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MERICHEV-SAIGO-MAEDA FRACTIONAL INTEGRATION OPERATOR ASSOCIATED WITH SRIVASTAVA'S POLYNOMIAL AND GAUSS HYPERGEOMETRIC FUNCTIONS

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Abstract. The object of the present paper is to establish seven theorems for the Marichev-Saigo-Maeda operators of fractional integration of the product for the Srivastava polynomial with Gauss hypergeometric function. The theorems established in this paper are of general character and provide extension of the recently given by results Saxena, Ram and Suthar [11] and Singh and Singh [12]. Some interesting special cases of our main results are also considered.

1. INTRODUCTION

We recall here the definitions of the generalized fractional integration operators of arbitrary order involving Appell function F_3 ([8], p.393, Eq.(4.12) and (4.13)) in the kernel introduced and studied by Saigo and Maeda in the following forms. Let $\alpha, \alpha', \beta, \beta', \gamma \in \mathbb{C}$ and $x > 0$, then

$$\left(I_{0+}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma} f\right)(x) = \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} \times F_3(\alpha, \alpha', \beta, \beta'; \gamma; 1-t/x, 1-x/t) f(t)dt, \quad \Re(\gamma) > 0. \tag{1.1}$$

$$\left(I_{0+}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma} f\right)(x) = \left(\frac{d}{dx}\right)^k \left(I_{0+}^{\alpha,\,\alpha',\,\beta+k,\,\beta',\,\gamma+k} f\right)(x), \quad (\Re(\gamma) \le 0; k = [-\Re(\gamma)] + 1). \tag{1.2}$$

The following results are required in the proofs:

$$\int_0^x t^{\rho-1}(x-t)^{\gamma-1} F_3(\alpha,\, \alpha',\, \beta,\, \beta';\, \gamma; 1-t/x, 1-x/t)dt = \Gamma\left[\begin{array}{c} \gamma,\, \rho+\alpha',\, \rho+\beta',\, \rho+\gamma+\alpha+\beta \\ \rho+\alpha'+\beta',\, \rho+\gamma-\alpha,\, \rho+\gamma-\beta \end{array}\right] x^{\rho+\gamma-1} \tag{1.3}$$

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where $\Re(\gamma) > 0$; $\Re(\rho) > \max[\Re(-\alpha'),\, Re\,(\beta'),\, \Re(\alpha+\beta-\gamma)]$.
The symbol $\Gamma\left[\begin{smallmatrix} \cdot & \cdot \\ \cdot & \cdot \end{smallmatrix}\right]$ occurring in (1.3) represents the quotient of the product of gamma functions.
An interesting extension of both the Riemann-Liouville and Erdélyi-Kober fractional integration operators was introduced by Saigo [7] in terms of Gausss hypergeometric function. Fractional integration operators associated with Gausss hypergeometric functions have been defined and studied by Saxena [9] and Kalla and Saxena [5]. Recently generalized fractional integral formulas for the H-function, Aleph function associated with genral class of polynomial are given by many authors, notably by Baleanu et al. [1], kumar et al. [2] and Ram and Kumar [6] and others due to their importance in problems associated with fractional integral equations where fractional calculus plays an importance role.

The series definition of hypergeometric function ${}_2F_1(.)$ is given by

$$ {}_2F_1(a,\, b;\, c;\, x) \;=\; \sum_{n\,=\,0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n\, n\,!} x^n, \tag{1.4}$$

where c is neither zero nor a negative integer, for convergence, $|x| < 1$; $x = 1$ and $\Re(c-a-b) > 0$, $x = -1$ and $\Re(c-a-b) >$ -1; and $(\alpha)_n$ is the Pochhammer symbol defined by

$$ (\alpha)_n \;=\; \alpha(\alpha+1)...(\alpha+n-1) \;=\; \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)}, \tag{1.5}$$

$$(\alpha)_0 = 1, \qquad (\alpha\, \in\, \mathbb{C}\,;\, n\, \in \mathbb{N}_0)$$

The hypergeometric function of two variables due to Srivastava and Karlsson [14] is defined as

$$ F^{p;\; q;\; k}_{l;\; m;\; n}\left[\begin{matrix} (a_p) : (b_q) : (c_k); \\ (\alpha_l) : (\beta_m) : (\gamma_n); \end{matrix}\; x,\; y\right] \tag{1.6}$$

$$= \sum_{r,\, s\, =\, 0}^{\infty} \frac{\prod_{j=1}^{p}(a_j)_{r+s}\; \prod_{j=1}^{q}(b_j)_r\; \prod_{j=1}^{k}(c_j)_s}{\prod_{j=1}^{l}(\alpha_j)_{r+s}\; \prod_{j=1}^{m}(\beta_j)_r\; \prod_{j=1}^{n}(\gamma_j)_s} \frac{x^r}{x\,!} \frac{y^s}{s\,!},$$

where, for convergence
(i) $p+q \;<\; l+m+1,\; p+k \;<\; l+n+1,\, |\,x\,| \;<\; \infty, |\,y\,| \;<\; \infty$ or
(ii) $p+q \;=\; l+m+1,\; p+k = \; l+n+1,$

$$|x\,|^{\frac{1}{(p\,-\,1)}} \,+\, |y\,|^{\frac{1}{(q\,-\,1)}} \,<\, 1\,, \quad if \quad p > 1,$$

$$max.\,\{|\,x\,|\,,\,|\,y\,|\} < 1,\;\; if \quad p \leq 1.$$

We also have

$$ (1-x)^{-a} = \sum_{n=0}^{\infty} \frac{(a)_n}{n\,!} x^n, \tag{1.7}$$

The beta function is defined by

$$ B(p\,,\, q) = \int_0^1 x^{p-1}(1-x)^{q-1} dx \;=\; \frac{\Gamma(p)\,\Gamma(q)}{\Gamma(p+q)}, \tag{1.8}$$

where $\Re(p) > 0\,,\, \Re(q) > 0$.

The general class of polynomials is defined by Srivastava ([13], p. 1, Eq. (1)) in the following manner :

$$S_w^u[x] = \sum_{s=0}^{[w/u]} \frac{(-w)_{u\,s}}{s\,!} A_{\,w,s}\, x^{\,s}, \qquad w = 0, 1, 2, \ldots \tag{1.9}$$

where u is an arbitrary positive integer and the coefficients $A_{\,w,\,s}$ (w, s ≥ 0) are arbitrary constants, real or complex.

Fractional integration and differentiation of the H- function under the operator (1.1) has been given by Saxena and Saigo [10]. This paper deals with derivation of Saigo-Maeda fractional transform of product of Srivastava polynomial with certain hypergeometric functions.

2. Main Results

Theorem 2.1. *If* $|x| < 1$,

$$ {}_2F_1\left[\begin{matrix} a,\, b\,; \\ c\,; \end{matrix}\; x\right] = \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n} \frac{x^n}{n\,!}, \tag{2.1}$$

where c is neither zero nor a negative integer, then we have

$$I_{0+}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma}\left[x^{\sigma-1} S_w^u\,[x]\; {}_2F_1\left[\begin{matrix} a,\, b\,; \\ c\,; \end{matrix}\; x\right]\right](x) = \sum_{s=0}^{[w/u]} \frac{(-w)_{us}}{s\,!} A_{w,s} \tag{2.2}$$

$$\times x^{\sigma-\alpha-\alpha'+\gamma+s-1} \frac{\Gamma(\sigma+s)\Gamma(\sigma-\alpha'+\beta'+s)\,\Gamma(\sigma-\alpha-\beta-\alpha'+\gamma+s)}{\Gamma(\sigma+\beta'+s)\,\Gamma(\sigma-\alpha-\alpha'+\gamma+s)\,\Gamma(\sigma-\alpha'-\beta+\gamma+s)}$$

$$\times {}_5F_4\left[\begin{matrix} \sigma+s,\, \sigma-\alpha'+\beta'+s,\, \sigma-\alpha-\beta-\alpha'+\gamma+s,\, a,\, b; \\ \sigma+\beta'+s,\, \sigma-\alpha-\alpha'+\gamma+s,\, \sigma-\alpha'-\beta+\gamma+s, c; \end{matrix}\; x\right].$$

Proof. Operating both sides of (2.1) and (1.9) by the generalized fractional integral operator $I_{0+}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma} x^{\sigma-1}$ and using equation (1.1), we obtain

$$= I_{0+}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma}\left[x^{\sigma-1}\sum_{s=0}^{[w/u]} \frac{(-w)_{us}}{s\,!} A_{w,s} x^s \sum_{n=0}^{\infty} \frac{(a)_n\,(b)_n}{(c)_n}\frac{x^n}{n\,!}\right],$$

$$= \sum_{s=0}^{[w/u]} \sum_{n=0}^{\infty} \frac{(-w)_{us}}{s!} A_{w,s} \frac{(a)_n\,(b)_n}{(c)_n} \frac{1}{n\,!} \frac{x^{-\alpha}}{\Gamma(\gamma)}$$

$$\times \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3(\alpha,\, \alpha',\, \beta,\, \beta';\, \gamma\,;\, 1-t/x, 1-x/t) t^{\sigma+n+s-1} dt,$$

$$= \sum_{s=0}^{[w/u]} \sum_{n=0}^{\infty} \frac{(-w)_{us}}{s!} A_{w,s} \frac{(a)_n\,(b)_n}{(c)_n\,n\,!}\, x^{\sigma-\alpha-\alpha'+\gamma+n+s-1}$$

$$\times \frac{\Gamma(\sigma+n+s)\,\Gamma(\sigma-\alpha'+\beta'+n+s)\,\Gamma(\sigma-\alpha-\beta-\alpha'+\gamma+n+s)}{\Gamma(\sigma+\beta'+n+s)\Gamma(\sigma-\alpha-\alpha'+\gamma+n+s)\Gamma(\sigma-\alpha'-\beta+\gamma+n+s)},$$

$$= x^{\sigma-\alpha-\alpha'+\gamma-1} \sum_{s=0}^{[w/u]} \frac{(-w)_{u\,s}}{s\,!} A_{w,s}\, x^s \tag{2.3}$$

$$\times \frac{\Gamma(\sigma+s)\,\Gamma(\sigma-\alpha'+\beta'+s)\,\Gamma(\sigma-\alpha-\beta-\alpha'+\gamma+s)}{\Gamma(\sigma+\beta'+s)\Gamma(\sigma-\alpha-\alpha'+\gamma+s)\Gamma(\sigma-\alpha'-\beta+\gamma+s)}$$

$$\times \sum_{n=0}^{\infty} \frac{(\sigma+s)_n(\sigma-\alpha'+\beta'+s)_n(\sigma-\alpha-\beta-\alpha'+\gamma+s)_n}{(\sigma+\beta'+s)_n(\sigma-\alpha-\alpha'+\gamma+s)_n(\sigma-\alpha'-\beta+\gamma+s)_n} \frac{(a)_n(b)_n}{(c)_n} \frac{x^n}{n!},$$

$$= x^{\sigma-\alpha-\alpha'+\gamma-1} \sum_{s=0}^{[w/u]} \frac{(-w)_{us}}{s\,!} A_{w,s}\, x^s \tag{2.4}$$

$$\times \frac{\Gamma(\sigma+s)\Gamma(\sigma-\alpha'+\beta'+s)\,\Gamma(\sigma-\alpha-\beta-\alpha'+\gamma+s)}{\Gamma(\sigma+\beta'+s)\,\Gamma(\sigma-\alpha-\alpha'+\gamma+s)\,\Gamma(\sigma-\alpha'-\beta+\gamma+s)}$$

$$\times {}_5F_4 \left[\begin{matrix} \sigma+s,\, \sigma-\alpha'+\beta'+s,\, \sigma-\alpha-\beta-\alpha'+\gamma+s,\, a,\, b; \\ \sigma+\beta'+s,\, \sigma-\alpha-\alpha'+\gamma+s,\, \sigma-\alpha'-\beta+\gamma+s, c; \end{matrix} \; x \right].$$

where $|x| < 1$. □

Theorem 2.2. *If $|x| < 1$ and if $\Re(c) > \Re(b) > 0$, [[3], p.47, Theorem 16]*

$${}_2F_1 \left[\begin{matrix} a,\, b\,; \\ c\,; \end{matrix} \; x \right] = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 k^{b-1}(1-k)^{c-b-1}(1-kx)^{-a} dk, \tag{2.5}$$

where c is neither zero nor a negative integer, then the following result holds

$$I_{0+}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma} \left[x^{\sigma-1} S_w^u\,[x]\; {}_2F_1 \left[\begin{matrix} a,\, b\,; \\ c\,; \end{matrix} \; x \right] \right](x) = \sum_{s=0}^{[w/u]} \frac{(-w)_{us}}{s\,!} A_{w,s} \tag{2.6}$$

$$\times x^{\sigma-\alpha-\alpha'+\gamma+s-1} \frac{\Gamma(\sigma+s)\Gamma(\sigma-\alpha'+\beta'+s)\,\Gamma(\sigma-\alpha-\beta-\alpha'+\gamma+s)}{\Gamma(\sigma+\beta'+s)\Gamma(\sigma-\alpha-\alpha'+\gamma+s)\Gamma(\sigma-\alpha'-\beta+\gamma+s)}$$

$$\times {}_5F_4 \left[\begin{matrix} \sigma+s,\, \sigma-\alpha'+\beta'+s,\, \sigma-\alpha-\beta-\alpha'+\gamma+s,\, a,\, b; \\ \sigma+\beta'+s,\, \sigma-\alpha-\alpha'+\gamma+s,\, \sigma-\alpha'-\beta+\gamma+s, c; \end{matrix} \; x \right].$$

Proof. From equation (1.9) and (2.5), we have

$$= I_{0+}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma} \left[x^{\sigma-1} \sum_{s=0}^{[w/u]} \frac{(-w)_{us}}{s!} A_{w,s}\, x^s \right.$$

$$\left. \times \frac{\Gamma(c)}{\Gamma(b)\,\Gamma(c-b)} \int_0^1 k^{b-1}(1-k)^{c-b-1}(1-kx)^{-a} dk \right](x),$$

Using equation (1.1) and (1.7) on the right side of the above equation, we obtain

$$= \sum_{s=0}^{[w/u]} \sum_{m=0}^{\infty} \frac{(-w)_{us}}{s!} A_{w,s} \frac{(a)_m}{m\,!} \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \tag{2.7}$$

$$\times \frac{x^{-\alpha}}{\Gamma(\gamma)} \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3(\alpha,\alpha',\beta,\beta';\gamma;1-t/x,1-x/t)$$

$$\times t^{\sigma+m+s-1} \left(\int_0^1 k^{b+m-1}(1-k)^{c-b-1} dk \right) dt,$$

If we use formula (1.8) in the right side of above equation (2.7), it gives

$$= \sum_{s=0}^{[w/u]} \sum_{m=0}^{\infty} \frac{(-w)_{us}}{s!} A_{w,s} \frac{(a)_m}{(c)_m} \frac{(b)_m}{m\,!} \frac{x^{-\alpha}}{\Gamma(\gamma)}$$

$$\times \int_0^x (x-t)^{\gamma-1} t^{-\alpha'} F_3(\alpha,\alpha',\beta,\beta';\gamma;1-t/x,1-x/t\;) t^{\sigma+m+s-1} dt,$$

$$= \sum_{s=0}^{[w/u]} \sum_{m=0}^{\infty} \frac{(-w)_{us}}{s!} A_{w,s} \frac{(a)_m}{(c)_m} \frac{(b)_m}{m\,!} x^{\sigma - \alpha - \alpha' + m + s + \gamma - 1}$$

$$\times \frac{\Gamma(\sigma + m + s)\Gamma(\sigma - \alpha' + \beta' + m + s)\Gamma(\sigma - \alpha - \beta - \alpha' + \gamma + m + s)}{\Gamma(\sigma + \beta' + m + s)\Gamma(\sigma - \alpha - \alpha' + \gamma + m + s)\Gamma(\sigma - \alpha' - \beta + \gamma + m + s)},$$

$$= x^{\sigma-\alpha-\alpha'+\gamma-1} \sum_{s=0}^{[w/u]} \frac{(-w)_{us}}{s!} A_{w,s}\, x^s \frac{\Gamma(\sigma + s)\Gamma(\sigma - \alpha' + \beta' + s)}{\Gamma(\sigma + \beta' + s)\Gamma(\sigma - \alpha - \alpha' + \gamma + s)}$$

$$\times \frac{\Gamma(\sigma - \alpha - \beta - \alpha' + \gamma + s)}{\Gamma(\sigma - \alpha' - \beta + \gamma + s)} \sum_{m=0}^{\infty} \frac{(\sigma + s)_m(\sigma - \alpha' + \beta' + s)_m}{(\sigma + \beta' + s)_m(\sigma - \alpha - \alpha' + \gamma + s)_m}$$

$$\times \frac{(\sigma - \alpha - \beta - \alpha' + \gamma + s)_m}{(\sigma - \alpha' - \beta + \gamma + s)_m} \frac{(a)_m(b)_m}{(c)_m} \frac{x^m}{m!},$$

$$= x^{\sigma-\alpha-\alpha'+\gamma-1} \sum_{s=0}^{[w/u]} \frac{(-w)_{us}}{s!} A_{w,s}\; x^s \tag{2.8}$$

$$\times \frac{\Gamma(\sigma + s)\Gamma(\sigma - \alpha' + \beta' + s)\Gamma(\sigma - \alpha - \beta - \alpha' + \gamma + s)}{\Gamma(\sigma + \beta' + S)\Gamma(\sigma - \alpha - \alpha' + \gamma + s)\Gamma(\sigma - \alpha' - \beta + \gamma + s)}$$

$$\times {}_5F_4 \left[\begin{array}{c} \sigma + s, \sigma - \alpha' + \beta' + s, \sigma - \alpha - \beta - \alpha' + \gamma + s, a, b; \\ \sigma + \beta' + s, \sigma - \alpha - \alpha' + \gamma + s, \sigma - \alpha' - \beta + \gamma + s, c; \end{array} \; x \right].$$

Which is same as equation (2.4) and where $|x| < 1$. □

Theorem 2.3. *If $|x| < 1$, then we have [[4], p.278, Eq. 8.13]*

$$\frac{d^n}{dx^n} {}_2F_1 \left[\begin{array}{c} a,\, b\,; \\ c\,; \end{array} \; x \right] = \frac{(a)_n\,(b)_n}{(c)_n} {}_2F_1 \left[\begin{array}{c} a + n, b + n; \\ c + n; \end{array} \; x \right], \tag{2.9}$$

where c is neither zero nor a negative integer then the following result is obtained

$$I_{0+}^{\alpha,\, \alpha',\, \beta,\, \beta',\, \gamma} \left[x^{\sigma-1}\; S_w^u[x]\; \frac{d^n}{dx^n}\, {}_2F_1 \left[\begin{array}{c} a,\, b; \\ c; \end{array} \; x \right] \right](x) \tag{2.10}$$

$$= x^{\sigma-\alpha-\alpha'+\gamma-1} \sum_{s=0}^{[w/u]} \frac{(-w)_{us}}{s!} A_{w,s}\, x^s$$

$$\times \frac{\Gamma(\sigma + s)\Gamma(\sigma - \alpha' + \beta' + s)\Gamma(\sigma - \alpha - \beta - \alpha' + \gamma + s)}{\Gamma(\sigma + \beta' + s)\Gamma(\sigma - \alpha - \alpha' + \gamma + s)\Gamma(\sigma - \alpha' - \beta + \gamma + s)} \frac{(a)_n(b)_n}{(c)_n}$$

$$\times {}_5F_4 \left[\begin{array}{c} \sigma + s, \sigma - \alpha' + \beta' + s, \sigma - \alpha - \beta - \alpha' + \gamma + s, a + n, b + n; \\ \sigma + \beta' + s, \sigma - \alpha - \alpha' + \gamma + s, \sigma - \alpha' - \beta + \gamma + s, c + n; \end{array} \; x \right].$$

where $\Re(\gamma) > 0$; $\Re(\rho) > \max[\Re(-\alpha'),\, \Re(\beta'),\, \Re(\alpha + \beta - \gamma)]$.

Proof. Operating both sides of (1.9) and (2.9) by the fractional integral operator $I_{0+}^{\alpha,\, \alpha',\, \beta,\, \beta',\, \gamma} x^{\sigma\, -1}$.

$$= I_{0+}^{\alpha,\, \alpha',\, \beta,\, \beta',\, \gamma} \left[x^{\sigma-1} \sum_{s=0}^{[w/u]} \frac{(-w)_{us}}{s!} A_{w,s} x^s \frac{(a)_n\,(b)_n}{(c)_n}\, {}_2F_1 \left[\begin{array}{c} a + n, b + n; \\ c + n; \end{array} \; x \right] \right],$$

$$= I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[x^{\sigma-1}\sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s!}A_{w,s}x^{s}\frac{(a)_n(b)_n}{(c)_n}\sum_{m=0}^{\infty}\frac{(a+n)_m(b+n)_m}{(c+n)_m m!}x^{m}\right],$$

Interchanging order of integration and summation which is valid under the conditions given in the theorem, the above equation

(2.11)
$$= \sum_{s=0}^{[w/u]}\sum_{m=0}^{\infty}\frac{(-w)_{us}}{s!}A_{w,s}\frac{(a)_n\,(b)_n}{(c)_n}\frac{(a+n)_m}{(c+n)_m}\frac{(b+n)_m}{m!}\,I_{0+}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma}[x^{\sigma+s+m-1}],$$

Now using equations (1.3) and (1.5), (2.11) becomes

$$= \sum_{s=0}^{[w/u]}\sum_{m=0}^{\infty}\frac{(-w)_{us}}{s!}A_{w,s}\frac{(a)_n\,(b)_n}{(c)_n}\frac{(a+n)_m}{(c+n)_m}\frac{(b+n)_m}{m!}\,x^{\sigma-\alpha-\alpha'+m+s+\gamma-1}$$

$$\times\frac{\Gamma(\sigma+m+s)\Gamma(\sigma-\alpha'+\beta'+m+s)\Gamma(\sigma-\alpha-\beta-\alpha'+\gamma+m+s)}{\Gamma(\sigma+\beta'+m+s)\Gamma(\sigma-\alpha-\alpha'+\gamma+m+s)\Gamma(\sigma-\alpha'-\beta+\gamma+m+s)},$$

$$= \sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s!}A_{w,s}\,x^{\sigma-\alpha-\alpha'+s+\gamma-1}$$

$$\times\frac{\Gamma(\sigma+s)\Gamma(\sigma-\alpha'+\beta'+s)\Gamma(\sigma-\alpha-\beta-\alpha'+\gamma+s)}{\Gamma(\sigma+\beta'+s)\Gamma(\sigma-\alpha-\alpha'+\gamma+s)\Gamma(\sigma-\alpha'-\beta+\gamma+s)}\frac{(a)_n\,(b)_n}{(c)_n}$$

$$\times\sum_{m=0}^{\infty}\frac{(\sigma+s)_m(\sigma-\alpha'+\beta'+s)_m(\sigma-\alpha-\beta-\alpha'+\gamma+s)_m}{(\sigma+\beta'+s)_m(\sigma-\alpha-\alpha'+\gamma+s)_m(\sigma-\alpha'-\beta+\gamma+s)_m}$$

$$\times\frac{(a+n)_m}{(c+n)_m}\frac{(b+n)_m}{m!}x^{m},$$

(2.12)
$$= \sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s!}A_{w,s}\,x^{\sigma-\alpha-\alpha'+s+\gamma-1}$$

$$\times\frac{\Gamma(\sigma+s)\Gamma(\sigma-\alpha'+\beta'+s)\Gamma(\sigma-\alpha-\beta-\alpha'+\gamma+s)}{\Gamma(\sigma+\beta'+s)\Gamma(\sigma-\alpha-\alpha'+\gamma+s)\Gamma(\sigma-\alpha'-\beta+\gamma+s)}\frac{(a)_n(b)_n}{(c)_n}$$

$$\times{}_5F_4\left[\begin{matrix}\sigma+s,\sigma-\alpha'+\beta'+s,\sigma-\alpha-\beta-\alpha'+\gamma+s,a+n,b+n;\\ \sigma+\beta'+s,\sigma-\alpha-\alpha'+\gamma+s,\sigma-\alpha'-\beta+\gamma+s,c+n;\end{matrix}\;x\right].$$

where $|x|<1$, which proves the required result. □

Theorem 2.4. *From [3], p. 60, eq.(5)], we have*

(2.13)
$${}_2F_1\left[\begin{matrix}a,\,b;\\ c;\end{matrix}\;x\right]=(1-x)^{c-a-b}\;{}_2F_1\left[\begin{matrix}c-a,c-b;\\ c;\end{matrix}\;x\right]\quad |x|<1,$$

then the following results holds:

(2.14)
$$I_{0+}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma}\left[x^{\sigma-1}S_w^u\,[x]\;{}_2F_1\left[\begin{matrix}a,\,b\,;\\ c\,;\end{matrix}\;x\right]\right](x)$$

$$= x^{\sigma-\alpha-\alpha'+\gamma-1}\sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s\,!}A_{w,s}\,x^{s}$$

$$\times\frac{\Gamma(\sigma+s)\Gamma(\sigma-\alpha'+\beta'+s)\Gamma(\sigma-\alpha-\beta-\alpha'+\gamma+s)}{\Gamma(\sigma+\beta'+s)\Gamma(\sigma-\alpha-\alpha'+\gamma+s)\Gamma(\sigma-\alpha'-\beta+\gamma+s)}$$

$$\times F\begin{smallmatrix}3;1;2\\3;0;1\end{smallmatrix}\left[\begin{matrix}\sigma+s,\sigma-\alpha'+\beta'+s,\sigma-\alpha-\beta-\alpha'+\gamma+s,\\ a+b-c,c-a,c-b;\\ \\ \sigma+\beta'+s,\sigma-\alpha-\alpha'+\gamma+s,\sigma-\alpha'-\beta+\gamma+s,\\ c;\end{matrix}\quad x,x\right].$$

where $\Re(\gamma)>0$; $\Re(\rho)>\max[\Re(-\alpha'),\Re(\beta'),\,\Re(\alpha+\beta-\gamma)]$, $|x|<1$,.

Proof. From (1.9) and (2.13), we have

$$=I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[x^{\sigma-1}\sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s!}A_{w,s}x^{s}(1-x)^{-(a+b-c)}{}_2F_1\left[\begin{matrix}c-a,c-b;\\c;\end{matrix}\;x\right]\right],$$

$$=I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[\sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s!}A_{w,s}x^{\sigma+s-1}\sum_{m,n=0}^{\infty}\frac{(a+b-c)_m(c-a)_n(c-b)_n}{(c)_n}\frac{x^{m+n}}{m!\,n!}\right],$$

Now changing the order of integration and summation valid under the conditions which is given with the theorem, we obtain

(2.15)

$$=\sum_{s=0}^{[w/u]}\sum_{m,n=0}^{\infty}\frac{(-w)_{us}}{s!}A_{w,s}\frac{(a+b-c)_m}{m!\,n!}\frac{(c-a)_n(c-b)_n}{(c)_n}I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[x^{\sigma+m+n+s-1}\right],$$

Now using equations (1.3) and (1.5), we obtain the right side of (2.15), which reduces to the following form

$$=\sum_{s=0}^{[w/u]}\sum_{m,n=0}^{\infty}\frac{(-w)_{us}}{s!}A_{w,s}\frac{(a+b-c)_m}{m!n!}\frac{(c-a)_n(c-b)_n}{(c)_n}$$

$$\times x^{\sigma-\alpha-\alpha'+m+n+\gamma+s-1}\frac{\Gamma(\sigma+m+n+s)\Gamma(\sigma-\alpha'+\beta'+m+n+s)}{\Gamma(\sigma+\beta'+m+n+s)\Gamma(\sigma-\alpha-\alpha'+\gamma+m+n+s)}$$

$$\times\frac{\Gamma(\sigma-\alpha-\beta-\alpha'+\gamma+m+n+s)}{\Gamma(\sigma-\alpha'-\beta+\gamma+m+n+s)},$$

$$=\sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s!}A_{w,s}\,x^{\sigma-\alpha-\alpha'+\gamma+s-1}$$

$$\times\frac{\Gamma(\sigma+s)\Gamma(\sigma-\alpha'+\beta'+s)\Gamma(\sigma-\alpha-\beta-\alpha'+\gamma+s)}{\Gamma(\sigma+\beta'+s)\Gamma(\sigma-\alpha-\alpha'+\gamma+s)\Gamma(\sigma-\alpha'-\beta+\gamma+s)}$$

$$\times\sum_{m,n=0}^{\infty}\frac{(\sigma+s)_{m+n}(\sigma-\alpha'+\beta'+s)_{m+n}(\sigma-\alpha-\beta-\alpha'+\gamma+s)_{m+n}}{(\sigma+\beta'+s)_{m+n}(\sigma-\alpha-\alpha'+\gamma+s)_{m+n}(\sigma-\alpha'-\beta+\gamma+s)_{m+n}}$$

$$\times\frac{(a+b-c)_m(c-a)_n(c-b)_n}{(c)_n}\frac{x^{m+n}}{m!\,n!},$$

Finally, using formula (1.6) we obtain that the above result

$$=x^{\sigma-\alpha-\alpha'+\gamma-1}\sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s\,!}A_{w,s}\,x^{s}\tag{2.16}$$

$$\times \frac{\Gamma(\sigma+s)\Gamma(\sigma-\alpha'+\beta'+s)\Gamma(\sigma-\alpha-\beta-\alpha'+\gamma+s)}{\Gamma(\sigma+\beta'+s)\Gamma(\sigma-\alpha-\alpha'+\gamma+s)\Gamma(\sigma-\alpha'-\beta+\gamma+s)}$$

$$\times F\begin{array}{c}3;1;2\\3;0;1\end{array}\left[\begin{array}{c}\sigma+s,\sigma-\alpha'+\beta'+s,\sigma-\alpha-\beta-\alpha'+\gamma+s,\\ a+b-c,c-a,c-b;\\ \\ \sigma+\beta'+s,\sigma-\alpha-\alpha'+\gamma+s,\sigma-\alpha'-\beta+\gamma+s,\\ c;\end{array} x,x\right].$$

where $|x|<1$. □

Theorem 2.5. *If* $|x|<1$*and* $\left|\frac{x}{1-x}\right|<1$, *we have [[3], p.60, eq-1]*

$$ {}_2F_1\left[\begin{array}{c}a,\ b\,;\\ c\,;\end{array} x\right] = (1-x)^{-a}\,{}_2F_1\left[\begin{array}{c}a\,,c-b;\\ c;\end{array} \frac{-x}{1-x}\right], \tag{2.17}$$

then

$$ I_{0+}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma}\left[x^{\sigma-1}\ S_w^u[x]\ {}_2F_1\left[\begin{array}{c}a,\ b\ ;\\ c;\end{array} x\right]\right] \tag{2.18}$$

$$= x^{\sigma-\alpha-\alpha'+\gamma+s-1}\sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s!}A_{w,s}\ x^s$$

$$\times \frac{\Gamma(\sigma+s)\,\Gamma(\sigma-\alpha'+\beta'+s)\,\Gamma(\sigma-\alpha-\beta-\alpha'+\gamma+s)}{\Gamma(\sigma+\beta'+s)\,\Gamma(\sigma-\alpha-\alpha'+\gamma+s)\,\Gamma(\sigma-\alpha'-\beta+\gamma+s)}$$

$$\times F\begin{array}{c}4;0;1\\3;0;1\end{array}\left[\begin{array}{c}\sigma+s,\ \sigma-\alpha'+\beta'+s,\ \sigma-\alpha-\beta-\alpha'+\gamma+s,\\ a,\ c-b;\\ \sigma+\beta'+s,\ \sigma-\alpha-\alpha'+\gamma+s,\ \sigma-\alpha'-\beta+\gamma+s,\\ c;\end{array} x,-x\right].$$

where $\Re(\gamma)>0$; $\Re(\rho)>\max[\Re(-\alpha'),\Re(\beta'),\ \Re(\alpha+\beta-\gamma)]$.

Proof. From (2.17), we have

$$I_{0+}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma}\left[x^{\sigma-1}\ S_w^u[x]\ {}_2F_1\left[\begin{array}{c}a,\ b;\\ c;\end{array} x\right]\right](x)$$

$$= I_{0+}^{\alpha,\alpha',\beta,\beta',\gamma}\left[x^{\sigma-1}\sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s!}A_{w,s}\,x^s(1-x)^{-a}{}_2F_1\left[\begin{array}{c}a,c-b;\\ c;\end{array}\frac{-x}{1-x}\right]\right],$$

$$= I_{0+}^{\alpha,\alpha',\,\beta,\,\beta',\,\gamma}\left[x^{\sigma-1}\sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s!}A_{w,s}\ x^s\right.$$

$$\left.\times\sum_{m,n=0}^{\infty}(-1)^n\frac{(a+n)_m(a)_n(c-b)_n}{(c)_n}\frac{x^{m+n}}{m!\,n!}\right],$$

Applying the formula $(a+n)_m(a)_n = (a)_{m+n}$ and changing the order of integration and summation which is permissible under the conditions stated with the theorem, we obtain

(2.19)

$$= \sum_{s=0}^{[w/u]}\sum_{m,n=0}^{\infty}\frac{(-w)_{us}}{s!}A_{w,s}(-1)^n\frac{(a)_{m+n}(c-b)_n}{(c)_n\ m!\,n!}I_{0+}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma}\left[x^{\sigma+m+n+s-1}\right],$$

Now using equations (1.3) and (1.5), we obtain the right side of (2.19), which reduces to the following result

$$= \sum_{s=0}^{[w/u]} \sum_{m,n=0}^{\infty} \frac{(-w)_{us}}{s!} A_{w,s} (-1)^n \frac{1}{m!\,n!} \frac{(a)_{m+n}(c-b)_n}{(c)_n}$$

$$\times x^{\sigma-\alpha-\alpha'+m+n+\gamma+s-1} \frac{\Gamma(\sigma+m+n+s)\Gamma(\sigma-\alpha'+\beta'+m+n+s)}{\Gamma(\sigma+\beta'+m+n+s)\Gamma(\sigma-\alpha-\alpha'+\gamma+m+n+s)}$$

$$\times \frac{\Gamma(\sigma-\alpha-\beta-\alpha'+\gamma+m+n+s)}{\Gamma(\sigma-\alpha'-\beta+\gamma+m+n+s)},$$

$$= \sum_{s=0}^{[w/u]} \frac{(-w)_{us}}{s!} A_{w,s}\ x^{\sigma-\alpha-\alpha'+\gamma+s-1}$$

$$\times \frac{\Gamma(\sigma+s)\Gamma(\sigma-\alpha'+\beta'+s)\Gamma(\sigma-\alpha-\beta-\alpha'+\gamma+s)}{\Gamma(\sigma+\beta'+s)\Gamma(\sigma-\alpha-\alpha'+\gamma+s)\Gamma(\sigma-\alpha'-\beta+\gamma+s)}$$

$$\times \sum_{m,n=0}^{\infty} \frac{(\sigma+s)_{m+n}(\sigma-\alpha'+\beta'+s)_{m+n}(\sigma-\alpha-\beta-\alpha'+\gamma+s)_{m+n}}{(\sigma+\beta'+s)_{m+n}(\sigma-\alpha-\alpha'+\gamma+s)_{m+n}(\sigma-\alpha'-\beta+\gamma+s)_{m+n}}$$

$$\times \frac{(a)_{m+n}(c-b)_n}{(c)_n} \frac{x^m}{m!} \frac{(-x)^n}{n!},$$

Using the formula $F \begin{matrix} l;\, m\,;\, n \\ p\,;\, q\,;\, r \end{matrix} [\ .\ ]$ (1.6), we obtain the right of the above equation

$$= \sum_{s=0}^{[w/u]} \frac{(-w)_{us}}{s!} A_{w,s}\ x^{\sigma-\alpha-\alpha'+\gamma+s-1} \tag{2.20}$$

$$\times \frac{\Gamma(\sigma+s)\,\Gamma(\sigma-\alpha'+\beta'+s)\,\Gamma(\sigma-\alpha-\beta-\alpha'+\gamma+s)}{\Gamma(\sigma+\beta'+s)\,\Gamma(\sigma-\alpha-\alpha'+\gamma+s)\,\Gamma(\sigma-\alpha'-\beta+\gamma+s)}$$

$$\times F\ \begin{matrix} 4;0;1 \\ 3;0;1 \end{matrix} \left[\begin{matrix} \sigma+s,\, \sigma-\alpha'+\beta'+s,\, \sigma-\alpha-\beta-\alpha'+\gamma+s, \\ a,\, c-b; \\ \sigma+\beta'+s,\, \sigma-\alpha-\alpha'+\gamma+s,\, \sigma-\alpha'-\beta+\gamma+s, \\ c; \end{matrix} \quad x, -x \right].$$

where $|\,x\,| < 1$ which is the desired result. □

Theorem 2.6. *For* $|x| < 1$ *and* $\left|\frac{x}{1-x}\right| < 1$, *we have*

$${}_2F_1 \left[\begin{matrix} a,\, b\,; \\ c\,; \end{matrix} \ x \right] = (1-x)^{-b}\ {}_2F_1 \left[\begin{matrix} c-a, b; \\ c; \end{matrix} \ \frac{-x}{1-x} \right], \tag{2.21}$$

then

$$I_{0+}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma} \left[x^{\sigma-1}\, S_w^u[x]\ {}_2F_1 \left[\begin{matrix} a,\, b; \\ c; \end{matrix} \ x \right] \right](x) = \sum_{s=0}^{[w/u]} \frac{(-w)_{us}}{s!} A_{w,s} \tag{2.22}$$

$$\times x^{\sigma-\alpha-\alpha'+\gamma+s-1} \frac{\Gamma(\sigma+s)\Gamma(\sigma-\alpha'+\beta'+s)\Gamma(\sigma-\alpha-\beta-\alpha'+\gamma+s)}{\Gamma(\sigma+\beta'+s)\Gamma(\sigma-\alpha-\alpha'+\gamma+s)\Gamma(\sigma-\alpha'-\beta+\gamma+s)}$$

$$\times F\ \begin{matrix} 4;0;1 \\ 3;0;1 \end{matrix} \left[\begin{matrix} \sigma+s, \sigma-\alpha'+\beta'+s, \sigma-\alpha-\beta-\alpha'+\gamma+s, b;\ c-a; \\ \sigma+\beta'+s, \sigma-\alpha-\alpha'+\gamma+s, \sigma-\alpha'-\beta+\gamma+s; c; \end{matrix} \ x, -x \right].$$

where $\Re(\gamma) > 0$; $\Re(\rho) > \max[\Re(-\alpha'), \Re(\beta'), \Re(\alpha+\beta-\gamma)]$.

Proof. From (2.21), (1.7), (1.9) and (1.4), we obtain

$$= I_{0+}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma}\left[x^{\sigma-1}\sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s!}A_{w,s}x^{s}(1-x)^{-b}\,{}_2F_1\left[\begin{matrix} c-a, b; \\ c; \end{matrix}\ \frac{-x}{1-x}\right]\right],$$

$$= I_{0+}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma}\left[x^{\sigma-1}\sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s!}A_{w,s}x^{s}\right.$$

$$\left.\times\sum_{m,n=0}^{\infty}(-1)^{n}\frac{1}{m!\,n!}\frac{(b+n)_m(b)_n(c-a)_n}{(c)_n}x^{m+n}\right],$$

(2.23)

$$= \sum_{s=0}^{[w/u]}\sum_{m,n=0}^{\infty}\frac{(-w)_{us}}{s!}A_{w,s}\frac{(-1)^n}{m!\,n!}\frac{(b)_{m+n}(c-a)_n}{(c)_n}I_{0+}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma}[x^{\sigma+m+n+s-1}],$$

$$= \sum_{s=0}^{[w/u]}\sum_{m,n=0}^{\infty}\frac{(-w)_{us}}{s!}A_{w,s}\frac{(-1)^n}{m!\,n!}\frac{(b)_{m+n}(c-a)_n}{(c)_n}x^{\sigma-\alpha-\alpha'+m+n+\gamma+s-1}$$

$$\times\frac{\Gamma(\sigma+m+n+s)\Gamma(\sigma-\alpha'+\beta'+m+n+s)}{\Gamma(\sigma+\beta'+m+n+s)\Gamma(\sigma-\alpha-\alpha'+\gamma+m+n+s)}$$

$$\times\frac{\Gamma(\sigma-\alpha-\beta-\alpha'+\gamma+m+n+s)}{\Gamma(\sigma-\alpha'-\beta+\gamma+m+n+s)},$$

$$= \sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s!}A_{w,s}x^{\sigma-\alpha-\alpha'+\gamma+s-1}$$

$$\times\frac{\Gamma(\sigma+s)\Gamma(\sigma-\alpha'+\beta'+s)\Gamma(\sigma-\alpha-\beta-\alpha'+\gamma+s)}{\Gamma(\sigma+\beta'+s)\Gamma(\sigma-\alpha-\alpha'+\gamma+s)\Gamma(\sigma-\alpha'-\beta+\gamma+s)}$$

$$\times\sum_{m,n=0}^{\infty}\frac{(\sigma+s)_{m+n}(\sigma-\alpha'+\beta'+s)_{m+n}(\sigma-\alpha-\beta-\alpha'+\gamma+s)_{m+n}}{(\sigma+\beta'+s)_{m+n}(\sigma-\alpha-\alpha'+\gamma+s)_{m+n}(\sigma-\alpha'-\beta+\gamma+s)_{m+n}}$$

$$\times\frac{(b)_{m+n}(c-a)_n}{(c)_n}\frac{x^m}{m!}\frac{(-x)^n}{n!},$$

Using the formula $F\begin{matrix} l;m;n \\ p;q;r \end{matrix}[.]$ (1.6) we obtain the right of the above equation as

(2.24)

$$= \sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s!}A_{w,s}\;x^{\sigma-\alpha-\alpha'+\gamma+s-1}$$

$$\times\frac{\Gamma(\sigma+s)\Gamma(\sigma-\alpha'+\beta'+s)\Gamma(\sigma-\alpha-\beta-\alpha'+\gamma+s)}{\Gamma(\sigma+\beta'+s)\Gamma(\sigma-\alpha-\alpha'+\gamma+s)\Gamma(\sigma-\alpha'-\beta+\gamma+s)}$$

$$\times F\begin{matrix} 4;0;1 \\ 3;0;1 \end{matrix}\left[\begin{matrix} \sigma+s, \sigma-\alpha'+\beta'+s, \sigma-\alpha-\beta-\alpha'+\gamma+s, b;\ c-a; \\ \sigma+\beta'+s, \sigma-\alpha-\alpha'+\gamma+s, \sigma-\alpha'-\beta+\gamma+s; c; \end{matrix}\ x, -x\right].$$

where $|x| < 1$, which is the desired result. □

Theorem 2.7. *If $|x| < 1$, then we have*

$$G(a,\, b; c;\, x) = \frac{\Gamma(1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)} F(a,\, b; c;\, x) + \frac{\Gamma(c-1)}{\Gamma(a)\Gamma(b)} x^{1-c} F(a+1-c,\, b+1-c;\, 2-c;\, x), \tag{2.25}$$

and

$$I_{0+}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma}\left[x^{\sigma-1}\, S_w^u[x]\, F(a,b;\, a+b+1-c; 1-x)\right]$$

$$= x^{\sigma-\alpha'-\alpha+\gamma-1} \sum_{s=0}^{[w/u]} \frac{(-w)_{us}}{s!} A_{w,s}\, x^s\, \frac{\Gamma(\sigma+s)}{\Gamma(\sigma+\beta'+s)} \tag{2.26}$$

$$\times \frac{\Gamma(\sigma-\alpha'+\beta'+s)\Gamma(\sigma-\alpha'-\alpha-\beta+\gamma+s)\Gamma(1-c)}{\Gamma(\sigma-\alpha'-\alpha+\gamma+s)\Gamma(\sigma-\alpha'+\gamma-\beta+s)\Gamma(a+1-c)\Gamma(b+1-c)}$$

$$\times {}_5F_4\left[\begin{array}{c} \sigma+s, \sigma-\alpha'+\beta'+s, \sigma-\alpha'-\alpha-\beta+\gamma+s, a\,,\, b\,; \\ \sigma+\beta'+s, \sigma-\alpha'-\alpha+\gamma+s, \sigma-\alpha'+\gamma-\beta+s,\, c; \end{array} x\right]$$

$$+ x^{\sigma-\alpha'-\alpha-c+\gamma} \sum_{s=0}^{[w/u]} \frac{(-w)_{us}}{s!} A_{w,s}\; x^s \frac{\Gamma(\sigma-c+1+s)}{\Gamma(\sigma+\beta'-c+1+s)}$$

$$\times \frac{\Gamma(\sigma-\alpha'+\beta'-c+1+s)\Gamma(\sigma-\alpha'-\beta-\alpha-c+1+s)\Gamma(1-c)}{\Gamma(\sigma-\alpha'+\gamma-\alpha-c+1+s)\Gamma(\sigma-\alpha'-\beta+\gamma-c+1+s)\Gamma(a)\Gamma(b)}$$

$$\times {}_5F_4\left[\begin{array}{c} \sigma-c+1+s,\ \sigma-\alpha'+\beta'-c+1+s, \\ \sigma-\alpha'-\alpha-\beta-c+1+s,\ a+1-c,\ b+1-c; \\ \\ \sigma+\beta'-c+1+s,\ \sigma-\alpha'-\alpha+\gamma-c+1+s, \\ \sigma-\alpha'-\beta+\gamma-c+1+s,\ 2-c; \end{array} x\right].$$

where $\Re(\gamma) > 0$; $\Re(\rho) >$ $\max[\Re(-\alpha'),\, \Re(\beta'),\, \Re(\alpha+\beta-\gamma)]$.

We note that (2.25) represents hypergeometric function of the second kind [[4], p. 285, Pb.28].

Proof. Operating both sides of (1.9) and (2.25) by the fractional integral operator $I_{0\,+}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma}\left[x^{\sigma\,-1}\right]$ and using expansion of hypergeometric function on the right side of the equation, we obtain

$$= I_{0+}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma}\left[x^{\sigma-1} \sum_{s=0}^{[w/u]} \frac{(-w)_{us}}{s!} A_{w,s} x^s \frac{\Gamma(1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)} \times \sum_{m=0}^{\infty} \frac{(a)_m}{m!} \frac{(b)_m}{(c)_m} x^m\right] + I_{0+}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma}\left[x^{\sigma-1} \sum_{s=0}^{[w/u]} \frac{(-w)_{us}}{s!} A_{w,s}\, x^s \times \frac{\Gamma(1-c)}{\Gamma(a)\Gamma(b)} x^{1-c} \sum_{m=0}^{\infty} \frac{(a+1-c)_m}{m!} \frac{(b+1-c)_m}{(2-c)_m} x^m\right],$$

Interchanging the order of integration and summation which is permissible under the conditions stated with the theorem, the above line becomes

$$= \frac{\Gamma(1-c)}{\Gamma(a+1-c)\Gamma(b+1-c)} \sum_{s=0}^{[w/u]} \sum_{m=0}^{\infty} \frac{(-w)_{us}}{s!} A_{w,s} \frac{(a)_m}{m!} \frac{(b)_m}{(c)_m} \tag{2.27}$$

$$\times I_{0+}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma}\left[x^{\sigma+m+s-1}\right]+\frac{\Gamma(1-c)}{\Gamma(a)\Gamma(b)}\sum_{s=0}^{[w/u]}\sum_{m=0}^{\infty}\frac{(-w)_{us}}{s!}A_{w,s}$$

$$\times\frac{(a+1-c)_m}{m!}\frac{(b+1-c)_m}{(2-c)_m}I_{0+}^{\alpha,\,\alpha',\,\beta,\,\beta',\,\gamma}\left[x^{\sigma+m-c+s}\right],$$

Now using the equation (1.3), we obtain

$$=\sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s!}A_{w,s}\,x^{\sigma-\alpha'-\alpha+\gamma+s-1}\frac{\Gamma(\sigma+s)\Gamma(\sigma-\alpha'+\beta'+s)}{\Gamma(\sigma+\beta'+s)\Gamma(\sigma-\alpha'-\alpha+\gamma+s)}$$

$$\times\frac{\Gamma(\sigma-\alpha'-\alpha-\beta+\gamma+s)\Gamma(1-c)}{\Gamma(\sigma-\alpha'+\gamma-\beta+s)\Gamma(a+1-c)\Gamma(b+1-c)}\sum_{m=0}^{\infty}\frac{(a)_m(b)_m}{(c)_m}\frac{(\sigma+s)_m}{(\sigma+\beta'+s)_m}$$

$$\times\frac{(\sigma-\alpha'+\beta'+s)_m(\sigma-\alpha-\beta-\alpha'+\gamma+s)_m}{(\sigma-\alpha-\alpha'+\gamma+s)_m(\sigma-\alpha'-\beta+\gamma+s)_m}\frac{x^m}{m!}+\sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s!}A_{w,s}$$

$$\times x^{\sigma-\alpha'-\alpha-c+\gamma+s}\frac{\Gamma(\sigma-c+1+s)\Gamma(\sigma-\alpha'+\beta'-c+1+s)}{\Gamma(\sigma+\beta'-c+1+s)\Gamma(\sigma-\alpha'+\gamma-\alpha-c+1+s)}$$

$$\times\frac{\Gamma(\sigma-\alpha'-\beta-\alpha-c+1+s)\Gamma(1-c)}{\Gamma(\sigma-\alpha'-\beta+\gamma-c+1+s)\Gamma(a)\Gamma(b)}\sum_{m=0}^{\infty}\frac{(\sigma-\alpha'+\beta'-c+1+s)_m}{(\sigma-\alpha-\alpha'+\gamma-c+1+s)_m}$$

$$\times\frac{(\sigma-c+1+s)_m(\sigma-\alpha-\beta-\alpha'-c+1+s)_m(a+1-c)_m(b+1-c)_m}{(\sigma+\beta'-c+1+s)_m(\sigma-\alpha'-\beta+\gamma-c+1+s)_m(2-c)_m}\frac{x^m}{m!},$$

$$=\sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s!}A_{w,s}\frac{\Gamma(\sigma+s)\Gamma(\sigma-\alpha'+\beta'+s)}{\Gamma(\sigma+\beta'+s)\Gamma(\sigma-\alpha'-\alpha+\gamma+s)} \tag{2.28}$$

$$\times\frac{\Gamma(\sigma-\alpha'-\alpha-\beta+\gamma+s)\Gamma(1-c)}{\Gamma(\sigma-\alpha'+\gamma-\beta+s)\Gamma(a+1-c)\Gamma(b+1-c)}\,x^{\sigma-\alpha'-\alpha+\gamma+s-1}$$

$$\times{}_5F_4\left[\begin{matrix}\sigma+s,\sigma-\alpha'+\beta'+s,\sigma-\alpha'-\alpha-\beta+\gamma+s,a\,,\,b\,;\\ \sigma+\beta'+s,\sigma-\alpha'-\alpha+\gamma+s,\sigma-\alpha'+\gamma-\beta+s,\,c;\end{matrix}\;x\right]$$

$$+\sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s!}A_{w,s}\;x^{\sigma-\alpha'-\alpha-c+\gamma+s}\frac{\Gamma(\sigma-c+1+s)}{\Gamma(\sigma+\beta'-c+1+s)}$$

$$\times\frac{\Gamma(\sigma-\alpha'+\beta'-c+1+s)\Gamma(\sigma-\alpha'-\beta-\alpha-c+1+s)\Gamma(1-c)}{\Gamma(\sigma-\alpha'+\gamma-\alpha-c+1+s)\Gamma(\sigma-\alpha'-\beta+\gamma-c+1+s)\Gamma(a)\Gamma(b)}$$

$$\times{}_5F_4\left[\begin{matrix}\sigma-c+1+s,\;\sigma-\alpha'+\beta'-c+1+s,\\ \sigma-\alpha'-\alpha-\beta-c+1+s,\;a+1-c,\;b+1-c;\\ \\ \sigma+\beta'-c+1+s,\;\sigma-\alpha'-\alpha+\gamma-c+1+s,\\ \sigma-\alpha'-\beta+\gamma-c+1+s,\;2-c;\end{matrix}\;x\right].$$

where $|x|<1$, which is the desired result. □

3. Special Cases

If we set $\alpha' = 0$ and use the identity

$$\left(I_{0+}^{\alpha+\beta,\,0\,,-\eta,\,\beta',\,\alpha}\right)(x) = \left(I_{0+}^{\alpha,\,\beta,\,\eta}\right)(x) \tag{3.1}$$

in theorem 1-7, where $\left(I_{0+}^{\alpha,\,\beta,\,\eta}\right)(x)$ is the Saigo operator [7]. Then we arrive at

Corollary 3.1. *If $|x| < 1$, and using equation (2.1), where c is neither zero nor a negative integer, then we have*

$$I_{0+}^{\alpha,\,\beta,\,\gamma}\left[x^{\sigma-1}\;S_w^u[x]\;{}_2F_1\left[\begin{array}{c}a,\,b;\\ c;\end{array}\;x\right]\right](x) \tag{3.2}$$

$$= x^{\sigma-\beta-1}\sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s!}A_{w,s}\frac{\Gamma(\sigma+s)\,\Gamma(\sigma-\beta+\gamma+s)}{\Gamma(\sigma+\alpha+\gamma+s)\,\Gamma(\sigma-\beta+s)}\;x^s$$
$$\times{}_4F_3\left[\begin{array}{c}\sigma+s,\,\sigma-\beta+\gamma+s,\,a,\,b;\\ \sigma+\alpha+\gamma+s,\,\sigma-\beta+s,\,c;\end{array}\;x\right]$$

Corollary 3.2. *If $|x| < 1$ and if $\Re(c) > \Re(b) > 0$, [[3], p.47, Theorem 16], then the following result holds*

$$I_{0+}^{\alpha,\,\beta,\,\gamma}\left[x^{\sigma-1}\;S_w^u[x]\;{}_2F_1\left[\begin{array}{c}a,\,b;\\ c;\end{array}\;x\right]\right](x) \tag{3.3}$$

$$= x^{\sigma-\beta-1}\sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s!}A_{w,s}x^s\frac{\Gamma(\sigma+s)\Gamma(\sigma-\beta+\gamma+s)}{\Gamma(\sigma+\alpha+\gamma+s)\,\Gamma(\sigma-\beta+s)}$$
$$\times{}_4F_3\left[\begin{array}{c}\sigma+s,\,\sigma-\beta+\gamma+s,\,a,\,b;\\ \sigma+\alpha+\gamma+s,\,\sigma-\beta\;+s,\,c;\end{array}\;x\right]$$

where $\Re(\gamma) > 0$; $\Re(\rho) >$ $\max[0, \Re(\beta-\gamma]$.

Corollary 3.3. *If $|x| < 1$, then we have [[4], p.278, Eq. (8.13)], the following result obtained*

$$I_{0+}^{\alpha,\,\beta,\,\gamma}\left[x^{\sigma-1}\;S_w^u[x]\;{}_2F_1\left[\begin{array}{c}a,\,b;\\ c;\end{array}\;x\right]\right](x) \tag{3.4}$$

$$= x^{\sigma-\beta-1}\sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s!}A_{w,s}\,x^s\frac{\Gamma(\sigma+s)\,\Gamma(\sigma-\beta+\gamma+s)}{\Gamma(\sigma+\beta'+s)\Gamma(\sigma+\alpha+\gamma+s)\Gamma(\sigma-\beta+s)}$$
$$\times\frac{(a)_n\,(b)_n}{(c)_n}{}_4F_3\left[\begin{array}{c}\sigma+s,\sigma-\beta+\gamma+s,\,a+n,\,b+n;\\ \sigma+\alpha+\gamma+s,\,\sigma-\beta+s,\,c+n;\end{array}\;x\right].$$

where $\Re(\gamma) > 0$; $\Re(\rho) > \max[0, \Re(\beta-\gamma]$.

Corollary 3.4. *From [[3], p. 60, eq.(5)], then the following result obtained*

$$I_{0+}^{\alpha,\,\beta,\,\gamma}\left[x^{\sigma-1}\;S_w^u[x]\;{}_2F_1\left[\begin{array}{c}a,\,b;\\ c;\end{array}\;x\right]\right](x) \tag{3.5}$$

$$= x^{\sigma-\beta-1}\sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s!}A_{w,s}\;x^s\frac{\Gamma(\sigma+s)\Gamma(\sigma-\beta+\gamma+s)}{\Gamma(\sigma+\alpha+\gamma+s)\Gamma(\sigma-\beta+s)}$$
$$\times F\begin{array}{c}2;1;2\\ 2;0;1\end{array}\left[\begin{array}{c}\sigma+s,\sigma-\beta+\gamma+s,\,a+b-c\,,\,c-a,\,c-b;\\ \sigma+\alpha+\gamma+s,\,\sigma-\beta+s;\,c;\end{array}\;x,\,x\right]$$

where $\Re(\gamma) > 0$; $\Re(\rho) >$ $\max[0, \max[0, \Re(\beta-\gamma](\beta-\gamma]$.

Corollary 3.5. *If* $|x| < 1$ *and* $\left|\frac{x}{1-x}\right| < 1$, *we have [[3], p.60, eq(1)], then the following result obtained*

$$I_{0+}^{\alpha,\,\beta,\,\gamma}\left[x^{\sigma-1}\;S_w^u[x]\;{}_2F_1\left[\begin{array}{c}a,\,b;\\ c;\end{array}\;x\right]\right](x) \tag{3.6}$$

$$= \sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s!}A_{w,s}x^{\sigma-\beta+s-1}\frac{\Gamma(\sigma+s)\,\Gamma(\sigma-\beta+\gamma+s)}{\Gamma(\sigma+\alpha+\gamma+s)\,\Gamma(\sigma-\beta+s)}$$

$$\times F\begin{array}{c}3;0;1\\2;0;1\end{array}\left[\begin{array}{c}\sigma+s,\sigma-\beta+\gamma+s,\,a,c-b;\\ \sigma+\alpha+\gamma+s,\sigma-\beta+s;\,c;\end{array}\;x,\,-x\right].$$

where $\Re(\gamma) > 0$; $\Re(\rho) > \max[0, \Re(\beta-\gamma]$.

Corollary 3.6. *If* $|x| < 1$ *and* $\left|\frac{x}{1-x}\right| < 1$, *using* (2.21), *then the result obtained*

$$I_{0+}^{\alpha,\,\beta,\,\gamma}\left[x^{\sigma-1}\;S_w^u[x]\;{}_2F_1\left[\begin{array}{c}a,\,b;\\ c;\end{array}\;x\right]\right](x) \tag{3.7}$$

$$= x^{\sigma-\beta-1}\sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s!}A_{w,s}\;x^{s}\frac{\Gamma(\sigma+s)\Gamma(\sigma-\beta+\gamma+s)}{\Gamma(\sigma+\alpha+\gamma+s)\,\Gamma(\sigma-\beta+s)}$$

$$\times F\begin{array}{c}3;0;1\\2;0;1\end{array}\left[\begin{array}{c}\sigma+s,\,\sigma-\beta+\gamma+s,\,b;\,c-a;\\ \sigma+\alpha+\gamma+s,\sigma-\beta+s;\,c;\end{array}\;x,-x\right].$$

where $\Re(\gamma) > 0$; $\Re(\rho) > \max[0, \Re(\beta-\gamma]$

Corollary 3.7. *If* $|x| < 1$ *and equation* (2.25), *then we have*

$$I_{0+}^{\alpha,\,\beta,\,\gamma}\left[x^{\sigma-1}S_w^u[x]F(a\,,b\,;\,a+b\,+\,1\,-\,c\,;\,1-x\,)\right] \tag{3.8}$$

$$= \sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s!}A_{w,s}\frac{\Gamma(\sigma+s)\Gamma(\sigma-\beta+\gamma+s)\Gamma(1-c)}{\Gamma(\sigma+\alpha+\gamma+s)\Gamma(\sigma-\beta+s)\Gamma(a+1-c)\Gamma(b+1-c)}$$

$$\times x^{\sigma-\beta+s-1}{}_4F_3\left[\begin{array}{c}\sigma+s,\sigma-\beta+\gamma+s,\,a,\,b\,;\\ \sigma+\alpha+\gamma+s,\,\sigma-\beta+s,\,c;\end{array}\;x\right]+\sum_{s=0}^{[w/u]}\frac{(-w)_{us}}{s!}A_{w,s}$$

$$\times x^{\sigma-\beta+s-1}\frac{\Gamma(\sigma-c+1+s)\,\Gamma(\sigma-\beta+\gamma-c+1+s)\,\Gamma(1c)}{\Gamma(\sigma+\gamma+\alpha-c+1+s)\Gamma(\sigma-\beta-c+1+s)\Gamma(a)\Gamma(b)}$$

$$\times{}_4F_3\left[\begin{array}{c}\sigma-c+1+s,\sigma-\beta+\gamma-c+1+s,a+1-c,\,b+1-c;\\ \sigma+\alpha+\gamma-c+1+s,\,\sigma-\beta-c+1+s,\,2-c;\end{array}\;x\right].$$

where $\Re(\gamma) > 0$; $\Re(\rho) > \max[0, \Re(\beta-\gamma]$.

We note that (2.25) represents hypergeometric function of the second kind [[4], p. 285, Pb. 28].

4. Conclusion

The Gauss hypergeometric functions defined by (1.4), possess the advantage that most of the known and widely-investigated special functions are expressible also in terms of the Gauss hypergeometric functions. Therefore, we conclude this paper by remarking that further special cases of the main result in Section 2 and Section 3 respectively as:

(i) It is interesting to observe that the results given by Saxena Ram and Suthar [11] follow from the results derived in this paper that if we set $w = 0$, $A_{0,0} = 1$ then $S_w^u = 1$, consequently the results (2.1), (2.5), (2.9), (2.13), (2.17), (2.21) and (2.25) yield respectively.

(ii) Also it is observe that the results given by Singh and Singh [[12], eq. (2.2), (3.2), (4.2), (5.2), (5.6), (5.10), (5.14)] follow from the corollary results derived in this paper, if we set $w = 0$, $A_{0,0} = 1$ then $S_w^u = 1$.

The results thus derived in this paper are general in character and likely to find certain applications in the theory of special functions.

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