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Periodic Korovkin Theorem via P_p^2-Statistical $\mathscr{A}$-Summation Process

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Abstract

In the current research, we investigate and establish Korovkin-type approximation theorems for linear operators defined on the space of all 2π-periodic and real valued continuous functions on $\mathbb{R}^2$ by means of $\mathscr{A}$-summation process via statistical convergence with respect to power series method. We demonstrate with an example how our theory is more strong than previously studied. Additionally, we research the rate of convergence of positive linear operators defined on this space.

1. Introduction and Preliminaries Notations

Before starting with the presentation of the definitions which will be used to prove approximation theorems, we recall the well-known notions.

A double sequence $x = (x_{ij})$ is convergent to L in Pringsheim's sense if, for every $\varepsilon > 0$, there exists $N = N(\varepsilon) \in \mathbb{N}$ such that $|x_{ij} - L| < \varepsilon$ whenever $i, j > N$ and denoted by $P - \lim\limits_{i,j} x_{ij} = L$ (see [1]). A double sequence is bounded if there exists a positive number M such that $|x_{ij}| \leq M$ for all $(i, j) \in \mathbb{N}^2 = \mathbb{N} \times \mathbb{N}$. As it is known that every single convergent sequence (in the usual sense) is bounded, while a convergent double sequence need not to be bounded.

Let us turn our attention to statistical convergence and power series method for double sequences.

Moricz [2] proposed and investigated the idea of statistical convergence for double sequences, which may be restated in terms of natural density. Let $E \subset \mathbb{N}^2$ be a two-dimensional subset of positive integers and let $E_{m,n} = \{(i, j) \in E : i \leq m,\ j \leq n\}$. Then the two-dimensional analogue of natural density can be defined as follows:

$$\delta_2(E) := P - \lim_{m,n} \frac{1}{mn} |E_{mn}|$$

if it exists. The number sequence $x = (x_{ij})$ is statistically convergent to L provided that for every $\varepsilon > 0$, the set $E := E_{mn}(\varepsilon) := \{i \leq m,\ j \leq n : \ |x_{ij} - L| \geq \varepsilon\}$ has natural density zero; in that case we write $st_2 - \lim\limits_{i,j} x_{ij} = L$. For all that a statistically convergent sequence need not be convergent in light of the above.

It is obvious that a double sequence that is P-convergent statistically converges to the same value, but the opposite is not always true. Additionally, Moricz [2] characterized the statistical convergence for double sequences as follows:

A double sequence $x = (x_{ij})$ is statistically convergent to L if and only if there exists a set $E \subset \mathbb{N}^2$ such that the natural density of E is 1 and

$$P - \lim_{\substack{i,j \to \infty \\ \text{and } (i,j) \in E}} x_{ij} = L.$$

Let (p_{ij}) be a double sequence of nonnegative numbers with $p_{00} > 0$ and such that the following power series

$$p(t, s) := \sum_{i,j=0}^{\infty} p_{ij} t^i s^j$$

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has radius of convergence R with $R \in (0,\infty]$ and $t,s \in (0,R)$. If for all $t,s \in (0,R)$,

$$\lim_{t,s\to R^-} \frac{1}{p(t,s)} \sum_{i,j=0}^{\infty} p_{ij} t^i s^j x_{ij} = L$$

then we say that the double sequence $x = (x_{ij})$ is convergent to L in the sense of power series method and denoted by $P_p^2 - \lim x_{ij} = L$ ([3]). Keep in mind that the method is regular if and only if

$$\lim_{t,s\to R^-} \frac{\sum_{i=0}^{\infty} p_{i\upsilon} t^i}{p(t,s)} = 0 \text{ and } \lim_{t,s\to R^-} \frac{\sum_{j=0}^{\infty} p_{\mu j} s^j}{p(t,s)} = 0, \text{ for any } \mu,\upsilon, \tag{1.1}$$

hold [3].

Remark 1.1. *In case of $R = 1$, if $p_{ij} = 1$ and $p_{ij} = \frac{1}{(i+1)(j+1)}$, the power series methods coincide with Abel summability method and logarithmic summability method, respectively. In the case of $R = \infty$ and $p_{ij} = \frac{1}{i!j!}$, the power series method coincides with Borel summability method.*

Here and throughout the paper power series method is always assumed to be regular.
Ünver and Orhan [4] have recently introduced P_p-density of $E \subset \mathbb{N}_0$ and the definition of P_p-statistical convergence for single sequences. A natural question is what about statistical convergence or P_p-statistical convergence of the sequence. Hence, they showed that statistical convergence and P_p-statistical convergence are incompatible. In view of their work, Yıldız, Demirci and Dirik [5] have introduced the definitions of P_p^2-density of $F \subset \mathbb{N}_0^2 = \mathbb{N}_0 \times \mathbb{N}_0$ and P_p^2-statistical convergence for double sequences:

Definition 1.2 ([5]). *Let $F \subset \mathbb{N}_0^2$. If the limit*

$$\delta_{P_p}^2(F) := \lim_{t,s\to R^-} \frac{1}{p(t,s)} \sum_{(i,j)\in F} p_{ij} t^i s^j$$

exists, then $\delta_{P_p}^2(F)$ is called the P_p^2-density of F. Note that, from the definition of a power series method and P_p^2-density it can be established that $0 \le \delta_{P_p}^2(F) \le 1$ whenever it exists.

Definition 1.3 ([5]). *Let $x = (x_{ij})$ be a double sequence. Then x is said to be statistically convergent with respect to power series method (P_p^2-statistically convergent) to L if for any $\varepsilon > 0$*

$$\lim_{t,s\to R^-} \frac{1}{p(t,s)} \sum_{(i,j)\in F_\varepsilon} p_{ij} t^i s^j = 0$$

where $F_\varepsilon = \left\{(i,j) \in \mathbb{N}_0^2 : |x_{ij} - L| \ge \varepsilon\right\}$, that is $\delta_{P_p}^2(F_\varepsilon) = 0$ for any $\varepsilon > 0$. In this case we write $st_{P_p}^2 - \lim x_{ij} = L$.

Let $A = [a_{klmn}]$, $k,l,m,n \in \mathbb{N}$, be a four-dimensional infinite matrix. The A-transform of a given double sequence $x = (x_{mn})$ is given by

$$(Ax)_{kl} = \sum_{(m,n)\in\mathbb{N}^2} a_{klmn} x_{mn}, \quad k,l \in \mathbb{N},$$

provided the double series converges in Pringsheim's sense for every $(k,l) \in \mathbb{N}^2$ and denoted by $Ax := ((Ax)_{kl})$. If the A-transform of x exists for all $k,l \in \mathbb{N}$ and convergent in the Pringsheim's sense i.e.,

$$P - \lim_{p,q} \sum_{m=1}^{p} \sum_{n=1}^{q} a_{klmn} x_{mn} = y_{kl} \text{ and } P - \lim_{k,l} y_{kl} = L$$

then we say that a sequence x is A-summable to L. A two-dimensional matrix transformation is referred to as regular in summability theory if it converts each convergent sequence into one with the same limit.
Now consider a sequence of four-dimensional infinite matrices with non-negative real elements $\mathscr{A} := \left(A^{(i,j)}\right) = \left(a_{klmn}^{(i,j)}\right)$. For a given double sequence of real numbers, $x = (x_{mn})$ is said to be *$\mathscr{A}$-summable* to L if

$$P - \lim_{k,l} \sum_{(m,n)\in\mathbb{N}^2} a_{klmn}^{(i,j)} x_{mn} = L$$

uniformly in i and j.
$\mathscr{A}$-summability is the A-summability for four-dimensional infinite matrix if $A^{(i,j)} = A$, four-dimensional infinite matrix. Some results regarding matrix summability method for double sequences may be found in the papers [6, 7]. $C^*(\mathbb{R}^2)$ stands for the space of all continuous functions on $\mathbb{R}^2$ that are real valued and have a period of 2π. If a function $h \in C^*(\mathbb{R}^2)$, then

$$h(x,y) = h(x+2k\pi, y) = h(x, y+2k\pi), \text{ for all } (x,y) \in \mathbb{R}^2,$$

holds for $k = 0, \pm1, \pm2, \ldots$. In what follows, this space is equipped with the supremum norm

$$\|f\|_* = \sup_{(x,y)\in\mathbb{R}^2} |h(x,y)|, \quad \left(h \in C^*\left(\mathbb{R}^2\right)\right).$$

A sequence $\mathbb{L} := (L_{mn})$ of positive linear operators from $C^*\left(\mathbb{R}^2\right)$ into itself is referred to as an $\mathscr{A}$-summation process on $C^*\left(\mathbb{R}^2\right)$ if $(L_{mn}h)$ is $\mathscr{A}$-summable to h for every $h \in C^*\left(\mathbb{R}^2\right)$, i.e.,

$$P - \lim_{k,l} \left\| S^{\mathbb{L}}_{klij}h - h \right\|_* = 0, \text{ uniformly in } i, j$$

where for all $k,l,i,j \in \mathbb{N}$, $h \in C^*\left(\mathbb{R}^2\right)$ the series

$$S^{\mathbb{L}}_{klij}h := \sum_{(m,n)\in\mathbb{N}^2} a^{(i,j)}_{klmn} L_{mn}h \tag{1.2}$$

and it is assumed that the series in (1.2) absolutely convergent for each $i, j, k, l \in \mathbb{N}$ and h.
For the rate of convergence, we need to recall the following modulus of continuity of h. Let $h \in C^*\left(\mathbb{R}^2\right)$, then

$$w(h;\gamma) = \sup\left\{ |h(u,v) - h(x,y)| : (u,v),(x,y) \in \mathbb{R}^2 \text{ and } \sqrt{(u-x)^2 + (v-y)^2} \leq \gamma \right\}$$

for $\gamma > 0$. This definition yields the following basic property for $h \in C^*\left(\mathbb{R}^2\right)$.
For any $a > 0$,

$$w(h;a\gamma) \leq (1+[a])\, w(h;\gamma)$$

where $[a]$ is defined to be the greatest integer less than or equal to a.
The paper of Korovkin [8] is an important issue. It can help us to understand the nature of approximation of sequences. This approximation problem has a rich history associated with the names of the different convergence methods on some spaces in the theory. For some recent research works in this direction, see [9–21]. In this paper, we investigate and establish Korovkin-type approximation theorems for linear operators defined on the space of all 2π-periodic and real valued continuous functions on $\mathbb{R}^2$ by means of $\mathscr{A}$-summation process via statistical convergence with respect to power series method. We demonstrate with an example how our theory is more strong than previously studied. Additionally, we research the rate of convergence of positive linear operators defined on this space.

2. The Second Theorem of Korovkin Type

The aim of this section is to deal with approximation of all 2π-periodic and real valued continuous functions on $\mathbb{R}^2$ by means of $\mathscr{A}$-summation process via statistical convergence with respect to power series method.
Our main result is the following.

Theorem 2.1. *Let $\mathscr{A} = \left(A^{(i,j)}\right)$ be a sequence of four-dimensional infinite matrices. Let $\mathbb{L} = (L_{mn})$ be a sequence of positive linear operators acting from $C^*\left(\mathbb{R}^2\right)$ into itself. Assume that (1.2) holds. Then, for all $h \in C^*\left(\mathbb{R}^2\right)$*

$$st^2_{P_p} - \lim \left\| S^{\mathbb{L}}_{klij}h - h \right\|_* = 0 \quad \text{uniformly in } i \text{ and } j \tag{2.1}$$

if and only if

$$st^2_{P_p} - \lim \left\| S^{\mathbb{L}}_{klij}h_r - h_r \right\|_* = 0 \quad \text{uniformly in } i \text{ and } j \quad (r = 0,1,2,3,4) \tag{2.2}$$

where $h_0(x,y) = 1$, $h_1(x,y) = \sin x$, $h_2(x,y) = \sin y$, $h_3(x,y) = \cos x$ and $h_4(x,y) = \cos y$.

Proof. Since 1, $\sin x$, $\sin y$, $\cos x$ and $\cos y$ belong to $C^*\left(\mathbb{R}^2\right)$, the necessity is clear. Suppose now that (2.2) holds. Let $h \in C^*\left(\mathbb{R}^2\right)$ and I, J be closed subinterval of length 2π of $\mathbb{R}$. Fix $(x,y) \in I \times J$. As in the proof of Theorem 2.1 in [22], it follows from the continuity of h that

$$|h(u,v) - h(x,y)| < \varepsilon + \frac{2M_h}{\sin^2 \frac{\delta}{2}} \varphi(u,v)$$

which gives,

$$\begin{aligned}
\left|\sum_{(m,n)\in\mathbb{N}^2} a_{klmn}^{(i,j)} L_{mn}(h;x,y)-h(x,y)\right| \leq & \sum_{(m,n)\in\mathbb{N}^2} a_{klmn}^{(i,j)} L_{mn}\left(|h(u,v)-h(x,y)|;x,y\right)+|h(x,y)|\left|\sum_{(m,n)\in\mathbb{N}^2} a_{klmn}^{(i,j)} L_{mn}(h_0;x)-h_0(x,y)\right| \\
\leq & \sum_{(m,n)\in\mathbb{N}^2} a_{klmn}^{(i,j)} L_{mn}\left(\varepsilon+\frac{2M_h}{\sin^2\frac{\delta}{2}}\varphi(u,v);x,y\right)+M_h\left|\sum_{(m,n)\in\mathbb{N}^2} a_{klmn}^{(i,j)} L_{mn}(h_0;x)-h_0(x,y)\right| \\
\leq & \varepsilon+(\varepsilon+M_h)\left|\sum_{(m,n)\in\mathbb{N}^2} a_{klmn}^{(i,j)} L_{mn}(h_0;x)-h_0(x,y)\right| \\
& +\frac{M_h}{\sin^2\frac{\delta}{2}}\left\{\begin{array}{l}
2\left|\sum_{(m,n)\in\mathbb{N}^2} a_{klmn}^{(i,j)} L_{mn}(h_0;x)-h_0(x,y)\right| \\
+|\sin x|\left|\sum_{(m,n)\in\mathbb{N}^2} a_{klmn}^{(i,j)} L_{mn}(h_1;x,y)-h_1(x,y)\right| \\
+|\sin y|\left|\sum_{(m,n)\in\mathbb{N}^2} a_{klmn}^{(i,j)} L_{mn}(h_2;x,y)-h_2(x,y)\right| \\
+|\cos x|\left|\sum_{(m,n)\in\mathbb{N}^2} a_{klmn}^{(i,j)} L_{mn}(h_3;x,y)-h_3(x,y)\right| \\
+|\cos y|\left|\sum_{(m,n)\in\mathbb{N}^2} a_{klmn}^{(i,j)} L_{mn}(h_4;x,y)-h_4(x,y)\right|
\end{array}\right\} \\
\leq & \varepsilon+N\sum_{r=0}^{4}\left|\sum_{(m,n)\in\mathbb{N}^2} a_{klmn}^{(i,j)} L_{mn}(h_r;x)-h_r(x,y)\right|
\end{aligned}$$

where $M_h=\|f\|_*$, $\varphi(u,v)=\sin^2\frac{u-x}{2}+\sin^2\frac{v-y}{2}$ and $N:=\varepsilon+M_h+\frac{2M_h}{\sin^2\frac{\delta}{2}}$. Then, taking supremum over $(x,y)\in\mathbb{R}^2$, we obtain

$$\left\|S_{klij}^{\mathbb{L}}h-h\right\|_*\leq\varepsilon+N\sum_{r=0}^{4}\left\|S_{klij}^{\mathbb{L}}h_r-h_r\right\|_*. \tag{2.3}$$

Now given $r>0$, choose $\varepsilon>0$ such that $\varepsilon<r$, and define

$$\begin{aligned}
D &:=\left\{(k,l):\left\|S_{klij}^{\mathbb{L}}h-h\right\|_*\geq r\right\}, \\
D_r &:=\left\{(k,l):\left\|S_{klij}^{\mathbb{L}}h_r-h_r\right\|_*\geq\frac{r-\varepsilon}{5N}\right\},\ r=0,1,2,3,4.
\end{aligned}$$

It is easy see that from (2.3)

$$D\subseteq\bigcup_{r=0}^{4}D_r.$$

Hence, we may write

$$\delta_{P_p}^2(D)\leq\sum_{r=0}^{4}\delta_{P_p}^2(D_r).$$

Then, according to (2.2), we have

$$\delta_{P_p}^2(D)=0,$$

and hence

$$st_{P_p}^2-\lim\left\|S_{klij}^{\mathbb{L}}h-h\right\|_*=0 \quad \text{uniformly in } i \text{ and } j$$

which is the desired result. □

3. An example

Now, we give an example that our theorem (Theorem 2.1) is stronger than Theorem 9 in [23].

Example 3.1. *Now assume that $\mathscr{A}=\left(A^{(i,j)}\right)$ is a sequence of four-dimensional infinite matrices defined by $a_{klmn}^{(i,j)}=\frac{1}{kl}$ if $i\leq m\leq k+i-1$, $j\leq n\leq l+j-1$ and $a_{klmn}^{(i,j)}=0$ otherwise. Let us consider the double sequence of Fejer operators on $C^*\left(\mathbb{R}^2\right)$ where*

$$L_{mn}(h;x,y)=\frac{1}{(m\pi)(n\pi)}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}h(u,v)F_m(u)F_n(v)\,du\,dv \tag{3.1}$$

where $F_m(u)=\frac{\sin^2\frac{m(u-x)}{2}}{2\sin^2\frac{u-x}{2}}$ *and* $\frac{1}{\pi}\int_{-\pi}^{\pi}F_m(u)\,du=1$. *Let* (p_{ij}) *be defined as follows*

$$p_{ij}=\begin{cases}0, & i \text{ and } j \text{ even}\\ 1, & i \text{ or } j \text{ odd}\end{cases},$$

and take the sequence (x_{ij}) *defined by*

$$x_{ij}=\begin{cases}ij, & i \text{ and } j \text{ even}\\ 0, & i \text{ or } j \text{ odd}\end{cases}. \tag{3.2}$$

It is easy to see that

$$st^2_{P_p}-\lim x_{ij}=0. \tag{3.3}$$

However, the sequence (x_{ij}) *neither statistically convergent to* 0 *nor Pringsheim convergent. Now using (3.1) and (3.2), we define the following double positive linear operators* $\mathbb{T}=(T_{mn})$ *on* $C^*(\mathbb{R}^2)$ *as follows:*

$$T_{mn}(f;x,y)=(1+x_{mn})L_{mn}(f;x,y). \tag{3.4}$$

We now claim that

$$st^2_{P_p}-\lim\left\|S^{\mathbb{T}}_{klij}h_r-h_r\right\|_*=0 \quad \text{uniformly in } i \text{ and } j,\ (r=0,1,2,3,4). \tag{3.5}$$

Observe that $L_{mn}(h_0;x,y)=h_0(x,y)$, $L_{mn}(h_1;x,y)=\frac{m-1}{m}h_1(x,y)$, $L_{mn}(h_2;x,y)=\frac{n-1}{n}h_2(x,y)$, $L_{mn}(h_3;x,y)=\frac{m-1}{m}h_3(x,y)$, $L_{mn}(h_4;x,y)=\frac{n-1}{n}h_4(x,y)$. *So, we can see,*

$$\left\|S^{\mathbb{T}}_{klij}h_0-h_0\right\|_*=\left\|\sum_{m=i}^{i+k-1}\sum_{n=j}^{j+l-1}\frac{1}{kl}(1+x_{mn})-1\right\|_*\leq\frac{1}{kl}\sum_{m=i}^{i+k-1}\sum_{n=j}^{j+l-1}x_{mn}.$$

It is well known that if a sequence is convergent, its arithmetic mean will also converge to the same value. Thus, by virtue of P^2_p*-statistical convergence and thanks to (3.3) it is clear that*

$$st^2_{P_p}-\lim\left(\sup_{i,j}\frac{1}{kl}\sum_{m=i}^{i+k-1}\sum_{n=j}^{j+l-1}x_{mn}\right)=0, \tag{3.6}$$

and hence

$$st^2_{P_p}-\lim\left\|S^{\mathbb{T}}_{klij}h_0-h_0\right\|_*=0, \ \text{uniformly in } i \text{ and } j,$$

which guarantees that (3.5) holds true for $r=0$. *Also, we compute*

$$\left\|S^{\mathbb{T}}_{klij}h_1-h_1\right\|_*=\left\|\sum_{m=i}^{i+k-1}\sum_{n=j}^{j+l-1}\frac{1}{kl}(1+x_{mn})\frac{m-1}{m}h_1-h_1\right\|_*\leq\left|\frac{1}{kl}\sum_{m=i}^{i+k-1}\sum_{n=j}^{j+l-1}\frac{m-1}{m}-1\right|+\frac{1}{kl}\sum_{m=i}^{i+k-1}\sum_{n=j}^{j+l-1}\frac{x_{mn}(m-1)}{m}.$$

Since $st^2_{P_p}-\lim\left(\sup_{i,j}\left(\frac{1}{kl}\sum_{m=i}^{i+k-1}\sum_{n=j}^{j+l-1}\frac{m-1}{m}-1\right)\right)=0$ *and from (3.6) we have,*

$$st^2_{P_p}-\lim\left\|S^{\mathbb{T}}_{klij}h_1-h_1\right\|_*=0, \ \text{uniformly in } i \text{ and } j.$$

So (3.5) valid for $r=1$. *Likewise, we have*

$$st^2_{P_p}-\lim\left\|S^{\mathbb{T}}_{klij}h_2-h_2\right\|_*=0, \ \text{uniformly in } i \text{ and } j,$$
$$st^2_{P_p}-\lim\left\|S^{\mathbb{T}}_{klij}h_3-h_3\right\|_*=0, \ \text{uniformly in } i \text{ and } j,$$
$$st^2_{P_p}-\lim\left\|S^{\mathbb{T}}_{klij}h_4-h_4\right\|_*=0, \ \text{uniformly in } i \text{ and } j.$$

So, our claim (3.5) is valid for each $r=0,1,2,3,4$. *Then, observe that the double sequence* $\mathbb{T}=(T_{mn})$ *defined by (3.4) satisfy all hypotheses of Theorem 2.1. Hence, we have, for all* $f\in C^*(\mathbb{R}^2)$,

$$st^2_{P_p}-\lim\left\|S^{\mathbb{T}}_{klij}h-h\right\|_*=0.$$

Also, since (x_{ij}) *is not statistically convergent to* 0, (T_{mn}) *does not satisfy Theorem 9 in [23].*

4. Rates of Convergence

In this section, via $\mathscr{A}$-summation process via statistical convergence with respect to power series method, we study the rates of convergence of a double sequence of positive linear operators mapping acting from $C^*\left(\mathbb{R}^2\right)$ into $C^*\left(\mathbb{R}^2\right)$ by means of the modulus of continuity.
We have the following result.

Theorem 4.1. *Let $\mathscr{A}=\left(A^{(i,j)}\right)$ be a sequence of four-dimensional infinite matrices. Let $\mathbb{L}=(L_{mn})$ be a double sequence of positive linear operators moving from $C^*\left(\mathbb{R}^2\right)$ into $C^*\left(\mathbb{R}^2\right)$. Suppose that (1.2) and the following conditions provided:*
(i) $st^2_{P_p}-\lim\left\|S^{\mathbb{L}}_{klij}h_0-h_0\right\|_*=0$, uniformly in i and j,
(ii) $st^2_{P_p}-\lim w(h;\gamma)=0$, uniformly in i and j,
where $\gamma:=\gamma^{(i,l)}_{(j,k)}:=\sqrt{\left\|S^{\mathbb{L}}_{klij}\varphi\right\|_*}$ with $\varphi(u,v)=\sin^2\frac{u-x}{2}+\sin^2\frac{v-y}{2}$. Then we have, for all $h\in C^*\left(\mathbb{R}^2\right)$,

$$st^2_{P_p}-\lim\left\|S^{\mathbb{L}}_{klij}h-h\right\|_*=0,\ \text{uniformly in } i \text{ and } j.$$

Proof. To prove this, we firstly suppose that $(x,y)\in[-\pi,\pi]\times[-\pi,\pi]$ and $h\in C^*\left(\mathbb{R}^2\right)$ be fixed, and that Let (i) and (ii) be provided.. Let γ be a positive number. As in the proof of Theorem 9 in [23], since the function h is continious, the following inequality is obtained:

$$|h(u,v)-h(x,y)|\leq\left(1+\pi^2\frac{\sin^2\frac{u-x}{2}+\sin^2\frac{v-y}{2}}{\gamma^2}\right)w(h;\gamma).$$

Using the definition of modulus of continuity and since the operators L_{mn} is linear and the positive, we have

$$\begin{aligned}\left|S^{\mathbb{L}}_{klij}(h;x,y)-h(x,y)\right|&=\left|\sum_{(m,n)\in\mathbb{N}^2}a^{(i,j)}_{klmn}L_{mn}(h;x,y)-h(x,y)\right|\\&\leq\sum_{(m,n)\in\mathbb{N}^2}a^{(i,j)}_{klmn}L_{mn}\left(|h(u,v)-h(x,y)|;x,y\right)+|h(x,y)|\left|\sum_{(m,n)\in\mathbb{N}^2}a^{(i,j)}_{klmn}L_{mn}(h_0;x,y)-h_0(x,y)\right|\\&\leq w(h;\gamma)\sum_{(m,n)\in\mathbb{N}^2}a^{(i,j)}_{klmn}L_{mn}(h_0;x,y)+\pi^2\frac{w(h;\gamma)}{\gamma^2}\sum_{(m,n)\in\mathbb{N}^2}a^{(i,j)}_{klmn}L_{mn}(\varphi;x,y)\\&\quad+|h(x,y)|\left|\sum_{(m,n)\in\mathbb{N}^2}a^{(i,j)}_{klmn}L_{mn}(h_0;x,y)-h_0(x,y)\right|\end{aligned}$$

where $\varphi(u,v)=\sin^2\frac{u-x}{2}+\sin^2\frac{v-y}{2}$. If supremum over (x,y) is taken on both sides of the above inequality and is chosen $\gamma:=\gamma^{(i,l)}_{(j,k)}:=\sqrt{\left\|S^{\mathbb{L}}_{klij}\varphi\right\|_*}$, then we obtain

$$\left\|S^{\mathbb{L}}_{klij}h-h\right\|_*\leq w\left(h;\gamma^{(i,l)}_{(j,k)}\right)\left\|S^{\mathbb{L}}_{klij}h_0-h_0\right\|_*+\left(1+\pi^2\right)w\left(h;\gamma^{(i,l)}_{(j,k)}\right)+M_h\left\|S^{\mathbb{L}}_{klij}h_0-h_0\right\|_* \tag{4.1}$$

where $M_h:=\|h\|_*$. Now, given $\varepsilon>0$, define the following sets:

$$\begin{aligned}D&:=\left\{(k,l):\left\|S^{\mathbb{L}}_{klij}h-h\right\|_*\geq\varepsilon\right\},\\D_1&:=\left\{(k,l):w\left(h;\gamma^{(i,l)}_{(j,k)}\right)\left\|S^{\mathbb{L}}_{klij}h_0-h_0\right\|_*\geq\frac{\varepsilon}{3}\right\},\\D_2&:=\left\{(k,l):w\left(h;\gamma^{(i,l)}_{(j,k)}\right)\geq\frac{\varepsilon}{3\left(1+\pi^2\right)}\right\},\\D_3&:=\left\{(k,l):\left\|S^{\mathbb{L}}_{klij}h_0-h_0\right\|_*\geq\frac{\varepsilon}{3M_h}\right\}.\end{aligned}$$

Then, it follows from (4.1) that $D\subset D_1\cup D_2\cup D_3$. Also, defining

$$\begin{aligned}D_4&:=\left\{(k,l):w\left(h;\gamma^{(i,l)}_{(j,k)}\right)\geq\sqrt{\frac{\varepsilon}{3}}\right\},\\D_5&:=\left\{(k,l):\left\|S^{\mathbb{L}}_{klij}h_0-h_0\right\|_*\geq\sqrt{\frac{\varepsilon}{3}}\right\},\end{aligned}$$

we have $D_1\subset D_4\cup D_5$, which yields

$$D\subseteq\bigcup_{i=2}^{5}D_i.$$

Hence, we may write

$$\delta_{P_p}^2(D) \leq \sum_{r=0}^{5} \delta_{P_p}^2(D_r).$$

Using the hypothesis (i) and (ii), we get

$$\delta_{P_p}^2(D) = 0,$$

and hence

$$st_{P_p}^2 - \lim \left\| S_{klij}^{\mathbb{L}} h - h \right\|_* = 0, \text{ uniformly in } i \text{ and } j.$$

Therefore, the proof is completed. □

5. Conclusion

The paper contains Korovkin-type approximation theorem and the rate of convergence for linear operators defined on the space of all 2π-periodic and real valued continuous functions on $\mathbb{R}^2$ by means of $\mathscr{A}$-summation process via statistical convergence with respect to power series method. Also, it is demonstrated with an example how the new theory is more stronger than previously studied.

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Author's contributions

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