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Research Article

Subcategories of the category of ⊤-convergence spaces

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Abstract

⊤-convergence structures serve as an important tool to describe fuzzy topology. This paper aims to give further investigations on ⊤-convergence structures. Firstly, several types of ⊤-convergence structures are introduced, including Kent ⊤-convergence structures, ⊤-limit structures and principal ⊤-convergence structures, and their mutual categorical relationships as well as their own categorical properties are studied. Secondly, by changing of the underlying lattice, the "change of base" approach is applied to ⊤-convergence structures and the relationships between ⊤-convergence structures with respect to different underlying lattices are demonstrated.

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1. Introduction

Filter convergence plays an important role in topology. Combing the axiomatic approach and filter convergence, the concept of axiomatic convergence structures is introduced. In the famous book [25], Preuss collected categorical properties of axiomatic convergence structures, which are also called generalized convergence structures. Several types of generalized convergence structures, including Kent convergence structures, limit structures, pseudo-topological convergence structures, and principal convergence structures, have closed categorical relationships and nice categorical properties. With the development of fuzzy set theory, Höhle suggested to develop an analogous convergence theory based on fuzzy filters [7,9]. Following this trend, Jäger [10] used stratified L-filters to propose the concept of stratified L-generalized convergence structures. Yao [29] introduced L-fuzzifying convergence structures via L-fuzzifying filters. Pang [20] proposed (L, M)-fuzzy convergence structures by means of (L, M)-fuzzy filters. Many scholars paid attention to these kinds of fuzzy convergence structures from different aspects (see, e.g., Jäger [12–15], Fang [3,4], Li et al. [17–19], Pang [21–23] and Zhang [32,34]).

In category theory, Borceux [2] summarized the effect of "a morphism of monoidal categories" on the corresponding enriched categories, functors and natural transformations.

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This allows us to see the importance of the basic monoidal category, which is used to define enriched categories. In order to reveal the essential characters of enriched categories from a high level, the basic monoidal category should not be fixed, but be changed. This leads to a new approach to enriched category called "change of base". Up to now, the "change of base" approach has been well developed in enriched category theory [2, 6, 28]. As an application, Höhle [8] proposed the "change of base" approach in many valued topology. Adopting this method, the relationships between classical topologies and fuzzy topologies can be demonstrated.

Considering subcategories of fuzzy convergence spaces, Pang and Zhao [24] studied several types of (L, M)-fuzzy convergence structures. Later, Pang [22] investigated subcategories of L-fuzzifying convergence spaces. It is worth noting that Jäger [11] not only studied various subcategories of stratified L-generalized convergence structures, but also applied the "change of base" approach to stratified L-generalized convergence structures. It is well known that $\top$-convergence structures via $\top$-filters are new type of fuzzy convergence structures and possess many nice categorical properties [30, 31]. Motivated by the nice categorical properties of stratified L-generalized convergence structures via stratified L-filters in [11], we will consider the following subjects in the framework of $\top$-convergence structures:

- Study various subcategories of $\top$-convergence structures.
- Investigate $\top$-convergence structures by the "change of base" approach.

The paper is organized as follows: In Section 2, we will recall some necessary concepts and notations. In Section 3, we will introduce the concepts of Kent $\top$-convergence structures, $\top$-limit structures and principal $\top$-convergence structures, and study their categorical relationships. In Section 4, we will consider $\top$-convergence structures from a "change of base" viewpoint and study the relationships between $\top$-convergence structures based on different lattices.

2. Preliminaries

For category theory we refer to Leinster [16] and Preuss [25]; for residuated lattices we refer to Bělohlávek [1]; for $\top$-filters we refer to Höhle [9] and Yu and Fang [30].

2.1. Basic concepts in category theory

For convenience, the class of all objects in a category $\mathscr{A}$ is denoted by $ob(\mathscr{A})$.

Definition 2.1 ([16])**.** Let $G : \mathscr{B} \longrightarrow \mathscr{A}$ be a functor and let $A \in ob(\mathscr{A})$. The comma category $(A \Rightarrow G)$ is the category defined as follows:

(1) objects are pairs $(B \in ob(\mathscr{B}), \varphi : A \longrightarrow G(B))$,

(2) morphisms $(B, \varphi) \longrightarrow (B', \varphi')$ in $(A \Rightarrow G)$ are morphisms $\psi : B \longrightarrow B'$ in $\mathscr{B}$ such that the triangle

$$\begin{tikzcd} A \arrow[r, "\varphi"] \arrow[dr, "\varphi'"'] & G(B) \arrow[d, "G(\psi)"] \\ & G(B') \end{tikzcd}$$

commutes.

Dually (reversing all the arrows), there is a comma category $(G \Rightarrow A)$, whose objects are pairs $(B \in ob(\mathscr{B}), \varphi : G(B) \longrightarrow A)$.

An object $I \in ob(\mathscr{A})$ is initial if there is exactly one morphism $I \longrightarrow A$ for all $A \in ob(\mathscr{A})$. An object $T \in ob(\mathscr{A})$ is terminal if there is exactly one morphism $A \longrightarrow T$ for all $A \in ob(\mathscr{A})$.

Theorem 2.2 ([16]). *Take categories and functors* $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$. *There is a one-to-one correspondence between:*

(1) *adjunctions between* F *and* G *(with* F *on the left and* G *on the right);*

(2) *natural transformations* $\eta : 1_{\mathscr{A}} \longrightarrow G \circ F$ *such that* $\eta_A : A \longrightarrow G \circ F(A)$ *is initial in* $(A \Rightarrow G)$ *for all* $A \in ob(\mathscr{A})$.

Definition 2.3 ([16]). An isomorphism between categories $\mathscr{A}$ and $\mathscr{B}$ consists of a pair of functors $\mathscr{A} \underset{G}{\overset{F}{\rightleftarrows}} \mathscr{B}$ such that $G \circ F = 1_{\mathscr{A}}$ and $F \circ G = 1_{\mathscr{B}}$.

Definition 2.4 ([25]). Let $\mathscr{A}$ be a full and isomorphism-closed subcategory of $\mathscr{B}$ and $I : \mathscr{A} \longrightarrow \mathscr{B}$ the inclusion functor. Then

(1) $\mathscr{A}$ is called reflective in $\mathscr{B}$ if I has a left adjoint $R : \mathscr{B} \longrightarrow \mathscr{A}$;

(2) $\mathscr{A}$ is called coreflective in $\mathscr{B}$ if I has a right adjoint $R_C : \mathscr{B} \longrightarrow \mathscr{A}$.

Remark 2.5. In Definition 2.4, R is called a bireflector when, the morphism $\eta_B : B \longrightarrow I \circ R(B)$ is a bimorphism for all $B \in ob(\mathscr{B})$. In this case, $\mathscr{A}$ is called a bireflective subcategory of $\mathscr{B}$. Of course, there is the dual notion of a bicoreflector (a bicoreflective subcategory).

Proposition 2.6 ([25]). *Let* $\mathscr{B}$ *be a Cartesian closed topological category and* $\mathscr{A}$ *a (full and isomorphism-closed) bicoreflective subcategory which is closed under formation of finite products in* $\mathscr{B}$. *Then* $\mathscr{A}$ *is a Cartesian closed topological category, and the power objects in* $\mathscr{A}$ *arise from the corresponding power objects in* $\mathscr{B}$ *by applying the bicoreflector.*

Proposition 2.7 ([25]). *If* $\mathscr{A}$ *is a bireflective (full and isomorphism-closed) subcategory of a Cartesian closed topological category* $\mathscr{B}$ *which is closed under formation of power objects in* $\mathscr{B}$, *then* $\mathscr{A}$ *is a Cartesian closed topological category, and the power objects in* $\mathscr{A}$ *are formed in* $\mathscr{B}$.

2.2. Basic concepts in lattice theory

Definition 2.8 ([9]). A complete residuated lattice is a triple $(L, \leq, *)$, where $(L, \leq)$ is a complete lattice with the top element $\top$ and the bottom element $\bot$, and $*$ is a commutative, associative binary operation such that

(1) $\top$ is the unit element for $*$;

(2) $*$ is distributive over arbitrary joins, i.e., $(\bigvee_{i\in I} \alpha_i) * \beta = \bigvee_{i\in I}(\alpha_i * \beta)$.

For a given complete residuated lattice L, the binary operation $\rightarrow$ on L can be computed by

$$\alpha \rightarrow \beta = \bigvee \{\gamma \in L \,|\, \alpha * \gamma \leq \beta\}.$$

The binary operation $\rightarrow$ is called the implication operation on L with respect to $*$. Further, $*$ and $\rightarrow$ form an adjoint pair in the sense of $\alpha * \gamma \leq \beta \Longleftrightarrow \gamma \leq \alpha \rightarrow \beta$ for all α, β, $\gamma \in L$. A complete residuated lattice L is an MV-algebra if it satisfies $\alpha \vee \eta = \alpha \rightarrow (\alpha \rightarrow \eta)$ for all α, $\eta \in L$.

An L-subset of X is a mapping from X to L, and the family of all L-subsets on X will be denoted by L^X, called the L-power set of X. $\top_X$ represents the constant L-subset with the value $\top$ and $\bot_X$ represents the constant L-subset with the value $\bot$. For a universal set X, the set of all subsets of X is denoted by $\mathcal{P}(X)$.

All algebraic operations on L can be extended to the L-power set L^X in a pointwise way. For each A, $B \in L^X$, $\alpha \in L$ and $x \in X$,

(1) $(A \vee B)(x) = A(x) \vee B(x)$;

(2) $(A \wedge B)(x) = A(x) \wedge B(x)$;

(3) $(A * B)(x) = A(x) * B(x)$ and $(\alpha * A)(x) = \alpha * A(x)$;

(4) $(A \rightarrow B)(x) = A(x) \rightarrow B(x)$ and $(\alpha \rightarrow B)(x) = \alpha \rightarrow B(x)$.

Let $\varphi : X \longrightarrow Y$ be a mapping. Define $\varphi^{\rightarrow} : L^X \longrightarrow L^Y$ and $\varphi^{\leftarrow} : L^Y \longrightarrow L^X$ by $\varphi^{\rightarrow}(A)(y) = \bigvee_{\varphi(x)=y} A(x)$ for all $A \in L^X$ and $y \in Y$, and $\varphi^{\leftarrow}(B)(x) = B(\varphi(x))$ for all $B \in L^Y$ and $x \in X$.

For a given set X, there is a binary mapping $\mathcal{S}_X(-,-) : L^X \times L^X \longrightarrow L$, defined by

$$\mathcal{S}_X(A,B) = \bigwedge_{x\in X} (A(x) \to B(x))$$

for all pair $(A,B) \in L^X \times L^X$. $\mathcal{S}_X(A,B)$ can be interpreted as the degree of A being a subset of B. $\mathcal{S}_X(-,-)$ is called the fuzzy inclusion order between L-subsets.

Lemma 2.9 ([1]). *For each* $A, B, C, D \in L^X$*, it holds that*
(1) $A \le B \Longleftrightarrow \mathcal{S}_X(A,B) = \top$;
(2) $\mathcal{S}_X(A,B) * \mathcal{S}_X(B,C) \le \mathcal{S}_X(A,C)$;
(3) $\mathcal{S}_X(A,B) * \mathcal{S}_X(C,D) \le \mathcal{S}_X(A * C, B * D)$;
(4) $\mathcal{S}_X(A,B) \wedge \mathcal{S}_X(C,D) \le \mathcal{S}_X(A \wedge C, B \wedge D)$;
(5) $\mathcal{S}_X(A,B) \wedge \mathcal{S}_X(C,D) \le \mathcal{S}_X(A \vee C, B \vee D)$;
(6) $A \le B$ *implies* $\mathcal{S}_X(C,A) \le \mathcal{S}_X(C,B)$ *and* $\mathcal{S}_X(B,D) \le \mathcal{S}_X(A,D)$.

Lemma 2.10 ([1]). *Let* $\varphi : X \longrightarrow Y$ *be a mapping. For each* $A, B \in L^X$ *and* $C, D \in L^Y$*, it holds that*
(1) $\mathcal{S}_X(A,B) \le \mathcal{S}_Y(\varphi^{\rightarrow}(A), \varphi^{\rightarrow}(B))$;
(2) $\mathcal{S}_Y(C,D) \le \mathcal{S}_X(\varphi^{\leftarrow}(C), \varphi^{\leftarrow}(D))$;
(3) $\mathcal{S}_Y(\varphi^{\rightarrow}(A), C) = \mathcal{S}_X(A, \varphi^{\leftarrow}(C))$.

2.3. $\top$-filters and $\top$-convergence spaces

Definition 2.11 ([9]). A $\top$-filter on X is a nonempty subset $\mathbb{F} \subseteq L^X$ with the following properties:
($\mathbb{F}$1) if $A \in L^X$ with $\bigvee_{C\in\mathbb{F}} S_X(C,A) = \top$, then $A \in \mathbb{F}$;
($\mathbb{F}$2) $A_1 \wedge A_2 \in \mathbb{F}$ for all $A_1, A_2 \in \mathbb{F}$;
($\mathbb{F}$3) $\bigvee_{x\in X} A(x) = \top$ for all $A \in \mathbb{F}$.

The family of all $\top$-filters on X is denoted by $\mathcal{F}_L^{\top}(X)$.

Definition 2.12 ([9]). A nonempty subset $\mathbb{B} \subseteq L^X$ is called a $\top$-filter base on X if it satisfies:
($\mathbb{B}$1) $\bigvee_{B\in\mathbb{B}} S_X(B, C \wedge D) = \top$ for all $C, D \in \mathbb{B}$;
($\mathbb{B}$2) $\bigvee_{x\in X} C(x) = \top$ for all $C \in \mathbb{B}$.

For a $\top$-filter base $\mathbb{B}$, a $\top$-filter can be generated in the following way:

$$\mathbb{F}_{\mathbb{B}} = \Big\{A \in L^X \mid \bigvee_{B\in\mathbb{B}} \mathcal{S}_X(B,A) = \top\Big\}.$$

Given a point $x \in X$, then $[x]_{\top} = \big\{A \in L^X \,|\, A(x) = \top\big\}$ is a $\top$-filter, and called the point $\top$-filter of x. Let $\mathbb{F} \in \mathcal{F}_L^{\top}(X)$ and $\mathbb{G} \in \mathcal{F}_L^{\top}(Y)$. Then

$$\mathbb{F} \times \mathbb{G} = \Big\{D \in L^{X\times Y} \mid \bigvee_{A\in\mathbb{F}, B\in\mathbb{G}} \mathcal{S}_{X\times Y}(A \times B, D) = \top\Big\}$$

is a $\top$-filter on $X \times Y$, which is called the product of $\mathbb{F}$ and $\mathbb{G}$ [33].

Let $\varphi : X \longrightarrow Y$ be a mapping and $\mathbb{F} \in \mathcal{F}_L^{\top}(X)$. Then

$$\varphi^{\Rightarrow}(\mathbb{F}) = \Big\{B \in L^Y \mid \bigvee_{A\in\mathbb{F}} S_Y(\varphi^{\rightarrow}(A), B) = \top\Big\}$$

is a $\top$-filter on Y, which is called the image of $\mathbb{F}$ under φ [5]. Obviously, $B \in \varphi^{\Rightarrow}(\mathbb{F})$ iff $\varphi^{\leftarrow}(B) \in \mathbb{F}$.

Definition 2.13 ([30]). A mapping $\lim : \mathcal{F}_L^\top(X) \longrightarrow \mathcal{P}(X)$ satisfying the following conditions:

(TC1) $\forall x \in X$, $x \in \lim[x]_\top$;

(TC2) $\forall \mathbb{F}, \mathbb{G} \in \mathcal{F}_L^\top(X)$, $\mathbb{F} \subseteq \mathbb{G}$ implies $\lim \mathbb{F} \subseteq \lim \mathbb{G}$,

is called a $\top$-convergence on X. The pair $(X, \lim)$ is called a $\top$-convergence space.

A mapping $\varphi : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ between $\top$-convergence spaces is said to be continuous provided that $x \in \lim \mathbb{F}$ implies $\varphi(x) \in \lim_Y \varphi^\Rightarrow(\mathbb{F})$ for all $x \in X$ and $\mathbb{F} \in \mathcal{F}_L^\top(X)$. The class of all $\top$-convergence spaces and continuous mappings forms a category, which is denoted by $\top$-**Conv**.

Yu and Fang [30] investigated some categorical properties of $\top$-**Conv**.

Theorem 2.14 ([30]). *$\top$-**Conv** is a topological category.*

Theorem 2.15 ([30]). *If L is a complete MV-algebra, then $\top$-**Conv** is Cartesian closed.*

Here we recall the power object in $\top$-**Conv**. Let $(X, \lim_X) \in ob(\top\text{-}\mathbf{Conv})$ and $(Y, \lim_Y) \in ob(\top\text{-}\mathbf{Conv})$. The set of all continuous mappings from $(X, \lim_X)$ to $(Y, \lim_Y)$ is denoted by $C_\top(X, Y)$. Then the mapping $\lim_C : \mathcal{F}_L^\top(C_\top(X, Y)) \longrightarrow \mathcal{P}(C_\top(X, Y))$ defined by

$$\lim_C \mathbb{H} = \left\{\varphi \in C_\top(X, Y) \mid \forall x \in X \text{ and } \mathbb{F} \in \mathcal{F}_L^\top(X), x \in \lim_X \mathbb{F} \Longrightarrow \varphi(x) \in \lim_Y ev_{X,Y}^\Rightarrow(\mathbb{H} \times \mathbb{F})\right\}$$

is the $\top$-convergence with respect to the power object, where $ev_{X,Y} : C_\top(X, Y) \times X \longrightarrow Y$ is the evaluation mapping (i.e., $ev_{X,Y}(\varphi, x) = \varphi(x)$ for all $(\varphi, x) \in C_\top(X, Y) \times X$).

3. Subcategories of $\top$-Conv

In this section, we will introduce several subcategories of $\top$-**Conv** and establish their categorical relationships.

3.1. Kent $\top$-convergence spaces

A $\top$-convergence $\lim$ on X is said to be Kent provided that

$$\text{(TCK)} \quad \lim \mathbb{F} \subseteq \lim(\mathbb{F} \cap [x]_\top) \quad \forall \mathbb{F} \in \mathcal{F}_L^\top(X), \forall x \in X.$$

The pair $(X, \lim)$ is called a Kent $\top$-convergence space. The full subcategory of $\top$-**Conv** consisting of Kent $\top$-convergence spaces is denoted by $\top$-**KConv**. For convenience, we use $I : \top\text{-}\mathbf{KConv} \longrightarrow \top\text{-}\mathbf{Conv}$ to denote the inclusion functor.

Proposition 3.1. *For each $(X, \lim) \in ob(\top\text{-}\mathbf{Conv})$, define $\lim^* : \mathcal{F}_L^\top(X) \longrightarrow \mathcal{P}(X)$ by*

$$\lim^* \mathbb{F} = \left\{x \in X \mid \exists \mathbb{H} \in \mathcal{F}_L^\top(X) \text{ such that } \mathbb{H} \cap [x]_\top \subseteq \mathbb{F} \text{ and } x \in \lim \mathbb{H}\right\} \quad \forall \mathbb{F} \in \mathcal{F}_L^\top(X).$$

Then $(X, \lim^*) \in ob(\top\text{-}\mathbf{KConv})$.

Proof. It suffices to verify that $\lim^*$ satisfies (TC1), (TC2) and (TCK). (TC1) and (TC2) are straightforward.

(TCK) Take any $x \in \lim^* \mathbb{F}$. Then there exists $\mathbb{H} \in \mathcal{F}_L^\top(X)$ such that $\mathbb{H} \cap [x]_\top \subseteq \mathbb{F}$ and $x \in \lim \mathbb{H}$. This implies that $\mathbb{H} \cap [x]_\top \subseteq \mathbb{F} \cap [x]_\top$. Thus, we obtain $x \in \lim^*(\mathbb{F} \cap [x]_\top)$. By the arbitrariness of x, we get $\lim^* \mathbb{F} \subseteq \lim^*(\mathbb{F} \cap [x]_\top)$. $\square$

Proposition 3.2. *If $\varphi : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ between $\top$-convergence spaces is continuous, then $\varphi : (X, \lim_X^*) \longrightarrow (Y, \lim_Y^*)$ between* Kent *$\top$-convergence spaces is continuous.*

Proof. Let $\mathbb{F} \in \mathcal{F}_L^\top(X)$. Take each $x \in \lim_X^* \mathbb{F}$. Then there exists $\mathbb{H} \in \mathcal{F}_L^\top(X)$ such that $\mathbb{H} \cap [x]_\top \subseteq \mathbb{F}$ and $x \in \lim_X \mathbb{H}$. Since $\varphi^\Rightarrow(\mathbb{H} \cap [x]_\top) = \varphi^\Rightarrow(\mathbb{H}) \cap [\varphi(x)]_\top$ and $\varphi : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ is continuous, it follows that $\varphi^\Rightarrow(\mathbb{H}) \cap [\varphi(x)]_\top \subseteq \varphi^\Rightarrow(\mathbb{F})$ and $\varphi(x) \in \lim_Y \varphi^\Rightarrow(\mathbb{H})$. Thus, we obtain $\varphi(x) \in \lim_Y^*(\mathbb{F})$. This proves that $\varphi : (X, \lim_X^*) \longrightarrow (Y, \lim_Y^*)$ is continuous. $\square$

By Propositions 3.1 and 3.2, we get a functor.

$$K : \begin{cases} \top\textbf{-Conv} & \longrightarrow & \top\textbf{-KConv} \\ (X, \lim) & \longmapsto & (X, \lim^*) \\ \varphi & \longmapsto & \varphi \end{cases}$$

Now let us establish the categorical relationships between $\top$-**KConv** and $\top$-**Conv**.

Proposition 3.3. *K is a left adjoint to I.*

Proof. It is enough to show that $\eta = id : 1_{\top\textbf{-Conv}} \longrightarrow I \circ K$ is a natural transformation and $id_X : (X, \lim_X) \longrightarrow I \circ K(X, \lim_X) = (X, \lim_X^*)$ is initial in $(X \Rightarrow I)$ for all $(X, \lim_X) \in ob(\top\textbf{-Conv})$.

(1) For each $(X, \lim_X) \in ob(\top\textbf{-Conv})$, take any $x \in X$ and $\mathbb{F} \in \mathcal{F}_L^\top(X)$ such that $x \in \lim \mathbb{F}$. Since $\mathbb{F} \cap [x]_\top \subseteq \mathbb{F}$, we have $x \in \lim^* \mathbb{F}$. This shows that $id_X : (X, \lim_X) \longrightarrow I \circ K(X, \lim_X)$ is continuous. Moreover, for each continuous mapping $\varphi : (X, \lim_X) \longrightarrow (Y, \lim_Y)$, the square

$$\begin{array}{ccc} (X, \lim_X) & \xrightarrow{\varphi} & (Y, \lim_Y) \\ {\scriptstyle id_X}\downarrow & & \downarrow{\scriptstyle id_Y} \\ (X, \lim_X^*) & \xrightarrow[I \circ K(\varphi) = \varphi]{} & (Y, \lim_Y^*) \end{array}$$

commutes. This shows $\{id_X\}_X$ is a natural transformation $1_{\top\textbf{-Conv}} \longrightarrow I \circ K$.

(2) Let $((Y, \lim_Y), \varphi)$ be an object in $(X \Rightarrow I)$. It suffices to show that there is exactly one morphism from $(K(X, \lim_X), id_X)$ to $((Y, \lim_Y), \varphi)$ in $(X \Rightarrow I)$. By Definition 2.1, we know that a morphism from $(K(X, \lim_X), id_X)$ to $((Y, \lim_Y), \varphi)$ in $(X \Rightarrow I)$ is a continuous mapping ψ from $K(X, \lim_X) = (X, \lim_X^*)$ to $(Y, \lim_Y)$ in $\top$-**KConv** such that the triangle

$$\begin{tikzcd} (X, \lim_X) \arrow[r, "id_X"] \arrow[dr, "\varphi"'] & I \circ K(X, \lim_X) \arrow[d, "I(\psi) = \psi"] \\ & I(Y, \lim_Y) \end{tikzcd}$$

commutes. Since $\varphi = \psi \circ id_X$, we obtain $\psi = \varphi$. Therefore we get the uniqueness of ψ. Now it suffices to verify that $\varphi : (X, \lim_X^*) \longrightarrow (Y, \lim_Y)$ is continuous.

Take each $\mathbb{F} \in \mathcal{F}_L^\top(X)$ and $x \in X$ such that $x \in \lim_X^* \mathbb{F}$. Then there exists $\mathbb{H} \in \mathcal{F}_L^\top(X)$ such that $\mathbb{H} \cap [x]_\top \subseteq \mathbb{F}$ and $x \in \lim_X \mathbb{H}$. Since $\varphi : (X, \lim_X) \longrightarrow I(Y, \lim_Y)$ is continuous in $\top$-**Conv**, it follows that there exists $\varphi^\Rightarrow(\mathbb{H}) \in \mathcal{F}_L^\top(Y)$ such that $\varphi^\Rightarrow(\mathbb{H}) \cap [\varphi(x)]_\top \subseteq \varphi^\Rightarrow(\mathbb{F})$ and $\varphi(x) \in \lim_Y \varphi^\Rightarrow(\mathbb{H})$. Since $(Y, \lim_Y) \in ob(\top\textbf{-KConv})$, we have

$$\varphi(x) \in \lim_Y \varphi^\Rightarrow(\mathbb{H}) \subseteq \lim_Y(\varphi^\Rightarrow(\mathbb{H}) \cap [\varphi(x)]_\top) \subseteq \lim_Y \varphi^\Rightarrow(\mathbb{F}).$$

This proves that $\varphi : (X, \lim_X^*) \longrightarrow (Y, \lim_Y)$ is continuous in $\top$-**KConv**. Thus, $id_X : (X, \lim_X) \longrightarrow I \circ K(X, \lim_X)$ is initial in $(X \Rightarrow I)$.

By Theorem 2.2, we obtain K is a left adjoint to I. □

Theorem 3.4. $\top$-**KConv** *is a bireflective subcategory of* $\top$-**Conv**.

Proof. Since $id_X : (X, \lim_X) \longrightarrow I \circ K(X, \lim_X)$ is a bijection and $\top$-**KConv** is a full and isomorphism-closed subcategory of $\top$-**Conv**. By Definition 2.4, we know $\top$-**KConv** is a bireflective subcategory of $\top$-**Conv**. □

Next we show the bicoreflectivity of $\top$-**KConv** in $\top$-**Conv**.

Proposition 3.5. *For each* $(X, \lim) \in ob(\top\textbf{-Conv})$*, define* $\lim_* : \mathcal{F}_L^\top(X) \longrightarrow \mathcal{P}(X)$ *by*

$$\lim_* \mathbb{F} = \{x \in X \mid x \in \lim(\mathbb{F} \cap [x]_\top)\} \quad \forall \mathbb{F} \in \mathcal{F}_L^\top(X).$$

Then $(X, \lim_*) \in ob(\top\text{-}\mathbf{KConv})$.

Proof. It suffices to verify that $\lim_*$ satisfies (TC1), (TC2) and (TCK). (TC1) and (TC2) are straightforward.

(TCK) Take any $x \in \lim_* \mathbb{F}$. Then $x \in \lim(\mathbb{F} \cap [x]_\top) = \lim(\mathbb{F} \cap [x]_\top \cap [x]_\top)$. Thus, we have $x \in \lim_*(\mathbb{F} \cap [x]_\top)$. By the arbitrariness of x, we obtain $\lim_* \mathbb{F} \subseteq \lim_*(\mathbb{F} \cap [x]_\top)$. □

Proposition 3.6. *If* $\varphi : (X, \lim^X) \longrightarrow (Y, \lim^Y)$ *between* $\top$*-convergence spaces is continuous, then* $\varphi : (X, \lim_*^X) \longrightarrow (Y, \lim_*^Y)$ *between* Kent $\top$*-convergence spaces is continuous.*

Proof. Take each $x \in X$ and $\mathbb{F} \in \mathcal{F}_L^\top(X)$ such that $x \in \lim_*^X \mathbb{F}$. Then we have $x \in \lim^X(\mathbb{F} \cap [x]_\top)$. Since $\varphi : (X, \lim^X) \longrightarrow (Y, \lim^Y)$ is continuous, we obtain

$$\varphi(x) \in \lim^Y \varphi^\Rightarrow(\mathbb{F} \cap [x]_\top) = \lim^Y(\varphi^\Rightarrow(\mathbb{F}) \cap [\varphi(x)]_\top).$$

Then it follows that $\varphi(x) \in \lim_*^Y \varphi^\Rightarrow(\mathbb{F})$. Thus, $\varphi : (X, \lim_*^X) \longrightarrow (Y, \lim_*^Y)$ is continuous. □

Therefore we obtain a functor.

$$K_* : \begin{cases} \top\text{-}\mathbf{Conv} & \longrightarrow & \top\text{-}\mathbf{KConv} \\ (X, \lim) & \longmapsto & (X, \lim_*) \\ \varphi & \longmapsto & \varphi \end{cases}$$

Proposition 3.7. K_* *is a right adjoint to* I.

Proof. It is enough to show that $\{id_X\}_X$ is a natural transformation $I \circ K_* \longrightarrow 1_{\top\text{-}\mathbf{Conv}}$ and $\epsilon_X = id_X : I \circ K_*(X, \lim^X) \longrightarrow (X, \lim^X)$ is terminal in $(I \Rightarrow X)$ for each $(X, \lim^X) \in ob(\top\text{-}\mathbf{Conv})$.

(1) For each $(X, \lim^X) \in ob(\top\text{-}\mathbf{Conv})$, take any $x \in X$ and $\mathbb{F} \in \mathcal{F}_L^\top(X)$ such that $x \in \lim_*^X \mathbb{F}$. Since $x \in \lim^X(\mathbb{F} \cap [x]_\top) \subseteq \lim^X \mathbb{F}$, we have $x \in \lim^X \mathbb{F}$. This shows $id_X : I \circ K_*(X, \lim^X) \longrightarrow (X, \lim^X)$ is continuous. Moreover, for each continuous mapping $\varphi : (X, \lim^X) \longrightarrow (Y, \lim^Y)$ in $\top$-**Conv**, the square

$$\begin{tikzcd} I \circ K_*(X, \lim^X) \arrow[r, "I \circ K_*(\varphi) = \varphi"] \arrow[d, "id_X"'] & I \circ K_*(Y, \lim^Y) \arrow[d, "id_Y"] \\ (X, \lim^X) \arrow[r, "\varphi"] & (Y, \lim^Y) \end{tikzcd}$$

commutes. This shows $\{id_X\}_X$ is a natural transformation $I \circ K_* \longrightarrow 1_{\top\text{-}\mathbf{Conv}}$.

(2) Let $((Y, \lim^Y), \varphi)$ be an object in $(I \Rightarrow X)$. It suffices to show that there is exactly one morphism from $((Y, \lim^Y), \varphi)$ to $(K_*(X, \lim^X), id_X)$ in $(I \Rightarrow X)$. We point out that a morphism from $((Y, \lim^Y), \varphi)$ to $(K_*(X, \lim^X), id_X)$ is a continuous mapping ψ from $(Y, \lim^Y)$ to $K_*(X, \lim^X) = (X, \lim_*^X)$ in $\top$-**KConv** such that the triangle

$$\begin{tikzcd} I(Y, \lim_Y) \arrow[r, "I(\psi) = \psi"] \arrow[dr, "\varphi"'] & I \circ K_*(X, \lim_X) \arrow[d, "id_X"] \\ & (X, \lim_X) \end{tikzcd}$$

commutes. Then it follows that $\psi = \varphi$. Next we will show $\varphi : (Y, \lim^Y) \longrightarrow K_*(X, \lim^X)$ is continuous.

Take any $x \in Y$ and $\mathbb{F} \in \mathcal{F}_L^\top(Y)$ such that $x \in \lim^Y \mathbb{F}$. Since $(Y, \lim^Y) \in ob(\top\text{-}\mathbf{KConv})$, we have $x \in \lim^Y(\mathbb{F} \cap [x]_\top)$. Since $\varphi : (Y, \lim_Y) \longrightarrow (X, \lim_X)$ is continuous, we get

$\varphi(x) \in \lim^X \varphi^{\Rightarrow}(\mathbb{F} \cap [x]_\top) = \lim^X(\varphi^{\Rightarrow}(\mathbb{F}) \cap [\varphi(x)]_\top)$. This shows $\varphi(x) \in \lim_*^X \varphi^{\Rightarrow}(\mathbb{F})$. Thus, $id_X : I \circ K_*(X, \lim^X) \longrightarrow (X, \lim^X)$ is terminal in $(I \Rightarrow X)$ for each $(X, \lim^X) \in ob(\top\text{-}\mathbf{Conv})$.

By Theorem 2.2, we obtain K_* is a right adjoint to I, as desired. □

By Proposition 3.7 and Definition 2.4, we get the following conclusion.

Theorem 3.8. $\top$-**KConv** *is a bicoreflective subcategory of* $\top$-**Conv**.

By Theorem 3.8 and Proposition 2.6, we obtain the following theorems.

Theorem 3.9. $\top$-**KConv** *is a topological category.*

Theorem 3.10. *If L is a complete MV-algebra, then* $\top$-**KConv** *is Cartesian closed.*

Proof. By Theorem 3.4, we know that $\top$-**KConv** is closed under formation of finite products in $\top$-**Conv**. Then by Theorem 3.8 and Proposition 2.6, we obtain that $\top$-**KConv** is Cartesian closed. Here we only provide the concrete form of its power objects. Let $(X, \lim_X), (Y, \lim_Y) \in ob\,(\top\text{-}\mathbf{KConv})$. Define $\lim_C : \mathcal{F}_L^\top(C_\top(X,Y)) \longrightarrow \mathcal{P}(C_\top(X,Y))$ by

$$\lim_C \mathbb{H} = \left\{ \varphi \in C_\top(X,Y) \;\middle|\; \begin{array}{l} \forall x \in X \text{ and } \mathbb{F} \in \mathcal{F}_L^\top(X), x \in \lim_X \mathbb{F} \\ \Longrightarrow \varphi(x) \in \lim_Y ev_{X,Y}^{\Rightarrow}[(\mathbb{H} \cap [\varphi]_\top) \times \mathbb{F}] \end{array} \right\} \quad \forall \mathbb{H} \in \mathcal{F}_L^\top(C_\top(X,Y)).$$

Then $(C_\top(X,Y), \lim_C)$ is the power object in $\top$-**KConv** with respect to $(X, \lim_X)$ and $(Y, \lim_Y)$. □

3.2. $\top$-limit spaces

A $\top$-convergence space $(X, \lim)$ is said to be $\top$-limit provided that

$$\text{(TCL)} \quad \lim \mathbb{F} \cap \lim \mathbb{G} = \lim(\mathbb{F} \cap \mathbb{G}) \quad \forall \mathbb{F},\ \mathbb{G} \in \mathcal{F}_L^\top(X).$$

By (TCL), we get that $x \in \lim \mathbb{F}$ implies $x \in \lim \mathbb{F} \cap \lim [x]_\top = \lim(\mathbb{F} \cap [x]_\top)$ for all $x \in X$ and $\mathbb{F} \in \mathcal{F}_L^\top(X)$. Thus, we obtain the full subcategory of $\top$-**KConv** consisting of $\top$-limit spaces, which is denoted by $\top$-**LConv**. We use $I : \top\text{-}\mathbf{LConv} \longrightarrow \top\text{-}\mathbf{KConv}$ to denote the inclusion functor.

Proposition 3.11. *For each* $(X, \lim) \in ob(\top\text{-}\mathbf{KConv})$*, define* $\lim^* : \mathcal{F}_L^\top(X) \longrightarrow \mathcal{P}(X)$ *by*

$$\lim^* \mathbb{F} = \left\{ x \in X \mid \exists \mathbb{F}_i \in \mathcal{F}_L^\top(X), i = 1, \cdots, n \text{ such that } \bigcap_{i=1}^n \mathbb{F}_i \subseteq \mathbb{F} \text{ and } \forall i, x \in \lim \mathbb{F}_i \right\} \quad \forall \mathbb{F} \in \mathcal{F}_L^\top(X).$$

Then $(X, \lim^*) \in ob(\top\text{-}\mathbf{LConv})$.

Proof. It suffices to verify that $\lim^*$ satisfies (TC1), (TC2) and (TCL). (TC1) and (TC2) are obvious. We only verify (TCL).

(TCL) It is enough to show $\lim^* \mathbb{F} \cap \lim^* \mathbb{G} \subseteq \lim^*(\mathbb{F} \cap \mathbb{G})$. Take any $x \in \lim^* \mathbb{F} \cap \lim^* \mathbb{G}$. Then there exists $\mathbb{F}_i \in \mathcal{F}_L^\top(X), \mathbb{G}_j \in \mathcal{F}_L^\top(X), i = 1, \cdots, n$ and $j = 1, \cdots, m$ such that $\bigcap_{i=1}^n \mathbb{F}_i \subseteq \mathbb{F}, x \in \lim \mathbb{F}_i$ and $\bigcap_{j=1}^m \mathbb{G}_j \subseteq \mathbb{G}, x \in \lim \mathbb{G}_j$. Let $q = i + j$. There exists $\mathbb{H}_p, p = 1, \cdots, q$ such that $\bigcap_{p=1}^q \mathbb{H}_p \subseteq \mathbb{F} \cap \mathbb{G}$ and $x \in \lim \mathbb{H}_p$. Then it follows that $x \in \lim^*(\mathbb{F} \cap \mathbb{G})$. By the arbitrariness of x, we obtain $\lim^* \mathbb{F} \cap \lim^* \mathbb{G} \subseteq \lim^*(\mathbb{F} \cap \mathbb{G})$. □

Proposition 3.12. *If* $\varphi : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ *between* Kent $\top$*-convergence spaces is continuous, then* $\varphi : (X, \lim_X^*) \longrightarrow (Y, \lim_Y^*)$ *between* $\top$-limit *spaces is continuous.*

Proof. Take any $x \in X$ and $\mathbb{F} \in \mathcal{F}_L^\top(X)$ such that $x \in \lim_X^* \mathbb{F}$. Then there exists $\mathbb{F}_i \in \mathcal{F}_L^\top(X), i = 1, \cdots, n$ such that $\bigcap_{i=1}^n \mathbb{F}_i \subseteq \mathbb{F}$ and $x \in \lim_X \mathbb{F}_i$. Since $\varphi : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ is continuous, we have $\varphi(x) \in \lim_Y \varphi^{\Rightarrow}(\mathbb{F}_i)$. Further, we obtain $\varphi^{\Rightarrow}(\mathbb{F}_i) \in \mathcal{F}_L^\top(Y)$, $i = 1, \cdots, n$ such that $\bigcap_{i=1}^n \varphi^{\Rightarrow}(\mathbb{F}_i) \subseteq \varphi^{\Rightarrow}(\mathbb{F})$ and $\varphi(x) \in \lim_Y \varphi^{\Rightarrow}(\mathbb{F}_i)$. By the definition of $\lim^*$, we have $\varphi(x) \in \lim_Y^* \varphi^{\Rightarrow}(\mathbb{F})$. This proves that $\varphi : (X, \lim_X^*) \longrightarrow (Y, \lim_Y^*)$ is continuous. □

Thus, we get a functor.

$$M^*\colon \begin{cases} \top\text{-}\mathbf{KConv} & \longrightarrow & \top\text{-}\mathbf{LConv} \\ (X, \lim) & \longmapsto & (X, \lim^*) \\ \varphi & \longmapsto & \varphi \end{cases}$$

Proposition 3.13. *M^* is a left adjoint to I.*

Proof. The proof of Proposition 3.2 can be adopted. □

By Definition 2.4 and Propositions 2.7 and 3.13, we obtain the following theorems.

Theorem 3.14. *$\top$-**LConv** is a bireflective subcategory of $\top$-**KConv**.*

Theorem 3.15. *$\top$-**LConv** is a topological category.*

Proposition 3.16 ([33])**.** *If L is distributive, then $(\mathbb{F} \cap \mathbb{G}) \times \mathbb{H} = (\mathbb{F} \times \mathbb{H}) \cap (\mathbb{G} \times \mathbb{H})$ for all $\mathbb{F}$, $\mathbb{G} \in \mathcal{F}_L^\top(X)$ and $\mathbb{H} \in \mathcal{F}_L^\top(Y)$.*

Theorem 3.17. *$\top$-**LConv** is Cartesian closed when L is a complete $MV-$algebra.*

Proof. By Theorems 3.4 and 3.14, we know that $\top$-**LConv** is a bireflective subcategory of $\top$-**Conv**. Then by Theorem 2.15 and Proposition 2.7, it is enough to show $\top$-**LConv** is closed under formation of power objects in $\top$-**Conv**. Let $(X, \lim_X) \in ob(\top\text{-}\mathbf{Conv})$ and $(Y, \lim_Y) \in ob(\top\text{-}\mathbf{LConv})$. Then for each $\mathbb{H} \in \mathcal{F}_L^\top(C_\top(X,Y))$,

$$\lim_C \mathbb{H} = \left\{\varphi \in C_\top(X,Y) \mid \forall x \in X, \forall \mathbb{F} \in \mathcal{F}_L^\top(X), x \in \lim_X \mathbb{F} \Longrightarrow \varphi(x) \in \lim_Y ev_{X,Y}^{\Rightarrow}(\mathbb{H} \times \mathbb{F})\right\}.$$

We need to check that $\lim_C \mathbb{H}$ satisfies (TCL). Let $\varphi \in \lim_C \mathbb{H}_1 \cap \lim_C \mathbb{H}_2$. Take any $x, y \in X$ and $\mathbb{F}, \mathbb{G} \in \mathcal{F}_L^\top(X)$. If $x \in \lim_X \mathbb{F}$ and $y \in \lim_X \mathbb{G}$, then $\varphi(x) \in \lim_Y ev_{X,Y}^{\Rightarrow}(\mathbb{H}_1 \times \mathbb{F})$ and $\varphi(y) \in \lim_Y ev_{X,Y}^{\Rightarrow}(\mathbb{H}_2 \times \mathbb{G})$. Take $x = y$ and $\mathbb{F} = \mathbb{G}$. Then

$$\varphi(x) \in \lim_Y ev_{X,Y}^{\Rightarrow}(\mathbb{H}_1 \times \mathbb{F}) \cap \lim_Y ev_{X,Y}^{\Rightarrow}(\mathbb{H}_2 \times \mathbb{F}).$$

Since $(Y, \lim_Y)$ satisfies (TCL), we obtain

$$\varphi(x) \in \lim_Y ev_{X,Y}^{\Rightarrow}((\mathbb{H}_1 \times \mathbb{F}) \cap (\mathbb{H}_2 \times \mathbb{F})).$$

By Proposition 3.16, we know $(\mathbb{H}_1 \cap \mathbb{H}_2) \times \mathbb{F} = (\mathbb{H}_1 \times \mathbb{F}) \cap (\mathbb{H}_2 \times \mathbb{F})$. Then it follows that

$$\varphi(x) \in \lim_Y ev_{X,Y}^{\Rightarrow}((\mathbb{H}_1 \cap \mathbb{H}_2) \times \mathbb{F}).$$

This shows $\varphi \in \lim_C(\mathbb{H}_1 \cap \mathbb{H}_2)$. Thus, $\lim_C \mathbb{H}$ satisfies (TCL). □

3.3. Principal $\top$-convergence spaces

Definition 3.18 ([30])**.** Let $\mathcal{U} = \{\mathbb{U}^x\}_{x \in X}$ be a family of $\top$-filters. Then $\mathcal{U}$ is called a system of $\top$-neighborhoods on X provided that $\mathcal{U}$ satisfies

$$\text{(N)} \quad B(x) = \top \quad \forall x \in X, \forall B \in \mathbb{U}^x.$$

For each $\top$-convergence sapce $(X, \lim)$, we denote $\mathcal{U}_{\lim} = \left\{\mathbb{U}_{\lim}^x\right\}_{x \in X}$, where $\mathbb{U}_{\lim}^x = \bigcap\left\{\mathbb{F} \in \mathcal{F}_L^\top(X) \mid x \in \lim \mathbb{F}\right\}$. Then $\mathcal{U}_{\lim}$ is a system of $\top$-neighborhoods.

A $\top$-convergence lim on X is said to be principal if $\mathcal{U}_{\lim} = \left\{\mathbb{U}_{\lim}^x\right\}_{x \in X}$ of $\top$-neighborhoods satisfies

$$\text{(TCP)} \quad x \in \lim \mathbb{U}_{\lim}^x \quad \forall x \in X.$$

The pair $(X, \lim)$ is called a principal $\top$-convergence space.

Proposition 3.19. *Let $(X, \lim)$ be a $\top$-convergence space. The following conditions are equivalent.*

(TCP) $x \in \lim \mathbb{U}_{\lim}^x \quad \forall x \in X$.
(TCP$'$) $x \in \lim \mathbb{F} \Longleftrightarrow \mathbb{U}_{\lim}^x \subseteq \mathbb{F} \quad \forall \mathbb{F} \in \mathcal{F}_L^\top(X)$.
(TCP$''$) $\lim \bigcap_{j \in J} \mathbb{F}_j = \bigcap_{j \in J} \lim \mathbb{F}_j \quad \forall \{\mathbb{F}_j\}_{j \in J} \subseteq \mathcal{F}_L^\top(X)$.

Proof. (TCP) $\Longrightarrow$ (TCP$'$) Straightforward.

(TCP$'$) $\Longrightarrow$ (TCP$''$) It is enough to show $\bigcap_{j\in J}\lim \mathbb{F}_j \subseteq \lim \bigcap_{j\in J}\mathbb{F}_j$. Take any $x \in \bigcap_{j\in J}\lim \mathbb{F}_j$. Then $x \in \lim \mathbb{F}_j$ for all $j \in J$. Thus, $\mathbb{U}^x_{\lim} \subseteq \bigcap_{j\in J}\lim \mathbb{F}_j$. By (TCP$'$), we have $x \in \bigcap_{j\in J}\lim \mathbb{F}_j$. By the arbitrariness of x, we obtain $\bigcap_{j\in J}\lim \mathbb{F}_j \subseteq \lim \bigcap_{j\in J}\mathbb{F}_j$.

(TCP$''$) $\Longrightarrow$ (TCP) It follows from

$$x \in \bigcap \{\lim \mathbb{F} \in \mathcal{P}(X) \,|\, x \in \lim \mathbb{F}\} = \lim \bigcap \left\{\mathbb{F} \in \mathcal{F}^{\top}_L(X) \,|\, x \in \lim \mathbb{F}\right\} = \lim \mathbb{U}^x_{\lim}. \qquad \square$$

By (TCP$''$), we know that every principal $\top$-convergence space is a $\top$-limit space. Then the full subcategory of $\top$-**LConv** consisting of principal $\top$-convergence spaces is denoted by $\top$-**PConv**. We use $I : \top$-**PConv** $\longrightarrow \top$-**LConv** to denote the inclusion functor.

Proposition 3.20. *For each* $(X, \lim) \in ob(\top\text{-}\mathbf{LConv})$, *define* $\lim^* : \mathcal{F}^{\top}_L(X) \longrightarrow \mathcal{P}(X)$ *by*

$$\lim{}^* \mathbb{F} = \{x \in X \,|\, \mathbb{U}^x_{\lim} \subseteq \mathbb{F}\} \qquad \forall \mathbb{F} \in \mathcal{F}^{\top}_L(X).$$

Then $(X, \lim^*) \in ob(\top\text{-}\mathbf{PConv})$.

Proof. It suffices to verity that $\lim^*$ satisfies (TC1), (TC2) and (TCP). (TC1) and (TC2) are straightforward. Next we check (TCP). By the definition of $\top$-neighborhood, we have

$$\mathbb{U}^x_{\lim^*} = \bigcap \left\{\mathbb{F} \in \mathcal{F}^{\top}_L(X) \,|\, x \in \lim{}^* \mathbb{F}\right\} = \bigcap \left\{\mathbb{F} \in \mathcal{F}^{\top}_L(X) \,|\, \mathbb{U}^x_{\lim} \subseteq \mathbb{F}\right\} = \mathbb{U}^x_{\lim}.$$

Then it follows that $x \in \lim^* \mathbb{U}^x_{\lim^*}$. $\square$

Proposition 3.21. *If* $\varphi : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ *between* $\top$-limit *spaces is continuous, then* $\varphi : (X, \lim^*_X) \longrightarrow (Y, \lim^*_Y)$ *between principal* $\top$*-convergence spaces is continuous.*

Proof. Let $\mathbb{F} \in \mathcal{F}^{\top}_L(X)$ and $x \in X$. Take any $x \in \lim^*_X \mathbb{F}$. Then

$$\bigcap \left\{\mathbb{F} \in \mathcal{F}^{\top}_L(X) \,|\, x \in \lim_X \mathbb{F}\right\} = \mathbb{U}^x_{\lim_X} \subseteq \mathbb{F}.$$

Since $\varphi : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ is continuous, we get

$$\bigcap \left\{\varphi^{\Rightarrow}(\mathbb{F}) \in \mathcal{F}^{\top}_L(Y) \,|\, \varphi(x) \in \lim_Y \varphi^{\Rightarrow}(\mathbb{F})\right\} \subseteq \varphi^{\Rightarrow}\left(\bigcap \left\{\mathbb{F} \in \mathcal{F}^{\top}_L(X) \,|\, x \in \lim_X \mathbb{F}\right\}\right) \subseteq \varphi^{\Rightarrow}(\mathbb{F}).$$

This implies that $\mathbb{U}^{\varphi(x)}_{\lim_Y} \subseteq \varphi^{\Rightarrow}(\mathbb{F})$. Thus, we obtain $\varphi(x) \in \lim^*_Y \varphi^{\Rightarrow}(\mathbb{F})$. $\square$

By Propositions 3.20 and 3.21, we obtain a functor.

$$P^* : \left\{ \begin{array}{ccc} \top\text{-}\mathbf{LConv} & \longrightarrow & \top\text{-}\mathbf{PConv} \\ (X, \lim) & \longmapsto & (X, \lim^*) \\ \varphi & \longmapsto & \varphi \end{array} \right.$$

Further, we can draw the following results.

Proposition 3.22. P^* *is a left adjoint to* I.

Theorem 3.23. $\top$-**PConv** *is a bireflective subcategory of* $\top$-**LConv**.

Theorem 3.24. $\top$-**PConv** *is a topological category.*

When $L = 2$, where 2 denotes the two-point chain $\{0, 1\}$, the category $\top$-**PConv** will reduce to the category **PrTop** of principle generalized convergence spaces. Since **PrTop** is not Cartesian closed [26], we know that $\top$-**PConv** is not always Cartesian closed for every complete MV-algebra L. However, it is unknown if there is a lattice $L \neq 2$ such that $\top$-**PConv** is Cartesian closed.

3.4. Strong L-topological $\top$-convergence spaces

A $\top$-convergence lim on X is said to be strong L-topological provided that $\{\mathbb{U}_{\lim}^x\}_{x\in X}$ satisfies (TCP) and (TT)

(TT) For each $x \in X$ and each $B \in \mathbb{U}_{\lim}^x$, there exists $B^* \in \mathbb{U}_{\lim}^x$ with $B^* \le B$ such that for each $y \in X$, there exists $B_y \in \mathbb{U}_{\lim}^y$ satisfying $B^*(y) \le \mathcal{S}_X(B_y, B)$.

The full subcategory of $\top$-**PConv** consisting of strong L-topological $\top$-convergence spaces is denoted by $\top$-**STConv**.

Yu and Fang [30] proved that the category $\top$-**STConv** is concretely isomorphic to the category SL-**Top**, whose objects are strong L-topological spaces (X,τ) and morphsims are continuous mappings $\varphi : (X,\tau_X) \longrightarrow (Y,\tau_Y)$.

Recall that for each $(X,\tau) \in ob(SL\text{-}\mathbf{Top})$ and $x \in X$, the $\top$-neighborhood with respect to x in (X,τ) is defined by

$$\mathbb{U}_\tau^x = \left\{B \in L^X \mid \bigvee_{U\in\tau} U(x) * \mathcal{S}_X(U,B) = \top\right\}.$$

Adopting the results presented by Yu and Fang (see Section 4 in [30]), we can easily distill the following propositions and omit the proofs.

Proposition 3.25. *Let* $(X,\tau) \in ob(SL\text{-}\mathbf{Top})$. *Define* $\lim_\tau : \mathcal{F}_L^\top(X) \longrightarrow \mathcal{P}(X)$ *by*

$$\lim_\tau \mathbb{F} = \{x \in X \mid \mathbb{U}_\tau^x \subseteq \mathbb{F}\} \quad \forall \mathbb{F} \in \mathcal{F}_L^\top(X).$$

Then $(X,\lim_\tau) \in ob(\top\text{-}\mathbf{PConv})$.

Proposition 3.26. *Let* $(X,\lim) \in ob(\top\text{-}\mathbf{PConv})$. *Define* $\tau_{\lim} \subseteq L^X$ *by*

$$\tau_{\lim} = \left\{U \in L^X \mid U(x) \le \bigvee_{B\in\mathbb{U}_{\lim}^x} \mathcal{S}_X(B,U), \forall x \in X\right\}.$$

Then $(X,\tau_{\lim}) \in ob(SL\text{-}\mathbf{Top})$.

Proposition 3.27. *Let* $(X \lim_X), (Y,\lim_Y) \in ob(SL\text{-}\mathbf{Top})$ *and* $(X,\tau_X),(Y,\tau_Y) \in ob(\top\text{-}\mathbf{PConv})$. *Then the following conditions hold.*

(1) *If* $\varphi : (X,\lim_X) \longrightarrow (Y,\lim_Y)$ *is continuous, then* $\varphi : (X,\tau_{\lim_X}) \longrightarrow (Y,\tau_{\lim_Y})$ *is continuous.*

(2) *If* $\varphi : (X,\tau_X) \longrightarrow (Y,\tau_Y)$ *is continuous, then* $\varphi : (X,\lim_{\tau_X}) \longrightarrow (Y,\lim_{\tau_Y})$ *is continuous.*

Thus, we get two functors.

$$I : \begin{cases} SL\text{-}\mathbf{Top} & \longrightarrow & \top\text{-}\mathbf{PConv} \\ (X,\tau) & \longmapsto & (X,\lim_\tau) \\ \varphi & \longmapsto & \varphi \end{cases} \text{ and } T : \begin{cases} \top\text{-}\mathbf{PConv} & \longrightarrow & SL\text{-}\mathbf{Top} \\ (X,\lim) & \longmapsto & (X,\tau_{\lim}) \\ \varphi & \longmapsto & \varphi \end{cases}$$

Proposition 3.28. T *is a left adjoint to* I.

Proof. As shown in [30], we know $(X,\tau_{\lim_\tau}) = (X,\tau)$ for each $(X,\tau) \in ob(SL\text{-}\mathbf{Top})$ and $\mathbb{U}_{\tau_{\lim}}^x \subseteq \mathbb{U}_{\lim}^x$ for all $x \in X$. Thus, we obtain $T \circ I = 1_{SL\text{-}\mathbf{Top}}$ and $1_{\top\text{-}\mathbf{PConv}} \subseteq I \circ T$. This proves that (T,I) is an adjunction. □

Theorem 3.29. $\top$-**STConv** *is a bireflective subcategory of* $\top$-**PConv**.

Proof. By Proposition 3.28, we know SL-**Top** is isomorphic to a reflctive subcategory of $\top$-**PConv**. As shown in [30], $\top$-**STConv** is isomorphic to SL-**Top**. Hence, we obtain $\top$-**STConv** is a reflective subcategory of $\top$-**PConv**. Further, it is easy to see that $\top$-**STConv** is bireflective in $\top$-**PConv**. □

Theorem 3.30. $\top$-**STConv** *is a topological category.*

When $L = 2$, the category ⊤-**STConv** is isomorphic to the category **Top** of topological spaces that is not Cartesian closed [27]. So ⊤-**STConv** is not always Cartesian closed for every complete MV-algebra L. However, it is also unknown if there is a lattice $L \neq 2$ such that ⊤-**STConv** is Cartesian closed.

Remark 3.31. The following graph summaries the results of this section.

$$\top\text{-}\mathbf{STConv} \xrightarrow{r} \top\text{-}\mathbf{PConv} \xrightarrow{r} \top\text{-}\mathbf{LConv} \xrightarrow{r} \top\text{-}\mathbf{KConv} \xrightarrow{r,c} \top\text{-}\mathbf{Conv}$$

4. Change of base

In this section, we will discuss the categorical relationships between ⊤-convergence spaces by changing of the underlying lattice.

Let $L_1 = (L_1, \leq, *)$ and $L_2 = (L_2, \leq, *)$ be two complete residuated lattices. Suppose that $h : L_1 \longrightarrow L_2$ and $k : L_2 \longrightarrow L_1$ are two mappings satisfying the following properties.

(L1) $h(\bot_1) = \bot_2$, $k(\bot_2) = \bot_1$, $h(\top_1) = \top_2$, $k(\top_2) = \top_1$.

(L2) $k(m \wedge n) = k(m) \wedge k(n)$.

(L3) $h(m \to n) = h(m) \to h(n)$, $h(\bigvee_{i\in I} m_i) = \bigvee_{i\in I} h(m_i)$.

(L4) $k \circ h = id_{L_1}$, $h \circ k \geq id_{L_2}$.

Obviously, the pair (k, h) is a Galois connection, where k is the left adjoint and h is the right adjoint.

Lemma 4.1. *Let $\mathbb{F} \in \mathcal{F}^{\top}_{L_2}(X)$. Then $\mathbb{B}_{\mathbb{F}} = \{k \circ B \in L_1^X \mid B \in \mathbb{F}\}$ is an L_1-⊤-filter base on X.*

Proof. It suffices to show $\mathbb{B}_{\mathbb{F}}$ satisfies ($\mathbb{B}$1) and ($\mathbb{B}$2).

($\mathbb{B}$1) Take any $k \circ B_1, k \circ B_2 \in \mathbb{B}_{\mathbb{F}}$. By (L2), we have

$$(k \circ B_1 \wedge k \circ B_2)(x) = k \circ B_1(x) \wedge k \circ B_2(x) = k \circ (B_1(x) \wedge B_2(x)) = k \circ (B_1 \wedge B_2)(x)$$

for all $x \in X$. By the arbitrariness of x, we get $k \circ (B_1 \wedge B_2) = k \circ B_1 \wedge k \circ B_2$. Since $B_1 \wedge B_2 \in \mathbb{F}$, we obtain $k \circ (B_1 \wedge B_2) \in \mathbb{B}_{\mathbb{F}}$. This shows that $\bigvee_{k\circ B\in\mathbb{B}_{\mathbb{F}}} \mathcal{S}_X(k \circ B, k \circ B_1 \wedge k \circ B_2) = \top_1$.

($\mathbb{B}$2) Let $k \circ B \in \mathbb{B}_{\mathbb{F}}$. Since $B \in \mathbb{F}$ and $k(\top_2) = \top_1$, we have $k\big(\bigvee_{x\in X} B(x)\big) = \top_1$. Then it follows that $\bigvee_{x\in X} k \circ B(x) = k\big(\bigvee_{x\in X} B(x)\big) = \top_1$. □

Lemma 4.2. *Let $\mathbb{G} \in \mathcal{F}^{\top}_{L_1}(X)$. Then $\mathbb{B}_{\mathbb{G}} = \{h \circ D \in L_2^X \mid D \in \mathbb{G}\}$ is an L_2-⊤-filter base on X.*

Proof. Since h is a right adjoint, we know $h(m \wedge n) = h(m) \wedge h(n)$. Then adopting the proof of Lemma 4.1, it is easy to show that $\mathbb{B}_{\mathbb{G}}$ satisfies ($\mathbb{B}$1) and ($\mathbb{B}$2). □

By Lemmas 4.1 and 4.2, we get the following results.

Proposition 4.3. *Let $\mathbb{F} \in \mathcal{F}^{\top}_{L_2}(X)$. Then $\mathbb{F}^k = \{A \in L_1^X \mid \bigvee_{B\in\mathbb{F}} \mathcal{S}_X(k \circ B, A) = \top_1\}$ is an L_1-⊤-filter.*

Proposition 4.4. *Let $\mathbb{G} \in \mathcal{F}^{\top}_{L_1}(X)$. Then $\mathbb{G}^h = \{C \in L_2^X \mid \bigvee_{D\in\mathbb{G}} \mathcal{S}_X(h \circ D, C) = \top_2\}$ is an L_2-⊤-filter.*

Proposition 4.5. (1) *If $\mathbb{F}_1 \subseteq \mathbb{F}_2 \in \mathcal{F}^{\top}_{L_2}(X)$, then $\mathbb{F}_1^k \subseteq \mathbb{F}_2^k$.*

(2) *If $\mathbb{G}_1 \subseteq \mathbb{G}_2 \in \mathcal{F}^{\top}_{L_1}(X)$, then $\mathbb{G}_1^h \subseteq \mathbb{G}_2^h$.*

(3) *If $\mathbb{F} \in \mathcal{F}^{\top}_{L_2}(X)$, then $(\mathbb{F}^k)^h \subseteq \mathbb{F}$.*

(4) *If $\mathbb{G} \in \mathcal{F}^{\top}_{L_1}(X)$, then $(\mathbb{G}^h)^k = \mathbb{G}$.*

(5) *If $\mathbb{F}_1, \mathbb{F}_2 \in \mathcal{F}^{\top}_{L_2}(X)$, then $(\mathbb{F}_1 \cap \mathbb{F}_2)^k = \mathbb{F}_1^k \cap \mathbb{F}_2^k$.*

(6) *If $\mathbb{G}_1, \mathbb{G}_2 \in \mathcal{F}^{\top}_{L_1}(X)$, then $(\mathbb{G}_1 \cap \mathbb{G}_2)^h = \mathbb{G}_1^h \cap \mathbb{G}_2^h$.*

(7) *For $[x]_{\top_2} \in \mathcal{F}^{\top}_{L_2}(X)$, we have $[x]^k_{\top_2} = [x]_{\top_1} \in \mathcal{F}^{\top}_{L_1}(X)$.*

(8) *For $[x]_{\top_1} \in \mathcal{F}^{\top}_{L_1}(X)$, we have $[x]^h_{\top_1} = [x]_{\top_2} \in \mathcal{F}^{\top}_{L_2}(X)$.*

Proof. (1) and (2) are straightforward.

(3) Take each $A \in (\mathbb{F}^k)^h$. Then $\bigvee_{B\in\mathbb{F}^k} \mathcal{S}_X(h\circ B, A) = \top_2$. For each $B \in \mathbb{F}^k$, we have $\bigvee_{C\in\mathbb{F}} \mathcal{S}_X(k\circ C, B) = \top_1$. Since h is a right adjoint, we have $h(\bigwedge_{i\in I} m_i) = \bigwedge_{i\in I} h(m_i)$. Further, by (L1) and (L3), we get

$$\bigvee_{C\in\mathbb{F}} \mathcal{S}_X(h\circ k\circ C, h\circ B) = h\Big(\bigvee_{C\in\mathbb{F}} \mathcal{S}_X(k\circ C, B)\Big) = \top_2.$$

Then it follows that

$$\begin{aligned}\top_2 &= \bigvee_{B\in\mathbb{F}^k} \mathcal{S}_X(h\circ B, A) * \bigvee_{C\in\mathbb{F}} \mathcal{S}_X(h\circ k\circ C, h\circ B)\\ &\le \bigvee_{B\in\mathbb{F}^k} \mathcal{S}_X(h\circ B, A) * \bigvee_{C\in\mathbb{F}} \mathcal{S}_X(C, h\circ B) \quad (h\circ k \ge id_{L_2})\\ &\le \bigvee_{C\in\mathbb{F}} \mathcal{S}_X(C, A),\end{aligned}$$

which implies $A \in \mathbb{F}$. Thus, we obtain $(\mathbb{F}^k)^h \subseteq \mathbb{F}$.

(4) Take each $C \in \mathbb{G}$. By the definition of $\mathbb{G}^h$ and $(\mathbb{G}^h)^k$, it follows immediately that

$$C = k\circ h\circ C \in (\mathbb{G}^h)^k.$$

This shows $\mathbb{G} \subseteq (\mathbb{G}^h)^k$. For the inverse inequality, take any $C \in (\mathbb{G}^h)^k$. Then $\bigvee_{B\in\mathbb{G}^h} \mathcal{S}_X(k\circ B, C) = \top_1$. For each $B \in \mathbb{G}^h$, we have $\bigvee_{D\in\mathbb{G}} \mathcal{S}_X(h\circ D, B) = \top_2$, which implies $k\big(\bigvee_{D\in\mathbb{G}} \mathcal{S}_X(h\circ D, B)\big) = \top_1$. By (L3) and (L4), we have

$$\begin{aligned}\top_1 &= \bigvee_{B\in\mathbb{G}^h} \mathcal{S}_X(k\circ B, C) * k\Big(\bigvee_{D\in\mathbb{G}} \mathcal{S}_X(h\circ D, B)\Big)\\ &= \bigvee_{B\in\mathbb{G}^h} \mathcal{S}_X(k\circ B, C) * \bigvee_{D\in\mathbb{G}} k\big(\mathcal{S}_X(h\circ D, B)\big)\\ &\le \bigvee_{B\in\mathbb{G}^h} \mathcal{S}_X(k\circ B, C) * \bigvee_{D\in\mathbb{G}} \bigwedge_{x\in X} k(h\circ D(x) \to B(x))\\ &\le \bigvee_{B\in\mathbb{G}^h} \mathcal{S}_X(k\circ B, C) * \bigvee_{D\in\mathbb{G}} \bigwedge_{x\in X} k(h\circ D(x) \to h\circ k\circ B(x))\\ &= \bigvee_{B\in\mathbb{G}^h} \mathcal{S}_X(k\circ B, C) * \bigvee_{D\in\mathbb{G}} \bigwedge_{x\in X} k\circ h(D(x) \to k\circ B(x))\\ &= \bigvee_{B\in\mathbb{G}^h} \mathcal{S}_X(k\circ B, C) * \bigvee_{D\in\mathbb{G}} \mathcal{S}_X(D, k\circ B)\\ &\le \bigvee_{D\in\mathbb{G}} \mathcal{S}_X(D, C),\end{aligned}$$

which implies $C \in \mathbb{G}$. This shows $(\mathbb{G}^h)^k \subseteq \mathbb{G}$. Thus, we obtain $\mathbb{G} = (\mathbb{G}^h)^k$.

(5) It is enough to show $\mathbb{F}_1^k \cap \mathbb{F}_2^k \subseteq (\mathbb{F}_1 \cap \mathbb{F}_2)^k$. Take each $A \in \mathbb{F}_1^k \cap \mathbb{F}_2^k$. Then

$$\begin{aligned}\top_1 &= \bigvee_{B_1\in\mathbb{F}_1} \mathcal{S}_X(k\circ B_1, A) * \bigvee_{B_2\in\mathbb{F}_2} \mathcal{S}_X(k\circ B_2, A)\\ &= \bigvee_{B_1\in\mathbb{F}_1} \bigvee_{B_2\in\mathbb{F}_2} \mathcal{S}_X(k\circ B_1, A) * \mathcal{S}_X(k\circ B_2, A)\\ &\le \bigvee_{B_1\in\mathbb{F}_1} \bigvee_{B_2\in\mathbb{F}_2} \mathcal{S}_X(k\circ B_1, A) \wedge \mathcal{S}_X(k\circ B_2, A)\\ &\le \bigvee_{B_1\vee B_2\in\mathbb{F}_1\cap\mathbb{F}_2} \mathcal{S}_X(k\circ B_1 \vee k\circ B_2, A)\\ &= \bigvee_{B_1\vee B_2\in\mathbb{F}_1\cap\mathbb{F}_2} \mathcal{S}_X(k\circ (B_1 \vee B_2), A)\\ &\le \bigvee_{B\in\mathbb{F}_1\cap\mathbb{F}_2} \mathcal{S}_X(k\circ B, A),\end{aligned}$$

which implies $A \in (\mathbb{F}_1 \cap \mathbb{F}_2)^k$. Thus, $(\mathbb{F}_1 \cap \mathbb{F}_2)^k \subseteq \mathbb{F}_1^k \cap \mathbb{F}_2^k$.

(6) The proof of (5) can be adopted.

(7) Take each $A \in [x]_{\top_1}$. Then $h \circ A(x) = \top_2$, i.e., $h \circ A \in [x]_{\top_2}$. It follows that

$$\bigvee_{B \in [x]_{\top_2}} \mathcal{S}_X(k \circ B, A) \geq \mathcal{S}_X(k \circ h \circ A, A) = \top_1.$$

Thus, we obtain $A \in [x]^k_{\top_2}$. This shows $[x]_{\top_1} \subseteq [x]^k_{\top_2}$. Conversely, take each $A \in [x]^k_{\top_2}$. Then $\bigvee_{B \in [x]_{\top_2}} \mathcal{S}_X(k \circ B, A) = \top_1$. This implies

$$\top_1 = \bigvee_{B \in [x]_{\top_2}} \mathcal{S}_X(k \circ B, A) \leq \bigvee_{B \in [x]_{\top_2}} k \circ B(x) \to A(x) = A(x),$$

which means $A \in [x]_{\top_1}$. This shows that $[x]^k_{\top_2} \subseteq [x]_{\top_1}$. Thus, $[x]^k_{\top_2} = [x]_{\top_1}$.

(8) Take each $B \in [x]_{\top_2}$. Define $\{x\}_{\top_1} \in L_1^X$ by $\{x\}_{\top_1}(x) = \top_1$ and $\{x\}_{\top_1}(y) = \bot_1$ when $y \neq x$. Then

$$\bigvee_{D \in [x]_{\top_1}} \mathcal{S}_X(h \circ D, B) \geq \mathcal{S}_X(h \circ \{x\}_{\top_1}, B) = \top_2 \to B(x) = B(x) = \top_2,$$

which implies $B \in [x]^h_{\top_1}$. This shows $[x]_{\top_2} \subseteq [x]^h_{\top_1}$. Conversely, take each $B \in [x]^h_{\top_1}$. Then $\bigvee_{D \in [x]_{\top_1}} \mathcal{S}_X(h \circ D, B) = \top_2$. Thus, we have

$$\top_2 = \bigvee_{D \in [x]_{\top_1}} \mathcal{S}_X(h \circ D, B) \leq \bigvee_{D \in [x]_{\top_1}} h \circ D(x) \to B(x) = \top_2 \to B(x) = B(x),$$

which implies $B \in [x]_{\top_2}$. This shows $[x]^h_{\top_1} \subseteq [x]_{\top_2}$. Thus, $[x]^h_{\top_1} = [x]_{\top_2}$. □

Proposition 4.6. (1) *Let* $\varphi : X \longrightarrow Y$ *be a mapping and* $\mathbb{F} \in \mathcal{F}^{\top}_{L_2}(X)$*. Then* $\varphi^{\Rightarrow}(\mathbb{F}^k) = (\varphi^{\Rightarrow}(\mathbb{F}))^k$*.*

(2) *Let* $\varphi : X \longrightarrow Y$ *be a mapping and* $\mathbb{G} \in \mathcal{F}^{\top}_{L_1}(X)$*. Then* $\varphi^{\Rightarrow}(\mathbb{G}^h) = (\varphi^{\Rightarrow}(\mathbb{G}))^h$*.*

Proof. (1) Take each $B \in (\varphi^{\Rightarrow}(\mathbb{F}))^k$. Then $\bigvee_{D \in \varphi^{\Rightarrow}(\mathbb{F})} \mathcal{S}_Y(k \circ D, B) = \top_1$. Thus, we have

$$\begin{aligned}
\top_1 &= \bigvee_{\varphi^{\leftarrow}(D) \in \mathbb{F}} \mathcal{S}_Y(k \circ D, B) \\
&\leq \bigvee_{\varphi^{\leftarrow}(D) \in \mathbb{F}} \mathcal{S}_X(\varphi^{\leftarrow}(k \circ D), \varphi^{\leftarrow}(B)) \\
&= \bigvee_{\varphi^{\leftarrow}(D) \in \mathbb{F}} \mathcal{S}_X(k \circ \varphi^{\leftarrow}(D), \varphi^{\leftarrow}(B)) \\
&\leq \bigvee_{E \in \mathbb{F}} \mathcal{S}_X(k \circ E, \varphi^{\leftarrow}(B)),
\end{aligned}$$

which implies $\varphi^{\leftarrow}(B) \in \mathbb{F}^k$, i.e., $B \in \varphi^{\Rightarrow}(\mathbb{F}^k)$. This shows $(\varphi^{\Rightarrow}(\mathbb{F}))^k \subseteq \varphi^{\Rightarrow}(\mathbb{F}^k)$. Conversely, take each $B \in \varphi^{\Rightarrow}(\mathbb{F}^k)$. Then $\varphi^{\leftarrow}(B) \in \mathbb{F}^k$, i.e., $\bigvee_{E \in \mathbb{F}} \mathcal{S}_X(k \circ E, \varphi^{\leftarrow}(B)) = \top_1$. Since k is a left adjoint, we have

$$\varphi^{\rightarrow}(k \circ E)(y) = \bigvee_{\varphi(x)=y} k \circ E(x) = k\Big(\bigvee_{\varphi(x)=y} E(x)\Big) = k \circ \varphi^{\rightarrow}(E)(y)$$

for all $x \in X$, $y \in Y$ and $E \in \mathbb{F}$. Thus, we have

$$\begin{aligned}
\top_1 &= \bigvee_{E \in \mathbb{F}} \mathcal{S}_X(k \circ E, \varphi^{\leftarrow}(B)) \\
&= \bigvee_{E \in \mathbb{F}} \mathcal{S}_Y(\varphi^{\rightarrow}(k \circ E), B) \quad \text{(by Lemma 2.10)} \\
&\leq \bigvee_{\varphi^{\rightarrow}(E) \in \varphi^{\Rightarrow}(\mathbb{F})} \mathcal{S}_Y(k \circ \varphi^{\rightarrow}(E), B) \\
&\leq \bigvee_{D \in \varphi^{\Rightarrow}(\mathbb{F})} \mathcal{S}_Y(k \circ D, B),
\end{aligned}$$

which implies $B \in (\varphi^{\Rightarrow}(\mathbb{F}))^k$. This shows $\varphi^{\Rightarrow}(\mathbb{F}^k) \subseteq (\varphi^{\Rightarrow}(\mathbb{F}))^k$. Thus, we obtain $(\varphi^{\Rightarrow}(\mathbb{F}))^k = \varphi^{\Rightarrow}(\mathbb{F}^k)$.

(2) It can be verified in a similar way. □

Next we will study the categorical relationships between L_1-$\top$-**Conv** and L_2-$\top$-**Conv**.

Proposition 4.7. *Let* $(X,\lim)\in ob(L_1\text{-}\top\text{-}\mathbf{Conv})$. *Define* $\lim^k:\mathcal{F}^{\top}_{L_2}(X)\longrightarrow\mathcal{P}(X)$ *by*

$$\lim^k\mathbb{F}=\{x\in X\,|\,x\in\lim\mathbb{F}^k\}\quad\forall\mathbb{F}\in\mathcal{F}^{\top}_{L_2}(X).$$

Then $(X,\lim^k)\in ob(L_2\text{-}\top\text{-}\mathbf{Conv})$.

Proof. It suffices to verify that $(X,\lim^k)$ satisfies (TC1) and (TC2).

(TC1) By Proposition 4.5, $x\in\lim[x]_{\top_1}=\lim[x]^k_{\top_2}$. This shows $x\in\lim^k[x]_{\top_2}$.

(TC2) Let $\mathbb{F}\in\mathcal{F}^{\top}_{L_2}(X)$ and $\mathbb{G}\in\mathcal{F}^{\top}_{L_2}(X)$ such that $\mathbb{F}\subseteq\mathbb{G}$ and $x\in\lim^k\mathbb{F}$. By Proposition 4.5, we have $x\in\lim\mathbb{F}^k\subseteq\lim\mathbb{G}^k$, which implies $x\in\lim^k\mathbb{G}$, as desired. □

Proposition 4.8. *If* $\varphi:(X,\lim_X)\longrightarrow(Y,\lim_Y)$ *between* L_1-$\top$-*convergence spaces is continuous, then* $\varphi:(X,\lim^k_X)\longrightarrow(Y,\lim^k_Y)$ *between* L_2-$\top$-*convergence spaces is continuous.*

Proof. Take each $\mathbb{F}\in\mathcal{F}^{\top}_{L_2}(X)$ and $x\in X$ such that $x\in\lim^k_X\mathbb{F}$. Then $x\in\lim\mathbb{F}^k$. Since $\varphi:(X,\lim_X)\longrightarrow(Y,\lim_Y)$ is continuous, we have $\varphi(x)\in\lim_Y\varphi^{\Rightarrow}(\mathbb{F}^k)$. By Proposition 4.6, it follows that $\varphi(x)\in\lim_Y\varphi^{\Rightarrow}(\mathbb{F})^k$. This implies $\varphi(x)\in\lim^k_Y\varphi^{\Rightarrow}(\mathbb{F})$, as desired. □

Proposition 4.9. *Let* $(X,\lim)\in ob(L_1\text{-}\top\text{-}\mathbf{Conv})$.

(1) *If* $(X,\lim)$ *satisfies* (TCK), *then* $(X,\lim^k)$ *satisfies* (TCK).

(2) *If* $(X,\lim)$ *satisfies* (TCL), *then* $(X,\lim^k)$ *satisfies* (TCL).

Proof. (1) Take any $x\in X$ and $\mathbb{F}\in\mathcal{F}^{\top}_{L_2}(X)$ such that $x\in\lim^k\mathbb{F}$. Then $x\in\lim\mathbb{F}^k$. Since $(X,\lim)$ satisfies (TCK), we have $x\in\lim(\mathbb{F}^k\cap[x]_{\top_1})$. By Proposition 4.5, we get $[x]^k_{\top_2}=[x]_{\top_1}$ and $(\mathbb{F}\cap[x]_{\top_2})^k=\mathbb{F}^k\cap[x]^k_{\top_2}$. Thus, $x\in\lim(\mathbb{F}^k\cap[x]_{\top_1})=\lim(\mathbb{F}\cap[x]_{\top_2})^k$, i.e., $x\in\lim^h(\mathbb{F}\cap[x]_{\top_2})$.

(2) Let $\mathbb{F}\in\mathcal{F}^{\top}_{L_2}(X)$ and $\mathbb{G}\in\mathcal{F}^{\top}_{L_2}(X)$. Take any $x\in\lim^k\mathbb{F}\cap\lim^k\mathbb{G}$. Since $(X,\lim)$ satisfies (TCL) and $\mathbb{F}^k\cap\mathbb{G}^k=(\mathbb{F}\cap\mathbb{G})^k$, we have

$$x\in\lim\mathbb{F}^k\cap\lim\mathbb{G}^k=\lim(\mathbb{F}^k\cap\mathbb{G}^k)=\lim(\mathbb{F}\cap\mathbb{G})^k.$$

By the arbitrariness of x, we obtain $\lim^k\mathbb{F}\cap\lim^k\mathbb{G}\subseteq\lim^k(\mathbb{F}\cap\mathbb{G})$, as desired. □

Paralleling to Propositions 4.7 and 4.8, we can easily obtain the following propositions and omit the proofs.

Proposition 4.10. *Let* $(X,\lim)\in ob(L_2\text{-}\top\text{-}\mathbf{Conv})$. *Define* $\lim^h:\mathcal{F}^{\top}_{L_1}(X)\longrightarrow\mathcal{P}(X)$ *by*

$$\lim^h\mathbb{G}=\{x\in X\,|\,x\in\lim\mathbb{G}^h\}\quad\forall\mathbb{G}\in\mathcal{F}^{\top}_{L_1}(X).$$

Then $(X,\lim^h)\in ob(L_1\text{-}\top\text{-}\mathbf{Conv})$.

Proposition 4.11. *If* $\varphi:(X,\lim_X)\longrightarrow(Y,\lim_Y)$ *between* L_2-$\top$-*convergence spaces is continuous, then* $\varphi:(X,\lim^k_X)\longrightarrow(Y,\lim^k_Y)$ *between* L_1-$\top$-*convergence spaces is continuous.*

Proposition 4.12. *Let* $(X,\lim)\in ob(L_2\text{-}\top\text{-}\mathbf{Conv})$.

(1) *If* $(X,\lim)$ *satisfies* (TCK), *then* $(X,\lim^h)$ *satisfies* (TCK).

(2) *If* $(X,\lim)$ *satisfies* (TCL), *then* $(X,\lim^h)$ *satisfies* (TCL).

By Propositions 4.7, 4.8, 4.10 and 4.11, we obtain two functors as follows.

$$L^k:\left\{\begin{array}{ccc}L_1\text{-}\top\text{-}\mathbf{Conv} & \longrightarrow & L_2\text{-}\top\text{-}\mathbf{Conv}\\ (X,\lim) & \longmapsto & (X,\lim^k)\\ \varphi & \longmapsto & \varphi\end{array}\right.\quad\text{and}\quad L^h:\left\{\begin{array}{ccc}L_1\text{-}\top\text{-}\mathbf{Conv} & \longrightarrow & L_2\text{-}\top\text{-}\mathbf{Conv}\\ (X,\lim) & \longmapsto & (X,\lim^h)\\ \varphi & \longmapsto & \varphi\end{array}\right.$$

Proposition 4.13. L^h *is a right adjoint to* L^k.

Proof. It is enough to show that $\eta = id : L^k \circ L^h \longrightarrow 1_{L_2\text{-}\top\text{-}\mathbf{Conv}}$ is a natural transformation and $id_X : L^k \circ L^h(X, \lim_X) \longrightarrow (X, \lim_X)$ is terminal in $(L^k \Rightarrow X)$ for each $(X, \lim_X) \in ob(L_2\text{-}\top\text{-}\mathbf{Conv})$.

(1) For each $(X, \lim_X) \in ob(L_2\text{-}\top\text{-}\mathbf{Conv})$, take any $x \in X$ and $\mathbb{F} \in \mathcal{F}^{\top}_{L_2}(X)$ such that $x \in (\lim^h_X)^k \mathbb{F}$, i.e., $x \in \lim_X (\mathbb{F}^k)^h$. Since $(\mathbb{F}^k)^h \subseteq \mathbb{F}$, we get $x \in \lim_X \mathbb{F}$. This shows that $id_X : (X, (\lim^h_X)^k) \longrightarrow (X, \lim_X)$ is continuous. Moreover, for each continuous mapping $\varphi : (X, \lim_X) \longrightarrow (Y, \lim_Y)$ in $L_2\text{-}\top\text{-}\mathbf{Conv}$, the square

$$\begin{tikzcd}
(X, (\lim^h_X)^k) \arrow[r, "\varphi"] \arrow[d, "id_X"'] & (Y, (\lim^h_Y)^k) \arrow[d, "id_Y"] \\
(X, \lim_X) \arrow[r, "\varphi"] & (Y, \lim_Y)
\end{tikzcd}$$

commutes.

(2) $id_X : L^k \circ L^h(X, \lim_X) \longrightarrow (X, \lim_X)$ is a terminal object in $(L^k \Rightarrow X)$ means that there is exactly one morphism from $((Y, \lim_Y), \varphi)$ to $(L^h(X, \lim_X), id_X)$ for each $((Y, \lim_Y), \varphi)$ in $(L^k \Rightarrow X)$. A morphism from $((Y, \lim_Y), \varphi)$ to $(L^h(X, \lim_X), id_X)$ is a continuous mapping ψ from $(Y, \lim_Y)$ to $L^h(X, \lim_X) = (X, \lim^h)$ in $L_1\text{-}\top\text{-}\mathbf{Conv}$ such that the triangle

$$\begin{tikzcd}
L^k(Y, \lim_Y) \arrow[r, "L^k(\psi)=\psi"] \arrow[dr, "\varphi"'] & L^k(L^h(X, \lim_X)) \arrow[d, "\eta_X = id_X"] \\
& (X, \lim_X)
\end{tikzcd}$$

commutes. Since $\varphi = \psi \circ id_X$, we have $\psi = \varphi$. This means ψ is unique and $\psi = \varphi$. It remains to prove $\varphi : (Y, \lim_Y) \longrightarrow L^h(X, \lim_X)$ in $L_1\text{-}\top\text{-}\mathbf{Conv}$ is continuous. Take any $y \in Y$ and $\mathbb{G} \in \mathcal{F}^{\top}_{L_1}(Y)$ such that $y \in (\lim^k_Y)^h \mathbb{G}$. Then

$$y \in (\lim^k_Y)^h \mathbb{G} \Longleftrightarrow y \in \lim^k_Y (\mathbb{G}^h) \Longleftrightarrow y \in \lim_Y (\mathbb{G}^h)^k \Longleftrightarrow y \in \lim_Y \mathbb{G}.$$

By the arbitrariness of y and $\mathbb{G}$, it follows that $L^h(L^k(Y, \lim_Y)) = (Y, \lim_Y)$. Since $\varphi : L^k(X, \lim_Y) \longrightarrow (X, \lim_X)$ in $L_2\text{-}\top\text{-}\mathbf{Conv}$ is continuous, we obtain

$$L^h(\varphi) = \varphi : L^h(L^k(Y, \lim_Y)) \longrightarrow L^h(X, \lim_X)$$

is continuous. Hence $\varphi : (Y, \lim_Y) \longrightarrow L^h(X, \lim_X)$ in $L_1\text{-}\top\text{-}\mathbf{Conv}$ is continuous.

By Definition 2.4, we obtain L^h is a right adjoint to L^k. □

Since $L^h(L^k(X, \lim)) = (X, \lim)$ for all $(X, \lim) \in ob(L_1\text{-}\top\text{-}\mathbf{Conv})$, we embed $L_1\text{-}\top\text{-}\mathbf{Conv}$ as a subcategory in $L_2\text{-}\top\text{-}\mathbf{Conv}$. In this case, $L_1\text{-}\top\text{-}\mathbf{Conv}$ is isomorphic to a subcategory of $L_2\text{-}\top\text{-}\mathbf{Conv}$. Hence we get the category $L_2\text{-}cr\top\text{-}\mathbf{Conv}$, whose objects are $(X, \lim^k)$ for all $(X, \lim) \in ob(L_1\text{-}\top\text{-}\mathbf{Conv})$ and morphisms are continuous mappings $\varphi : (X, \lim^k_X) \longrightarrow (Y, \lim^k_Y)$.

Theorem 4.14. (1) $L_2\text{-}cr\top\text{-}\mathbf{Conv}$ *is a bicoreflective subcategory of* $L_2\text{-}\top\text{-}\mathbf{Conv}$.

(2) $L_1\text{-}\top\text{-}\mathbf{Conv}$ *is isomorphic to* $L_2\text{-}cr\top\text{-}\mathbf{Conv}$.

Proof. (1) By the definition of $L_2\text{-}cr\top\text{-}\mathbf{Conv}$, we obtain two functors as follows.

$$I : \left\{ \begin{array}{ccc} L_2\text{-}cr\top\text{-}\mathbf{Conv} & \longrightarrow & L_2\text{-}\top\text{-}\mathbf{Conv} \\ (X, \lim^k) & \longmapsto & (X, \lim^k) \\ \varphi & \longmapsto & \varphi \end{array} \right. \quad \text{and} \quad R : \left\{ \begin{array}{ccc} L_2\text{-}\top\text{-}\mathbf{Conv} & \longrightarrow & L_2\text{-}cr\top\text{-}\mathbf{Conv} \\ (X, \lim) & \longmapsto & (X, (\lim^h)^k) \\ \varphi & \longmapsto & \varphi \end{array} \right.$$

Take any $(X, \lim^k) \in ob(L_2\text{-}cr\top\text{-}\mathbf{Conv})$. Then $R \circ I(X, \lim^k) = (X, ((\lim^k)^h)^k) = (X, \lim^k)$. Let $(X, \lim) \in ob(L_2\text{-}\top\text{-}\mathbf{Conv})$. Then $I \circ R = (X, (\lim^h)^k) \le (X, \lim)$. This proves that (I, R) is an adjunction. Thus, $L_2\text{-}cr\top\text{-}\mathbf{Conv}$ is a bicoreflective subcategory of $L_2\text{-}\top\text{-}\mathbf{Conv}$.

(2) There are two functors.

$$L_c^k : \begin{cases} L_1\text{-}\top\text{-}\mathbf{Conv} & \longrightarrow & L_2\text{-}cr\top\text{-}\mathbf{Conv} \\ (X, \lim) & \longmapsto & (X, \lim^k) \\ \varphi & \longmapsto & \varphi \end{cases} \quad \text{and} \quad L_c^h : \begin{cases} L_2\text{-}cr\top\text{-}\mathbf{Conv} & \longrightarrow & L_1\text{-}\top\text{-}\mathbf{Conv} \\ (X, \lim^k) & \longmapsto & (X, (\lim^k)^h) \\ \varphi & \longmapsto & \varphi \end{cases}$$

It is easy to check that

$$L_c^k \circ L_c^h(X, \lim^k) = (X, ((X, \lim^k)^h)^k) = (X, \lim^k)$$

and

$$L_c^h \circ L_c^k(X, \lim) = (X, (\lim^k)^h) = (X, \lim).$$

This means $L_c^k \circ L_c^h = 1_{L_1\text{-}\top\text{-}\mathbf{Conv}}$ and $L_c^h \circ L_c^k = 1_{L_2\text{-}cr\top\text{-}\mathbf{Conv}}$. By Definition 2.3, $L_1\text{-}\top\text{-}\mathbf{Conv}$ is isomorphic to $L_2\text{-}cr\top\text{-}\mathbf{Conv}$. ☐

Corollary 4.15. *Let $h : L_1 \longrightarrow L_2$ and $k : L_2 \longrightarrow L_1$ be mappings satisfying* (L1)–(L4). *Then*

(1) $L_1\text{-}\top\text{-}\mathbf{KConv}$ *is isomorphic to a bicoreflective subcategory of* $L_2\text{-}\top\text{-}\mathbf{KConv}$;

(2) $L_1\text{-}\top\text{-}\mathbf{LConv}$ *is isomorphic to a bicoreflective subcategory of* $L_2\text{-}\top\text{-}\mathbf{LConv}$.

As an application of Theorem 4.14, we present the categorical relationships between classical convergences spaces and $\top$-convergence spaces. For convenience, let **GConv** [25] denote the category of classical convergence spaces with classical convergence spaces as objects and continuous mappings as morphisms. Then the full subcategory of **GConv** consisting of Kent convergence spaces is denoted by **KConv**, and the full subcategory of **GConv** consisting of limit spaces is denoted by **Lim**.

Corollary 4.16. (1) **GConv** *is isomorphic to a bicoreflective subcategory of* $L_2\text{-}\top\text{-}\mathbf{Conv}$.

(2) **KConv** *is isomorphic to a bicoreflective subcategory of* $L_2\text{-}\top\text{-}\mathbf{KConv}$.

(3) **Lim** *is isomorphic to a bicoreflective subcategory of* $L_2\text{-}\top\text{-}\mathbf{LConv}$.

Proof. Let $L_1 = \{\top, \bot\}$. Define $h : L_1 \longrightarrow L_2$ by $h(\top) = \top_2$, $h(\bot) = \bot_2$ and $k : L_2 \longrightarrow L_1$ by $k(m) = \top$ when $m \ne \bot_2$, $k(n) = \bot$ when $n = \bot_2$. It is easy to see the pair (k, h) satisfies (L1)–(L4). By Theorem 4.14 and Corollary 4.15, we obtain (1)–(3). ☐

5. Conclusions

In this paper, we studied the categorical relationships between various subcategories of $\top$-convergence spaces. We got $\top$-**LConv**, $\top$-**PConv** and $\top$-**STConv** are bireflective subcategories of $\top$-**Conv** and $\top$-**KConv** is a bicoreflective and bireflective subcategory of $\top$-**Conv**. Further, we showed that $\top$-**KConv** and $\top$-**LConv** are Cartesian closed, and $\top$-**KConv**, $\top$-**LConv**, $\top$-**PConv** and $\top$-**STConv** are topological categories. Moreover, we investigated the categorical relationships between different $\top$-convergence spaces by changing the underlying lattice. In the future, we will consider the following problems:

- Choquet convergence structure is also an important type of generalized convergence structures. This motivates us to define $\top$-ultrafilters and to introduce the concept of $\top$-Choquet convergence structures via $\top$-ultrafilters.
- As a further application of $\top$-ultrafilters, we can define compactness of a $\top$-convergence space. Motivated by the compactification of stratified L-generalized convergence spaces [12], we will also consider the compactification of $\top$-convergence spaces.

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