcd}
R \arrow[r, "\phi"] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & R' \arrow[d, shift right, "t'"'] \arrow[d, shift left, "s'"] \\
S \arrow[r, "\varphi"'] \arrow[u, bend left=60, "e"] & S' \arrow[u, bend right=60, "e'"']
\end{tikzcd}$$

Consequently, we have the category of cat-1 racks.

## 3. Pullback Cat-1 Racks

Given a cat-1 rack $C = (e; s, t : R \to S)$ and a rack homomorphism $\phi : Q \to S$, the pullback cat-1 rack $C^* = (e^*; s^*, t^* : \phi^*(R) \to Q)$ is a cat-1 rack, such that satisfying the following universal property:

Given a morphism of cat-1 racks,

$$(\varphi, \phi) : C' = (e'; s', t' : P \to Q) \to C = (e; s, t : R \to S)$$

there is a unique cat-1 rack morphism

$$(\psi, id_Q) : C' = (e'; s', t' : P \to Q) \to C^* = (e^*; s^*, t^* : \phi^*(R) \to Q)$$

such that the following diagram commutes:

$$\begin{tikzcd}
P \arrow[rr, "\varphi"] \arrow[dr, dotted, "\psi"] \arrow[dd, shift right, "t'"'] \arrow[dd, shift left, "s'"] & & R \arrow[dd, shift right, "t"'] \arrow[dd, shift left, "s"] \\
& \phi^*(R) \arrow[ur, "\pi"'] \arrow[dl, shift right, "t^*"'] \arrow[dl, shift left, "s^*"] & \\
Q \arrow[rr, "\phi"'] & & S
\end{tikzcd}$$

3.1. **Construction of pullback cat-1 racks.** Let $C = (e; s, t : R \to S)$ be a cat-1 rack and $\phi : Q \to S$ be a rack homomorphism. Define:

$$C^* = (e^*; s^*, t^* : \phi^*(R) \to Q)$$

where:

$$\phi^*(R) = \{(q_1, r, q_2) \in Q \times R \times Q \mid \phi(q_1) = s(r),\ \phi(q_2) = t(r)\}.$$

with the rack operation:

$$(q_1, r, q_2) \lhd (q_1', r', q_2') = (q_1 \lhd q_1', r \lhd r', q_2 \lhd q_2').$$

Then with the rack homomorphisms:

$$\begin{aligned} s^*(q_1, r, q_2) &= q_1 \\ t^*(q_1, r, q_2) &= q_2 \\ e^*(q) &= (q, e\phi(q), q) \end{aligned}$$

we get a cat-1 rack $(e^*; s^*, t^* : \phi^*(R) \to Q)$ since:

**i**)

$$\begin{aligned} s^*e^*(q) &= s^*(q, e\phi(q), q) \\ &= q \\ &= id_Q(q) \end{aligned}$$

and

$$\begin{aligned} t^*e^*(q) &= t^*(q, e\phi(q), q) \\ &= q \\ &= id_Q(q). \end{aligned}$$

for all $q \in Q$, so that $s^*e^* = t^*e^* = id_Q$.

**ii**) Let $(q_1', r_1, q_1) \in \ker s^*$ and $(q_2, r_2, q_2') \in \ker t^*$. Then:

$$s^*\left(q_1', r_1, q_1\right) = 1 \quad \text{and} \quad t^*\left(q_2, r_2, q_2'\right) = 1$$

which implies:

$$q_1' = q_2' = 1$$

Therefore:

$$\begin{aligned}
\left(q_1', r_1, q_1\right) \lhd \left(q_2, r_2, q_2'\right) &= \left(1 \lhd q_2, r_1 \lhd r_2, q_1 \lhd 1\right) \\
&= \left(1, r_1, q_1\right) \\
&= \left(q_1', r_1, q_1\right).
\end{aligned}$$

Define the rack homomorphism:

$$\begin{array}{rccl}
\pi: & \phi^*(R) & \to & R \\
& (q_1, r, q_2) & \mapsto & \pi\left(q_1, r, q_2\right) = r.
\end{array}$$

Then $(\pi, \phi)$ is a cat-1 rack morphism since:

$$\begin{aligned}
\phi s^*\left(\left(q_1, r, q_2\right)\right) &= \phi\left(q_1\right) \\
&= s\left(r\right) \\
&= s\pi\left(q_1, r, q_2\right),
\end{aligned}$$

$$\begin{aligned}
\phi t^*\left(\left(q_1, r, q_2\right)\right) &= \phi\left(q_2\right) \\
&= t\left(r\right) \\
&= t\pi\left(q_1, r, q_2\right),
\end{aligned}$$

$$\begin{aligned}
\pi e^*\left(q\right) &= \pi\left(q, e\phi\left(q\right), q\right) \\
&= e\phi\left(q\right),
\end{aligned}$$

for all $(q_1, r, q_2) \in \phi^*(R)$, $q \in Q$; namely the following diagram commutes:

$$\begin{tikzcd}
\phi^*(R) \arrow[r, "\pi"] \arrow[d, shift right, "t^*"'] \arrow[d, shift left, "s^*"] & R \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
Q \arrow[r, "\phi"'] \arrow[u, bend left=60, "e^*"] & S \arrow[u, bend right=60, "e"']
\end{tikzcd}$$

This construction satisfies the universal property as follows:

Let $C' = (e'; s', t' : P \to Q)$ be a cat-1 rack and

$$(\varphi, \phi) : C' = (e'; s', t' : P \to Q) \to C = (e; s, t : R \to S)$$

be a cat-1 rack morphism where the following diagram commutes:

$$\begin{tikzcd}
P \arrow[r, "\varphi"] \arrow[d, shift right, "t'"'] \arrow[d, shift left, "s'"] & R \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
Q \arrow[r, "\phi"'] \arrow[u, bend left=60, "e'"] & S \arrow[u, bend right=60, "e"']
\end{tikzcd}$$

Then there exist a unique cat-1 rack morphism:

$$(\psi, id_Q) : C' = (e'; s', t' : P \to Q) \to C^* = (e^*; s^*, t^* : \phi^*(R) \to Q)$$

such that:

$$\pi\psi = \varphi.$$

Firstly, we need to prove that $(\psi, id_Q) : C' \to C^*$ is a cat-1 rack morphism. Define $\psi : P \to \phi^*(R)$ by:

$$\psi\left(p\right) = \left(s'\left(p\right), \varphi\left(p\right), t'\left(p\right)\right).$$

Then we get:

$$\begin{aligned} s^*\psi\left(p\right) &= s^*\left(s'\left(p\right), \varphi\left(p\right), t'\left(p\right)\right) \\ &= s'\left(p\right) \\ &= id_Q s'\left(p\right), \end{aligned}$$

$$\begin{aligned} t^*\psi\left(p\right) &= t^*\left(s'\left(p\right), \varphi\left(p\right), t'\left(p\right)\right) \\ &= t'\left(p\right) \\ &= id_Q t'\left(p\right), \end{aligned}$$

and also

$$\begin{aligned} \psi e'\left(q\right) &= \left(s'e'\left(q\right), \varphi e'\left(q\right), t'e'\left(q\right)\right) \\ &= \left(q, e\phi\left(q\right), q\right) \\ &= e^* id_Q\left(q\right). \end{aligned}$$

which means that $(\psi, id_Q) : C' \to C^*$ is a cat-1 rack morphism.

Furthermore, for all $p \in P$ we have:

$$\begin{aligned} \pi\psi\left(p\right) &= \pi\left(s'\left(p\right), \varphi\left(p\right), t'\left(p\right)\right) \\ &= \varphi\left(p\right) \end{aligned}$$

which makes the following commutative diagram and completes the construction:

$$\begin{tikzcd}
P \arrow[rr, "\varphi"] \arrow[dr, dotted, "\psi"] \arrow[dd, shift right, "t'"'] \arrow[dd, shift left, "s'"] & & R \arrow[dd, shift right, "t"'] \arrow[dd, shift left, "s"] \\
& \phi^*(R) \arrow[ur, "\pi"'] \arrow[dl, shift right, "t^*"'] \arrow[dl, shift left, "s^*"] & \\
Q \arrow[rr, "\phi"'] & & S.
\end{tikzcd}$$

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