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FRACTIONAL ORDER MATHEMATICAL MODELING OF LUMPY SKIN DISEASE

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Abstract. In this article, we study the fractional-order SEIR mathematical model of Lumpy Skin Disease (LSD) in the sense of Caputo. The existence, uniqueness, non-negativity and boundedness of the solutions are established using fixed point theory. Using a next-generation matrix, the reproduction number R_0 is determined for the disease's prognosis and durability. Using the fractional Routh-Hurwitz stability criterion, the evolving behaviour of the equilibria is investigated. Generalized Adams–Bashforth–Moulton approach is applied to arrive at the solution of the proposed model. Furthermore, to visualise the efficiency of our theoretical conclusions and to track the impact of arbitrary-order derivative, numerical simulations of the model and their graphical presentations are carried out using MATLAB(R2021a).

1. Introduction

Lumpy skin disease mainly spread to ruminants such as cattle and water buffaloes (Bubalus bubalis), making it a non-zoonotic viral disease that develop and reproduce entirely in non-human hosts via arthropod vectors such as biting flies, mosquitoes, and ticks. Contagious sustenance such as contaminated fodder, water and animal semen during artificial insemination are also responsible for the spread. It is a trans-boundary disease brought on by the Lumpy skin disease virus (LSDV) which go by names Pseudo-urticaria, Neethling viral disease belonging to the Poxviridae family, and genus Capripoxvirus ([6], [19], [28], [51], [59]).

Zambia marked the presence of LSD in 1929 [38], propagating to Zimbabwe and South Africa in 1949, Ethiopia in 1983, Israel in 1989, and then spreading

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throughout the Middle East, West Asia, and Europe. It produced a massive economic calamity in South Africa for about 30 years (1950-1980) [3], [30]. The year 2018-19 recorded infections in Greece, Georgia, and Russia. Cattle in various Asian nations are currently suffering from LSD including Nepal [4], Thailand [7], [37], Malaysia, Laos, Cambodia [8], Mynmar [18], Bangladesh [25], India [23], China [32], Sri Lanka, Bhutan and Vietnam [58].

Though cattle are the prime species to be infected by LSD but experimental infections show that the virus can also infect sheep, goat, giraffe, gazalles and impalas [23]. The name LSD is attributed to the fact that lymph nodes of the infected animal grows and resemble lumps on the skin. Large cutaneous nodules emerge on the head, neck, arms, legs, udder, abdomen, and private parts of the infected cattle subsequently evolve into ulcers and finally convert into skin scabs [51]. According to the FAO [24], it is a high morbidity(2-45 percent) and low mortality disease (less than 10 percent). The disease evolve in 4 to 14 days.

August 2019 marked the initial outbreak of LSD in the Indian states of Odisha and West Bengal [56]. Within a few months, other LSD outbreaks were recorded across the country causing the dairy industry to incur significant financial losses. With the most cow and buffalo in the world, India is the largest milk producer and ranks first in the world, producing twenty-four percent of global milk output in 2021-22. According to government data, lumpy skin disease has infected millions of cattle and killed more than 1,84,000 in India, causing less milk production due to weakness and appetite loss caused by mouth ulcers, inadequate development, decreasing draught power, and reproductive difficulties such as abortions, infertility, and a lack of sperm for artificial insemination. As a result, LSD has been identified in India as a potentially lethal disease for cattle.

1.1. **Motivation and Research Background.** Modelling of epidemic diseases is of utmost importance to understand the behaviour of the ailment across time and to devise appropriate safeguards for the same. Numerous epidemic models have been developed for various diseases, including dengue and chikungunya [1], typhoid [2], cholera [10], HIV/AIDS [11], Covid-19 [14], [15], [57], leptospirosis, H1N1, measles [17], and others. But to our surprise there is not enough research on transmission dynamics and LSD control using a compartmental modelling technique; by the time this study was completed, there had only been one work [46], to examine the effects of vaccination on LSD and the spread of the illness in Ethiopia. Butt et al. [15] had also researched the SVEIR epidemic model and examined it for the presence of a unique positive and bounded solution at the end of initial revision. The authors of both of these studies, however, relied on the traditional integer-order derivatives, which are frequently unable to foresee the remembrance and inheritance characteristics of substances and phenomenons, leading to erroneous depictions of dynamic real-world events. Due to the significant amount of unidentified, uncertainties, and misinformation, developing a mathematical model that accurately captures LSD using classical differentiation is a difficult task. The use of non-local operators is

encouraged by coincidences and diminishing retention effects, the argument being supported by plenty of scholarly articles [12], [13], [42], [43], [60].

Fractional derivatives come in a wide range of forms, both with and without singular kernels. For singular kernels, we've got the derivatives of Caputo, Riemann-Liouville, and Katugampola [27], [53]. The Caputo-Fabrizio fractional derivative [16], which has an exponential kernel, and the Atangana-Baleanu fractional derivative [9], which has a Mittag-Leffler kernel, are the two types of fractional derivatives without singular kernels. It is crucial to work with fractional-order derivatives because they provide a more accurate way to describe LSD outbreaks, even while memory and genetic features are implicated. We offer and examine the fractional order SEIR mathematical model in Caputo sense in light of the recent research to comprehend the evaluation, existence, stability, and control of LSD and to the best of our knowledge, this is the first paper to use fractional order derivative for modeling the transmission dynamics of LSD, which is critical for understanding the epidemiology and dynamic nature of exotic disease for timely disease management and planning because of the global character of the fractional derivatives which improves the system's consistency domain. The Caputo derivative serves best as a base model and is preferred over Riemann–Liouville fractional derivative for formulating epidemiological models for the obvious reasons concerning the use of initial and boundary conditions and the differentiation of a constant being zero. For more details one can refer to the following researches [4], [7], [8], [18], [20], [23], [25], [26], [36], [40], [47], [52], [54], [58], [62].

1.2. **Structure of the Paper.** The following is how rest of the paper is set up: Section 2 presents auxiliary results and essential notions from fractional calculus. The LSD propagation model is devised in Section 3, along with a schematic diagram for the same. Section 4 provides us with the insights of the model by providing the existence, uniqueness, positivity, and feasible region for the proposed system's solution, along with the analysis of the equilibrium points, reproduction number, and stability of the proposed model. Computational simulations are executed in Section 5 to backup the qualitative analysis results of the model. The findings and discussions required for the policy implications are covered in Section 6.

2. Auxiliary Results

Definition 1 ([31])**.** *The Caputo fractional derivative of a continuous function g on $[0,T]$ is defined as:*

$$\mathfrak{D}^{\alpha} g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} \frac{d^n}{ds^n} g(s) ds,$$

where $0 < \alpha \leq 1$, $n = [\alpha] + 1$, and $[\alpha]$ represents the integer part of α.

Definition 2 ([31]). *The fractional integral of a continuous function g on $L^1([0,T],\mathbb{R})$ of order $0<\alpha\leq 1$ corresponding to t is defined as:*

$$I^{\alpha}g(t)=\frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}g(s)ds.$$

Definition 3 ([29]). *The Laplace transform is defined by*

$$F(s)=L[f(t)]=\int_0^{\infty} e^{-st}f(t)dt,$$

where $f(t)$ is n-dimensional vector-valued function.

Definition 4 ([49]). *The Mittag-Leffler function in two parameters is defined as*

$$E_{\alpha,\beta}(z)=\sum_{k=0}^{\infty}\frac{z^k}{\Gamma(\alpha k+\beta)},\ z\in\mathbb{C},$$

where $\alpha>0$, $\beta>0$, $\mathbb{C}$ denotes the complex plane.

Lemma 1 ([29]). *Let $\mathbb{C}$ be a complex plane, for any $\alpha>0$, $\beta>0$ and $A\in C^{n\times n}$,*

$$L[t^{\beta-1}E_{\alpha,\beta}(At^{\alpha})]=\frac{s^{\alpha-\beta}}{s^{\alpha}-A}$$

holds for $Re(s)>\|A\|^{\frac{1}{\alpha}}$, where $Re(s)$ represents the real part of the complex number s.

Lemma 2 ([39]). *Let $F(s)$ be the Laplace transform of the function $f(t)$, n being an integer then the Laplace transform of the Caputo fractional derivative of order α is given by*

$$L(\mathfrak{D}^{\alpha}f(t))=s^{\alpha}F(s)-\sum_{k=1}^{n}s^{\alpha-k}f^{(k-1)}(0),\ n-1<\alpha\leq n.$$

Lemma 3 ([44], Generalized Mean Value Theorem). *Let $g(t)\in C[a,b]$ and $\mathfrak{D}^{\alpha}g(t)\in C[a,b]$ for $0<\alpha\leq 1$, then*

$$g(t)=g(a)+\frac{1}{\Gamma(\alpha)}(\mathfrak{D}^{\alpha}g)(s)(t-a)^{\alpha}$$

with $0\leq s\leq t$, $\forall\ t\in(a,b]$. Thus, we can deduce that for $g(t)\in C[0,b]$ and Caputo fractional derivative $\mathfrak{D}^{\alpha}g(t)\in C[0,b]$ for $0<\alpha\leq 1$, if $\mathfrak{D}^{\alpha}g(t)\geq 0$, $\forall\ t\in[0,b]$, then the function $g(t)$ is non-decreasing and if $\mathfrak{D}^{\alpha}g(t)\leq 0$, $\forall\ t\in[0,b]$, then the function $g(t)$ is non-increasing $\forall\ t\in[0,b]$.

Theorem 1 ([55]). *Consider the fractional differential equation:*

$$\begin{aligned}&\mathfrak{D}^{\alpha}\boldsymbol{x}(t)=f(t,\boldsymbol{x}(t)),\\ &\boldsymbol{x}^{(k)}(t_0)=\boldsymbol{x}_0^{(k)},\ k=0,1,\ldots,n-1,\end{aligned}\tag{1}$$

where $\mathfrak{D}^\alpha$ represents the Caputo fractional derivative. Let $L > 0$ and $f : [0, L] \times \mathbb{R} \to \mathbb{R}$ is continuous and suppose that there exists a real number $l > 0$ such that $|f(t, x) - f(t, y)| \leq l|x - y|$ for $t \in [0, L]$ and $x, y \in \mathbb{R}$. Then, the initial value problem has a unique solution in $AC[0, L]$.

Theorem 2 ([48]). *Consider the following fractional-order system:*

$$\mathfrak{D}^\alpha X(t) = \mathcal{F}(X); \tag{2}$$

with $0 < \alpha < 1$, $X(t) = [x^1(t), x^2(t), \ldots, x^n(t)]$ and $\mathcal{F}(X) : [t_0, \infty) \to \mathbb{R}^{n\times n}$. The equilibrium points of system (2) are evaluated by solving system of equations $\mathcal{F}(X) = 0$. These equilibrium points are locally asymptotically stable if each eigenvalue λ of the Jacobian matrix $J(X)$ calculated at the equilibrium points satisfies $|arg(\lambda_i)| > \frac{\alpha\pi}{2}$.

3. LSD Propagation Model

A lumpy skin disease propagation model is proposed by categorising the entire cattle population $\mathcal{N}$ into system four different classes: $\mathcal{S}$, $\mathcal{E}$, $\mathcal{I}$ and $\mathcal{R}$ susceptible, exposed, infected and recovered cattle population respectively. $\mathcal{S}$ reflects the cattle population that is prone to infection, $\mathcal{E}$ displays livestock that have previously been exposed to disease-causing germs (LSDV), $\mathcal{I}$ comprises of those cattle who have been identified and confirmed positive for LSD, and finally, the recovered cattle are placed in the category $\mathcal{R}$. According to the model, cattle enter the susceptible population at the rate of Ξ either by migration from some other state or by birth. Susceptible cattle become infected by interacting with the diseased cattle at a contact rate of β per cattle per time(morbidity rate). η, ρ, σ denotes the incubation, recovery, mortality rate of the disease respectively.

$$\begin{aligned}
\mathfrak{D}_t^\alpha \mathcal{S}_t &= \Xi - \beta \mathcal{S}_t \mathcal{I}_t - \sigma \mathcal{S}_t \,, \\
\mathfrak{D}_t^\alpha \mathcal{E}_t &= \beta \mathcal{S}_t \mathcal{I}_t - (\sigma + \eta)\mathcal{E}_t \,, \\
\mathfrak{D}_t^\alpha \mathcal{I}_t &= \eta \mathcal{E}_t - (\rho + \sigma)\mathcal{I}_t \,, \\
\mathfrak{D}_t^\alpha \mathcal{R}_t &= \rho \mathcal{I}_t - \sigma \mathcal{R}_t
\end{aligned} \tag{3}$$

along with the initial conditions $\mathcal{S}_{t=0} = \mathcal{S}_0$, $\mathcal{E}_{t=0} = \mathcal{E}_0$, $\mathcal{I}_{t=0} = \mathcal{I}_0$, $\mathcal{R}_{t=0} = \mathcal{R}_0$. Here, $\mathfrak{D}_t^\alpha$ is the Caputo fractional derivative of order α; $0.5 < \alpha < 1$.

Table 1. Meaning of various parameters

Parameter	Significance
Ξ	influx rate or birth/migration rate
β	morbidity rate/number of bites
η	incubation rate
ρ	recovery rate
σ	death rate

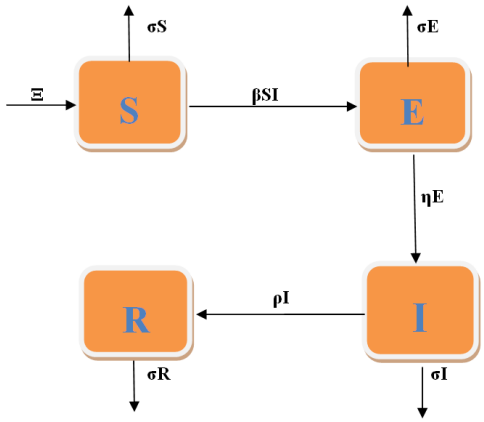


FIGURE 1. An illustration of the model's scheme.

4. Model Analysis

This section marks the discussion about the uniqueness of the solution along with its non-negative and bounded nature, the equilibrium points and basic reproduction number are also obtained for the model.

Theorem 3. *There is a unique solution* $\mathcal{U}(t) = [\mathcal{S}(t), \mathcal{E}(t), \mathcal{I}(t), \mathcal{R}(t)]^T$ *for the initial value problem given by the system of equations in* (3) *on* $t \geq 0$ *in* $(0, \theta)$ *and the solution will remain in* $\mathbb{R}^4_+$. *Furthermore, the solutions are all bounded.*

Proof. Here, Lemma 2 is used to establish the uniqueness of solution for the given system of initial value problems on $(0, \infty)$. Firstly, we shall establish the non-negativity and boundedness of solution. From model (3), we find

$$\begin{aligned}
\mathfrak{D}_t^\alpha \mathcal{S}_t|_{\mathcal{S}=0} &= \Xi > 0\,,\\
\mathfrak{D}_t^\alpha \mathcal{E}_t|_{\mathcal{E}=0} &= \beta \mathcal{S}_t \mathcal{I}_t \geq 0\,,\\
\mathfrak{D}_t^\alpha \mathcal{I}_t|_{\mathcal{I}=0} &= \eta \mathcal{E}_t \geq 0\,,\\
\mathfrak{D}_t^\alpha \mathcal{R}_t|_{\mathcal{R}=0} &= \rho \mathcal{I}_t \geq 0\,.
\end{aligned}$$

The vector field on each hyperplane enclosing the non-negative orthant points into $\mathbb{R}^4_+$. Furthermore, from system (3)

$$\begin{aligned}
\mathfrak{D}^\alpha \mathcal{N}(t) &= \Xi - \sigma \mathcal{N}(t) \geq 0\,,\\
i.e.\ \mathfrak{D}^\alpha \mathcal{N}(t) &+ \sigma \mathcal{N}(t) \leq \Xi\,. \qquad (4)
\end{aligned}$$

Thus, from equation (4) and deduction of Lemma 3, in the case of LSD infection, the total population and hence the sub populations are all bounded. Consequently,

the IVP's biologically viable region (3) is

$$\Omega = \left\{ (\mathcal{S}_t, \mathcal{E}_t, \mathcal{I}_t, \mathcal{R}_t) \in \mathbb{R}^4_+ : \mathcal{S}, \mathcal{E}, \mathcal{I}, \mathcal{R} \geq 0;\ 0 \leq \mathcal{S}_t + \mathcal{E}_t + \mathcal{I}_t + \mathcal{R}_t \leq \frac{\Xi}{\sigma} \right\}. \quad (5)$$

The next step is to demonstrate the uniqueness of solution in $\Omega\ \forall\ t \geq 0$. As we know, $\mathcal{N}(t)$ is the sum $\mathcal{S}(t)$, $\mathcal{E}(t)$, $\mathcal{I}(t)$, $\mathcal{R}(t)$ populations. The Caputo fractional derivative of order α of this equation, gives

$$\mathfrak{D}^\alpha \mathcal{N}(t) = \mathfrak{D}^\alpha \mathcal{S}_t + \mathfrak{D}^\alpha \mathcal{E}_t + \mathfrak{D}^\alpha \mathcal{I}_t + \mathfrak{D}^\alpha \mathcal{R}_t$$

which gives

$$\mathfrak{D}^\alpha \mathcal{N}(t) = \Xi - \sigma \mathcal{N}(t).$$

Now, by taking Laplace transformation using Lemma 2, we have

$$\mathcal{N}(s) = \frac{\Xi\ s^{-1} + s^{\alpha-1}\mathcal{N}(0)}{s^\alpha + \sigma}\,.$$

Using Lemma 1 to obtain inverse Laplace transformation, we get

$$\mathcal{N}(t) = \frac{\Xi}{\sigma}[1 - E_\alpha(-\sigma t^\alpha)] + \mathcal{N}(0) E_\alpha(-\sigma t^\alpha)\,.$$

From the complete monotonicity of $E_\alpha(-t)$ for $t > 0$ and $0 \leq E_\alpha(-\sigma t^\alpha) \leq 1$ on $0 < \alpha \leq 1$ [35], [50], we obtain

$$N(t) \leq \frac{\Xi}{\sigma}\,. \quad (6)$$

To explore the presence of unique solution, we assume the model (3), where all the functions on right hand side of system of equation (3) are continuous and bounded for $t \geq 0$ as $\mathcal{S}(t)$, $\mathcal{E}(t)$, $\mathcal{I}(t)$, $\mathcal{R}(t)$ bounded by equation (6). Also, they satisfy Lipschitz condition. Thus, there exists a bounded and unique solution of the proposed model on $(0, \infty)$ owing to Theorem 1. □

4.1. **Equilibrium Points.**

4.1.1. *LSD-free equilibrium.* When there are no infected cattle i.e. $\mathcal{I}_t = 0$. The LSD-free equilibrium point (E_0) is attained when we take $\mathcal{E} = 0$, $\mathcal{I} = 0$, $\mathcal{R} = 0$. Thus, the steady state for LSD-free equilibrium is $\left(\frac{\Xi}{\sigma}, 0, 0, 0\right)$.

4.1.2. *Reproduction Number:* The number of cattle infected by a single sick cattle throughout the course of the incubation period in the population of entirely susceptible cattle is known as the reproduction number (R_0). The largest eigenvalue of $\mathcal{F}^*\mathcal{V}^{*^{-1}}$ at E_0 is used to calculate the reproduction number (R_0) of the given model [61].

$$\left[\mathfrak{D}^\alpha \mathcal{S}_t, \mathfrak{D}^\alpha \mathcal{E}_t, \mathfrak{D}^\alpha \mathcal{I}_t, \mathfrak{D}^\alpha \mathcal{R}_t\right]^T = \mathcal{F}(t) - \mathcal{V}(t)\,, \quad (7)$$

where $\mathcal{F}$ represents the rate at which new infections appear in different classes, $\mathcal{V}^-$ is the pace of shifting individual cattle into various classes using all other methods,

and $\mathcal{V}^+$ is the pace at which individual cattle are transferred between classes. Also, $\mathcal{V}(t)$=$\mathcal{V}^-(t) - \mathcal{V}^+(t)$ such that

$$\mathcal{F}(t) = \begin{bmatrix} 0 \\ \beta \mathcal{S}_t \mathcal{I}_t \\ 0 \\ 0 \end{bmatrix}, \ \mathcal{V}^+(t) = \begin{bmatrix} \Xi \\ 0 \\ \eta \mathcal{E}_t \\ \rho \mathcal{I}_t \end{bmatrix}, \ \mathcal{V}^-(t) = \begin{bmatrix} \beta \mathcal{S}_t \mathcal{I}_t + \sigma \mathcal{S}_t \\ (\sigma + \eta)\mathcal{E}_t \\ (\rho + \sigma)\mathcal{I}_t \\ \sigma \mathcal{R}_t \end{bmatrix}.$$

At E_0, the Jacobian matrix of $\mathcal{F}(t)$ is given by

$$\mathcal{F}^*(t) = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{\beta\ \Xi}{\sigma} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The Jacobian matrix of $\mathcal{V}(t)$ is

$$\mathcal{V}^*(t) = \mathcal{V}^{*-}(t) - \mathcal{V}^{*+}(t) = \begin{bmatrix} \sigma & 0 & \frac{\beta\ \Xi}{\sigma} & 0 \\ 0 & (\sigma + \eta) & 0 & 0 \\ 0 & -\eta & (\rho + \sigma) & 0 \\ 0 & 0 & -\rho & \sigma \end{bmatrix}.$$

$\mathcal{F}^*{\mathcal{V}^*}^{-1}$ is the next generation matrix for the model. And, R_0 is the spectral radius of this matrix. Now, the eigenvalues of $\mathcal{F}^*{\mathcal{V}^*}^{-1}$ are $0, 0, 0$ and $\frac{\beta\ \Xi\ \eta}{\sigma(\sigma+\eta)(\sigma+\rho)}$. Thus, the reproduction number is given by

$$R_0 = \frac{\beta\ \Xi\ \eta}{\sigma(\sigma + \eta)(\sigma + \rho)}. \tag{8}$$

Analyzing R_0:
To determine how sensitive each of R_0's parameters is,

$$\frac{\partial R_0}{\partial \beta} = \frac{\Xi\ \eta}{\sigma(\sigma + \eta)(\sigma + \rho)} > 0, \tag{9}$$

$$\frac{\partial R_0}{\partial \Xi} = \frac{\beta \eta}{\sigma(\sigma + \eta)(\sigma + \rho)} > 0, \tag{10}$$

$$\frac{\partial R_0}{\partial \eta} = \frac{\beta\ \Xi\ \sigma}{\sigma(\sigma + \eta)^2(\sigma + \rho)} > 0, \tag{11}$$

$$\frac{\partial R_0}{\partial \rho} = \frac{-\beta\ \Xi\ \sigma}{\sigma(\sigma + \eta)(\sigma + \rho)^2} < 0, \tag{12}$$

$$\frac{\partial R_0}{\partial \sigma} = \frac{-\beta\ \Xi\ \eta}{\sigma(\sigma + \eta)(\sigma + \rho)} \left\{ \frac{1}{\sigma} + \frac{1}{(\sigma + \eta)} + \frac{1}{(\sigma + \rho)} \right\} < 0. \tag{13}$$

Thus, R_0 is increasing with β, Ξ, η and decreasing with ρ and σ.

4.1.3. *LSD-Persistent Equilibrium.* When the number of infected cattle i.e. $\mathcal{I}_t \neq 0$. The LSD-persistent equilibrium point (E_1) is attained when the number of infected cattle is not zero i.e ($\mathcal{I} \neq 0$). Therefore, the disease persistent equilibrium point is given by $(\mathcal{S}_1, \mathcal{E}_1, \mathcal{I}_1, \mathcal{R}_1)$, where

$$\mathcal{S}_1 = \frac{(\sigma+\eta)(\sigma+\rho)}{\beta\eta},\ \mathcal{E}_1 = \frac{\beta\ \Xi\ \eta - \sigma(\sigma+\eta)(\sigma+\rho)}{\beta\eta(\sigma+\eta)},$$
$$\mathcal{I}_1 = \frac{\sigma(R_0-1)}{\beta} \text{ and } \mathcal{R}_1 = \frac{\rho(R_0-1)}{\beta}$$

which implies $(\mathcal{S}_1, \mathcal{E}_1, \mathcal{I}_1, \mathcal{R}_1) > 0$ iff $R_0 > 1$. So, the LSD-persistent steady state exists iff $R_0 > 1$. For $R_0 = 1$, LSD-persistent steady state becomes LSD-free steady state.

4.2. **Stability Analysis.**

Theorem 4. *LSD-free equilibrium point* $E_0 = \left(\frac{\Xi}{\sigma}, 0, 0, 0\right)$ *of the system is locally asymptotically stable when* $R_0 < 1$*, unstable otherwise.*

Proof. The Jacobian matrix at E_0 is

$$\begin{bmatrix} -\sigma & 0 & -\frac{\beta\ \Xi}{\sigma} & 0 \\ 0 & -(\sigma+\eta) & \frac{\beta\ \Xi}{\sigma} & 0 \\ 0 & \eta & -(\rho+\sigma) & 0 \\ 0 & 0 & \rho & -\sigma \end{bmatrix}.$$

Now, two of the eigenvalues are $-\sigma$. The characteristic equation for finding the remaining two eigenvalues is given by

$$P(\lambda) = \lambda^2 + P_1\lambda + P_2\,, \tag{14}$$

where

$$P_1 = (2\sigma + \eta + \rho),$$
$$P_2 = (\eta+\sigma)(\rho+\sigma) - \frac{\beta\ \Xi\ \eta}{\sigma} = (\eta+\sigma)(\rho+\sigma)[1-R_0].$$

Now, $P_1 > 0$ always and $P_2 > 0$ for $R_0 < 1$. Thus, for $R_0 < 1$, by using Routh-Hurwitz criteria [5], all the eigenvalues of the Jacobian matrix at E_0 have negative real parts, it implies from Theorem 2 that the LSD-free equilibrium point is locally asymptotically stable when $R_0 < 1$ and unstable otherwise. □

Theorem 5. *The LSD-persistent equilibrium point* $E_1 = (\mathcal{S}_1, \mathcal{E}_1, \mathcal{I}_1, \mathcal{R}_1)$ *exists and is locally asymptotically stable iff* $R_0 > 1$.

Proof. The Jacobian matrix at E_1 is

$$\begin{bmatrix} -\sigma R_0 & 0 & -\frac{(\eta+\sigma)(\rho+\sigma)}{\eta} & 0 \\ \sigma(R_0-1) & -(\sigma+\eta) & \frac{(\eta+\sigma)(\rho+\sigma)}{\eta} & 0 \\ 0 & \eta & -(\rho+\sigma) & 0 \\ 0 & 0 & \rho & -\sigma \end{bmatrix}.$$

Thus, on observation we see that one of the eigenvalues is $-\sigma$. The characteristic equation to obtain the remaining eigenvalues is

$$P(\lambda) = \lambda^3 + P_1\lambda^2 + P_2\lambda + P_3\,,$$

where

$$\begin{aligned} P_1 &= (\sigma(R_0+2) + \eta + \rho)\,,\\ P_2 &= \sigma R_0(2\sigma + \eta + \rho)\,,\\ P_3 &= (R_0-1)\sigma(\eta+\sigma)(\rho+\sigma)\,. \end{aligned}$$

Clearly, $P_1 > 0$ and $P_3 > 0$ whenever $R_0 > 1$. Also, $P_1P_2 - P_3 > 0$. Thus, by Routh-Hurwitz criterion, all the eigenvalues of the Jacobian matrix of the system of equations defining the model have negative real parts at LSD-persistent equilibrium point E_1 for $R_0 > 1$, which ensures the locally asymptotic stability of the LSD-persistent equilibrium point for $R_0 > 1$ and unstable elsewhere using Theorem 2. □

5. Numerical Simulations

Computing findings that highlight the fluctuating nature of the lumpy skin disease propagation model and to verify the analytical outcomes for multiple derivative orders are presented in this section. Using a MATLAB programme supplied by Roberto Garappa in [22], the proposed model is solved using the Adams-Bashforth-Moulton predictor-corrector method. Table 2 carries the variables and parameters used for simulation. According to 19th livestock census-2012 and 20th livestock census-2019 all India report the total Cattle population in the country was 190.90 and 192.50 million respectively [41]. This shows that there has been an approximate increase of 0.0114 percent per year giving us the birth rate or the influx rate (Ξ). The morbidity rate (β) can be retrieved from [43] by making a few necessary changes to it. As per the 20th livestock census-2019, the total cattle population in the state of Gujarat is 10,165,000. Therefore, the total susceptible cattle population is 10,165,000/232. Similarly as in the case of COVID-19 (there it was 250 for the Wuhan city with a population of 11 million), the denominator was chosen early in the epidemic and later proven to be a reasonable figure. It is a suitable parameter for limiting the movement of cattle that were imposed by the respective state governments on different dates between July to September, 2022 as reported by various newspapers [34]. Now, assuming the average number of bites per cattle per day to be 5, this gives us $\beta = 5 * 10,165,000/232$ [43]. The incubation period is between 4 to 14 days [33]. Since, there is no or a little information available about the mortality rate and recovery period(reciprocal of the recovery rate), we assume them to be 0.0057 (half of the birth rate) and 7 days (keeping a positive view), respectively.

TABLE 2. Parameter Values

Parameter	**Value**	**Source**
Ξ	0.0114	[41]
β	1.1412×10^{-4}	[43]
η	$1/6$	[33]
ρ	$1/7$	Assumed
σ	0.0057	Assumed

Population	$\mathcal{S}$	$\mathcal{E}$	$\mathcal{I}$	$\mathcal{R}$
Initial Values	43815	1	1	0

For the initial populations, the initial susceptible population along with restricted cattle movement is assumed to be $\mathcal{S}_0 = 10,165,000/232$, we assume that initially exposed and infected cattle are 1 each, no recovered cattle. In the event that $R_0 > 1$, the cattle population cannot be free of disease. Following the start of the pandemic, the number of susceptible cattle continued to decline, while the exposed and infected cattle classes show a rapid rise in population density, as seen by Figures 2, 3 and 4, respectively. The rapid rise in the number of recovered cattle population in Figure 5 can be attributed to the massive vaccination drive in the state of Gujarat, steps were made to control disease causing vectors and restrict bovine movement. Regardless of the order, the plots in Figure 6 for each class of cattle population indicates that the proposed model is asymptotically stable for the LSD-persistent equilibrium points the population swiftly approaches its equilibria when we increase the value of α. Since the susceptible and infected cattle populations are reduced to negative populations, which is something we all know is not conceivable, we can plainly state that the fractional order models are far superior than the conventional integer order model with $\alpha = 1$. The equations (9), (10) and (13) support the findings of Figure 7(a), (b)and (c), respectively. Equation (11) demonstrates that R_0 rises with an increase in the incubation rate, η, and falls with an increase in the incubation duration $(1/\eta)$, as shown by Figure 8(a). In a similar vein, equation (12) reveals that R_0 drops as the recovery rate, (ρ) rises. The recovery period $(1/\rho)$ grows as R_0 does, as shown by Figure 8(b).

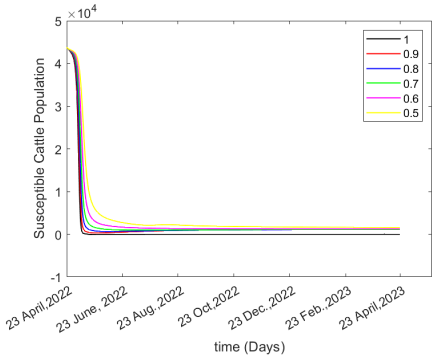


FIGURE 2. Graphical display of the susceptible class at various fractional orders.

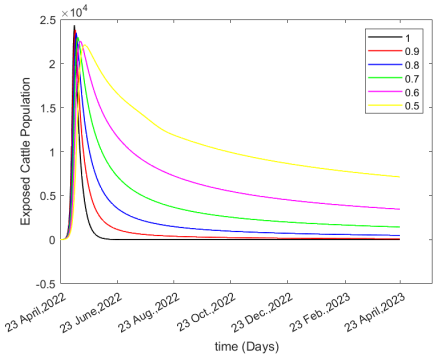


FIGURE 3. Graphical display of the exposed class at various fractional orders.

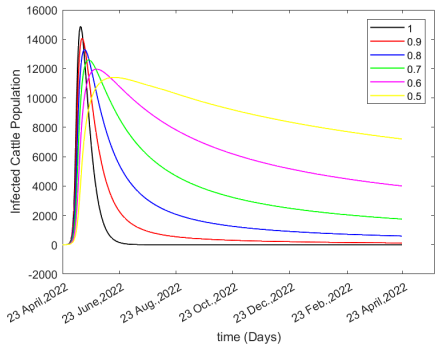


FIGURE 4. Graphical display of the infected class at various fractional orders.

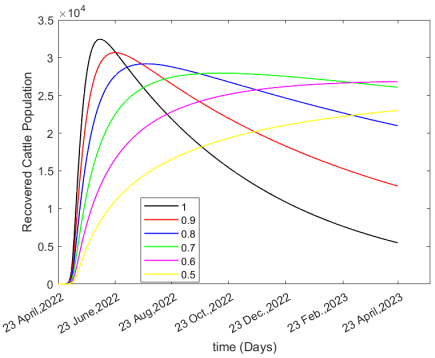
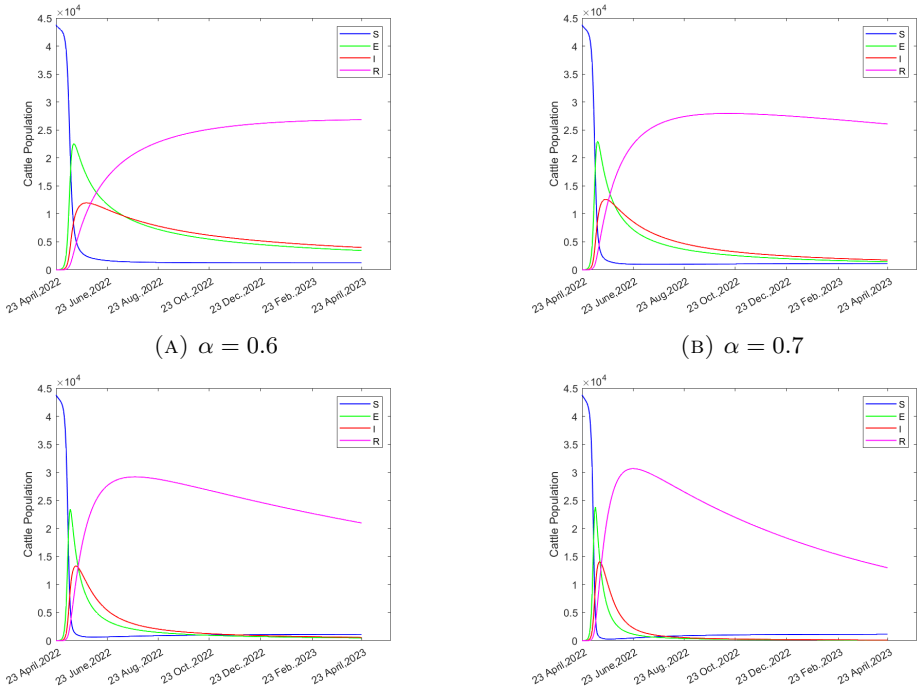


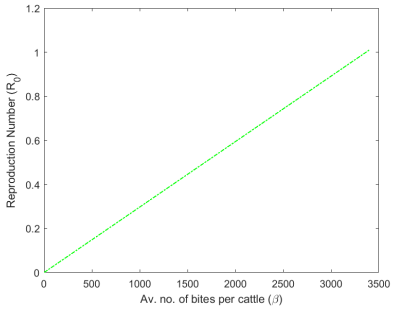
FIGURE 5. Graphical display of the recovered class at various fractional orders.



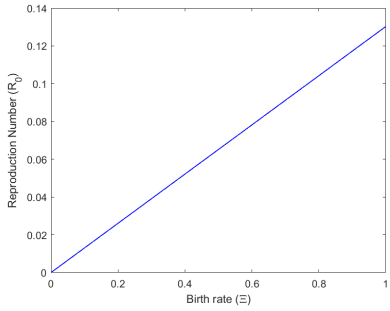
(A) $\alpha = 0.6$ (B) $\alpha = 0.7$

(C) $\alpha = 0.8$ (D) $\alpha = 0.9$

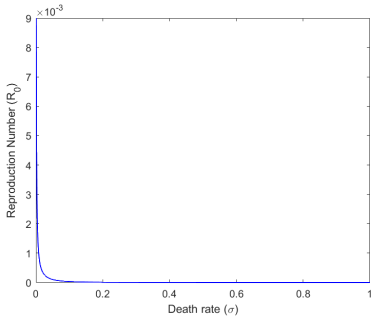
FIGURE 6. Variations of susceptible, exposed, infected and recovered cattle populations with different values of α



(A) R_0 increases with increase in morbidity rate (β)

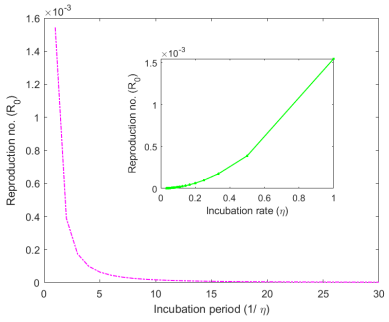


(B) R_0 increases with increase in birth rate (Ξ)

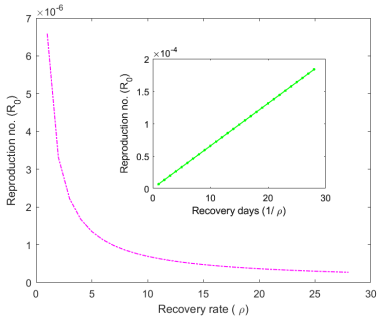


(C) R_0 decreases with increase in death rate (σ)

FIGURE 7. Variation of R_0 with β, Ξ, σ



(A) R_0 increases with increase in incubation rate (η)



(B) R_0 decreases with increase in recovery rate (ρ)

FIGURE 8. Variation of R_0 with η and ρ

6. Concluding Remarks And Future Strategy

For $R_0 < 1$, the diffusion of the virus can be controlled, and the equilibrium free of LSD can be preserved across Gujarat. The susceptible cattle population keeps on decreasing with time. The exposed and infected cattle population regularly rises over time until it reaches a peak, after which it starts to decline until it attains equilibrium. We can see that the best results are shown by taking $\alpha = 0.5$ as it shows the infected cases reach an all-time high in 56 days following the discovery of the first case on April 23 of this year in the hamlet of Kaiyari, located on the Indo-Pak border in the Kutch district's Lakhpat taluka. Mosquito and housefly infestations continue at their peak during the monsoon season, and veterinary scientists and government officials blame a very wet July for the infection's quick spread in Gujarat this year. So far, Gujarat has experienced 1010 mm of rain, which is 20 percent higher than the state normal of 850 mm. The four-month south-west monsoon season began in June and ended in September. There is also an issue with feral cattle in Gujarat, a state where cow slaughter is outlawed, and experts believe these free-roaming cattle may be a factor in the quick spread of LSD. The dearth of knowledge about the sickness may also contribute to the rapid spread of LSD. As can be seen, the peak does not last long, which might be attributed to the state animal husbandry department treating diseased cattle and administering goat pox vaccine to healthy animals in surrounding regions.

This current investigation suggests the following policy changes to assist, isolate, and stop the further spread: import restrictions on domestic cattle and water buffaloes, as well as their products; surveillance beyond the containment zone of goods, trash, and disease spreading vectors; restriction on movement of cattle; pest control measures; incineration; and cleaning and disinfection of the surroundings.

Effective LSD treatment with complete coverage is required. Given that LSD is in close relation to the sheep pox and goat pox viruses, vaccine against same is used to treat LSD. New animals should be inoculated before being introduced to the afflicted farm. Calves reared from vaccinated or naturally infected moms should be inoculated at the age of 3 to 4 months. Bulls used for breeding and pregnant cows can both receive annual vaccinations [21]. The R_0 may be used to calculate the amount of vaccine needed to suppress an epidemic (i.e. to reduce R_0 below one). The study also emphasised the need of starting immunisation efforts ahead of viral entrance.

Author Contribution Statements The authors contributed equally to this article.

Declaration of Competing Interests The authors declare that they have no competing interests.

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