Proceedings of International Mathematical Sciences
ISSN: 2717-6355, URL: https://dergipark.org.tr/tr/pub/pims
Volume 5 Issue 1 (2023), Pages 1-4.
Doi: https://doi.org/10.47086/pims.1205921

SUZUKI TYPE P-CONTRACTIVE MAPPINGS

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Abstract. We introduce Suzuki type P-contractive mappings by taking into account the concepts of contractive, P-contractive, and Suzuki type contractive mappings. Then, for such mappings on compact metric spaces, we present a fixed point theorem that is more general than the well-known Edelstein fixed point theorem.

1. Introduction

Metric fixed point theory, as it is known, investigates the conditions that guarantee the existence and even uniqueness of fixed point of a self mapping on a metric space. These conditions are typically comprised of completeness of space and some type of contraction inequality. It is difficult to obtain a new result when the completeness of space is ignored. As a result, studies are conducted to ensure the existence of the fixed point by weakening the contraction inequalities. However, in complete metric space generalizations, the sum of the coefficients of the terms on the right side of the linear contraction inequalities is less than 1. Nonlinear contraction inequalities are subject to a similar constraint. Edelstein [4] introduced the concept of contractivity to overcome the coefficient problem and obtained a fixed point theorem. Although Edelstein extended the relevant class of mappings, he had to consider compactness of the space, which is a more strong condition than completeness. Many studies, covering Edelstein's fixed theorem, have been obtained by generalizing the concept of contractivity in the literature (for example see [2, 3, 5]). For the sake of completeness we recall the following:

Let (X, d) be a metric space and $T : X \to X$ be a mapping. Then, T is said to be contractive if

$$d(Tx, Ty) < d(x, y) \tag{C}$$

for all $x, y \in X$ with $x \neq y$. Hence, Edelstein presented the following theorem:

Theorem 1.1 ([4]). *Let (X, d) be a compact metric space and $T : X \to X$ be a contractive mapping. Then, T has a unique fixed point.*

2020 *Mathematics Subject Classification.* Primary: 81Q05, 33C20, 35Q40.
Key words and phrases. Asymptotic iteration method; Yukawa potential; Hulthén potential.

Submitted on 16.11.2022, Published on 13.02.2023.
Communicated by Hakan ŞAHİN.

Suzuki obtained a new fixed point theorem by weakening the concept of contractivity in 2009.

Theorem 1.2 ([5]). *Let (X, d) be a compact metric space and $T : X \to X$ be a mapping such that*

$$\frac{1}{2}d(x, Tx) < d(x, y) \text{ implies } d(Tx, Ty) < d(x, y) \tag{SC}$$

for all $x, y \in X$. Then, T has a unique fixed point.

For the sake of simplicity, we will refer to the mappings that provide the (SC) inequality as Suzuki type contractive mappings. In 2018, Altun et al. [2] defined the concept of P-contractivity. A self mapping T on X is said to be P-contractive if

$$d(Tx, Ty) < d(x, y) + |d(x, Tx) - d(y, Ty)| \tag{PC}$$

for all $x, y \in X$ with $x \neq y$. Then, the following theorem has been presented.

Theorem 1.3 ([2]). *Let (X, d) be a compact metric space and $T : X \to X$ be a continuous P-contractive mapping. Then, T has a unique fixed point.*

It is clear that every contractive (C) mapping is Suzuki type contractive (SC), also every contractive (C) mapping is P-contractive (PC). The following examples demonstrate that the converse of both propositions are not true.

Example 1.1 ([5]). *Let $X = [-11, -10] \cup \{0\} \cup [10, 11]$ with the usual metric d and $T : X \to X$, defined by*

$$Tx = \begin{cases} \frac{11x+100}{x+9} & , \quad x \in [-11, -10) \\ 0 & \quad x \in \{-10, 0, 10\} \\ -\frac{11x-100}{x-9} & , \quad x \in (10, 11] \end{cases} .$$

Then, T is Suzuki type contractive, but it is not contractive.

Example 1.2 ([3]). *Let $X = [0, 1]$ with the usual metric d and $T : X \to X$, defined by*

$$Tx = \begin{cases} \frac{1}{2} & , \quad x = 0 \\ \frac{x}{2} & , \quad x \neq 0 \end{cases} .$$

Then, T is P-contractive, but it is not contractive.

The classes of Suzuki type contractive (SC) mappings and P-contractive (PC) mappings, on the other hand, are distinct. The following examples demonstrate this fact.

Example 1.3 ([2]). *Let $X = [0, 2]$ with the usual metric d and $T : X \to X$, defined by*

$$Tx = \begin{cases} 1 & , \quad x \leq 1 \\ 0 & , \quad x > 1 \end{cases} .$$

Then, T is P-contractive, but it is not Suzuki type contractive.

Example 1.4 ([2]). *Let $X = \{(0, 0), (4, 0), (0, 4), (4, 5), (5, 4)\} \subset \mathbb{R}^2$ with the metric*

$$d(x, y) = d((x_1, x_2), (y_1, y_2)) = |x_1 - y_1| + |x_2 - y_2|$$

for $x = (x_1, x_2), y = (y_1, y_2) \in X$. *Define a mapping* $T : X \to X$ *by*

$$T = \begin{pmatrix} (0,0) & (4,0) & (0,4) & (4,5) & (5,4) \\ \\ (0,0) & (0,0) & (0,0) & (4,0) & (0,4) \end{pmatrix}.$$

Then, T *is Suzuki type contractive, but it is not* P*-contractive.*

Remark. *Although contractive mappings are continuous, neither Suzuki type contractive nor* P*-contractive mappings are continuous. Note that Suzuki did not need the continuity in Theorem 1.2. However, in Theorem 1.3 the continuity of the mapping has been assumed. Example 1.2 above shows that the condition of continuity can not be removed in Theorem 1.3.*

In this paper, we introduce Suzuki type P-contractive mappings, which are inspired by the concepts of contractive, P-contractive, and Suzuki type contractive mappings. Then, we present a fixed point theorem that is more general than Theorem 1.1 and Theorem 1.3.

The following lemma will be used in our second theorem.

Lemma 1.4 ([1]). *Let* X *be a compact topological space and* $f : X \to \mathbb{R}$ *be a lower semicontinuous function. Then, there exists an element* $x_0 \in X$ *such that* $f(x_0) = \inf\{f(x) : x \in X\}$.

2. Main Result

First, we introduce a new concept for self mapping T on a metric space (X, d).

Definition 2.1. *Let* (X, d) *be a metric space and* $T : X \to X$ *be a mapping. Then* T *is said to be Suzuki type* P*-contractive if*

$$\frac{1}{2}d(x, Tx) < d(x, y) \text{ implies } d(Tx, Ty) < d(x, y) + |d(x, Tx) - d(y, Ty)| \quad \text{(SPC)}$$

for all $x, y \in X$.

Remark. *For the aforementioned contractivity concepts, we can draw the diagram below:*

$$\begin{array}{ccc} C & \Longrightarrow & PC \\ \Downarrow & & \Downarrow \\ SC & \Longrightarrow & SPC \end{array}.$$

Examples 1.1, 1.2, 1.3, 1.4 show that the converse of all implications are not true.

Now, we are ready to state our main result.

Theorem 2.1. *Let* (X, d) *be a compact metric space and* $T : X \to X$ *be a continuous Suzuki type* P*-contractive mapping. Then,* T *has a unique fixed point in* X.

Proof. Since X is compact and T is continuous, then there exists $u \in X$ such that

$$d(u, Tu) = \inf\{d(x, Tx) : x \in X\}.$$

We claim that $d(u, Tu) = 0$. Assume the contrary. In this case, since $0 < \frac{1}{2}d(u, Tu) < d(u, Tu)$, we have

$$\begin{aligned} d(Tu, T^2u) &< d(u, Tu) + \left|d(u, Tu) - d(Tu, T^2u)\right| \\ &= d(u, Tu) + d(Tu, T^2u) - d(u, Tu) \\ &= d(Tu, T^2u), \end{aligned}$$

which is a contradiction. Therefore, $d(u, Tu) = 0$ and so u is a fixed point of T. Now, assume v is another fixed point of T. In this case, since $0 = \frac{1}{2}d(u, Tu) < d(u, v)$, we have

$$\begin{aligned} d(u,v) &= d(Tu,Tv) \\ &< d(u,v) + |d(u,Tu) - d(v,Tv)| \\ &= d(u,v), \end{aligned}$$

which is a contradiction. Hence, the fixed point of T is unique. □

To see that the continuity condition in this theorem cannot be removed, one can refer to Example 1.2 again. However, a result can be obtained by assuming the lower semicontinuity of the function f defined by $f(x) = d(x, Tx)$ instead of the continuity of T. It is well known that if T is continuous, then f is also continuous (and so it is lower semicontinuous). However, if f is lower semicontinuous, then T may not be continuous (see Remark 2.8 in [2]).

Hence, by Lemma 1.4, we can state the following result:

Theorem 2.2. *Let (X, d) be a compact metric space and $T : X \to X$ be a Suzuki type P-contractive mapping. Then T has a unique fixed point in X provided that the function f defined by $f(x) = d(x, Tx)$ is lower semicontinuous.*

Proof. Since X is compact and $f : X \to \mathbb{R}$ is lower semicontinuous, then by Lemma 1.4, there exists $u \in X$ such that $f(u) = \inf f(X)$, that is, we have

$$d(u, Tu) = \inf\{d(x, Tx) : x \in X\}.$$

Therefore, the proof can be completed as in the proof of Theorem 2.1. □

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