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Applications of Several Minimal Point Principles

Sehie Park[a]

[a] *The National Academy of Sciences, Republic of Korea; Seoul 06579 and Department of Mathematical Sciences, Seoul National University, Seoul 08826, Korea.*

Abstract

In our previous works, a Metatheorem in ordered fixed point theory showed that certain maximal element principles can be reformulated to various types of fixed point theorems for progressive maps and conversely. Therefore, there should be the dual principles related to minimality, anti-progressive maps, and others. In the present article, we derive several minimal element principles particular to Metatheorem and their applications. One of such applications is the Brøndsted-Jachymski Principle. We show that known examples due to Zorn (1935), Kasahara (1976), Brézis-Browder (1976), Tasković (1989), Zhong (1997), Khamsi (2009), Cobzaş (2011) and others can be improved and strengthened by our new minimal element principles.

Keywords: The 2023 Metatheorem Brøndsted-Jachymski Principle Zorn's Lemma Caristi fixed point theorem Ekeland variational principle preorder fixed point stationary point minimum principle.
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1. Introduction

In 1985-87, we introduced a Metatheorem [10-12] for some equivalent statements on maximality, fixed points, stationary points, common fixed points, common stationary points, and others. In 1985-2000, we had published several articles mainly related to the Ekeland variational principle for approximate solutions of minimization problems and its equivalent formulations with some applications. From the beginning of such study, we applied Metatheorem for various occasions. However, for a long period it was not attracted by any other peoples.

In 2022, we found the extended version of Metatheorem in [12-15, 17] and applied its several particular forms to various results in ordered fixed point theory, nonlinear analysis, and other fields. In fact, we applied Metatheorem to Zorn's lemma, Banach contraction principle, Nadler's fixed point theorem, Ekeland's

Email address: park35@snu.ac.kr; sehiepark@gmail.com; parksehie.com (Sehie Park)

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variational principle, Brézis-Browder principle, Caristi's fixed point theorem, Takahashi's nonconvex minimization theorem, some others and their variants, generalizations or equivalent formulations. Consequently, many new theorems of other authors after 2000 can be reformulated equivalently to more useful ones.

While we were studying such topics in 2022, we found the Brøndsted principle for preordered sets [16] and the Brøndsted-Jachymski principle for partially ordered sets [18,19] with several applications of them. Later, we found that those two principles are consequences of Metatheorem. Moreover, in [22], we obtained the expanded 2013 Metatheorem and applied it to the previous manuscripts [18-21] and others.

Note that those maximal element principles are closely related to fixed points of progressive maps and others, and have scores of examples as we have seen in our previous works. Therefore, there should be the dual principles related to minimality, anti-progressive maps, and others. We found such dual principles which are also consequences of Metatheorem. In the present article, we obtain several principles related to minimality particular to Metatheorem. We show that known examples given by Zorn, Kasahara [8], Brézis-Browder [1], Tasković [23], Zhong [24], Khamsi [9], Cobzaş [3,4], and Park [21] can be reformulated, improved and strengthened by our new miniml element principles.

In the present article, Section 2 devotes to the 2023 Metatheorem with the proof for completeness. In Section 3, by applying Metatheorem to preordered sets (Theorem 3.1), we obtain logically equivalent formulations of existence of maximal (or minimal) elements, fixed points of (anti-)progressive maps, collectively fixed or stationary points, etc. Section 4 deals with dual forms of the Brøndsted principle for preordered sets and the Brøndsted-Jachymski principle for partially ordered sets. In Section 5, as the first example of existence of minimal elements, we obtain equivalent formulations and a strengthened form of Zorn's Lemma and a result of Kasahara [8]. This is applied to the well-known Brézis-Browder principle [1] in Section 6 and to a result of Tasković [23] in Section 7. Section 8 devotes to an application of our method to Zhong's example [24] in 1998 on the maps in Banach spaces satisfying the weak P.-S. condition. In Section 9, we improve a result of Khamsi [9] in 2009 and show two examples given by Park [21] in 2022. Section 10 devotes to apply Metatheorem to some results of Cobzaş [4] in 2011. Finally, Section 11 deals with some conclusion.

2. The Metatheorem in Ordered Fixed Point Theory

In order to get some equivalents of the well-known central result of I. Ekeland [5, 6] on the variational principle for approximate solutions of minimization problems, we obtained a Metatheorem in [10-12] and its applications in 1983-2000. Later in 2022 we found an extended version of the Metatheorem [13-15, 17]. Now the following is the 2023 version in [22].

Metatheorem. *Let X be a set, A its nonempty subset, and $G(x,y)$ a sentence formula for $x,y \in X$. Then the following are equivalent:*

(α) *There exists an element $v \in A$ such that $G(v,w)$ for any $w \in X\backslash\{v\}$.*

($\beta 1$) *If $f : A \to X$ is a map such that, for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X\backslash\{x\}$ satisfying $\neg G(x,y)$, then f has a fixed element $v \in A$, that is, $v = f(v)$.*

($\beta 2$) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ such that, for any $x \in A$ with $x \neq f(x)$, there exists a $y \in X\backslash\{x\}$ satisfying $\neg G(x,y)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

($\gamma 1$) *If $f : A \to X$ is a map such that $\neg G(x, f(x))$ for any $x \in A$, then f has a fixed element $v \in A$, that is, $v = f(v)$.*

($\gamma 2$) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ satisfying $\neg G(x, f(x))$ for all $x \in A$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

($\delta 1$) *If $F : A \multimap X$ is a multimap such that, for any $x \in A\backslash F(x)$, there exists $y \in X\backslash\{x\}$ satisfying $\neg G(x,y)$, then F has a fixed element $v \in A$, that is, $v \in F(v)$.*

($\delta 2$) *Let $\mathfrak{F}$ be a family of multimaps $F : A \multimap X$ such that, for any $x \in A\backslash F(x)$, there exists $y \in X\backslash\{x\}$ satisfying $\neg G(x,y)$. Then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v \in F(v)$ for all $F \in \mathfrak{F}$.*

(ϵ1) *If* $F : A \multimap X$ *is a multimap satisfying* $\neg G(x, y)$ *for any* $x \in A$ *and any* $y \in F(x)\backslash\{x\}$*, then* F *has a stationary element* $v \in A$*, that is,* $\{v\} = F(v)$*.*

(ϵ2) *If* $\mathfrak{F}$ *is a family of multimaps* $F : A \multimap X$ *such that* $\neg G(x, y)$ *holds for any* $x \in A$ *and any* $y \in F(x)\backslash\{x\}$*, then* $\mathfrak{F}$ *has a common stationary element* $v \in A$*, that is,* $\{v\} = F(v)$ *for all* $F \in \mathfrak{F}$*.*

(ζ1) *If a multimap* $F : A \multimap X$ *satisfies, for all* $x \in A$ *with* $F(x) \neq \emptyset$*, there exists* $y \in X\backslash\{x\}$ *such that* $\neg G(x, y)$ *holds, then there exists* $v \in A$ *such that* $F(v) = \emptyset$*.*

(ζ2) *Let* $\mathfrak{F}$ *be a family of multimaps* $F : A \multimap X$ *such that, for all* $x \in A$ *with* $F(x) \neq \emptyset$*, there exists* $y \in X\backslash\{x\}$ *satisfying* $\neg G(x, y)$*. Then there exists* $v \in A$ *such that* $F(v) = \emptyset$ *for all* $F \in \mathfrak{F}$*.*

(η) *If* Y *is a subset of* X *such that, for each* $x \in A\backslash Y$*, there exists a* $z \in X\backslash\{x\}$ *satisfying* $\neg G(x, z)$*, then there exists a* $v \in A \cap Y$*.*

Here, $\neg$ denotes the negation. From now on, this version will be called the 2023 Metatheorem.

3. Maximal or Minimal Element Principles

Recall that a preorder is the one satisfying reflexivity and transitivity; and a partial order is the one satisfying additional antisymmetry. A partially ordered set is abbreviated to a poset, sometimes.

As the first application of an abridged form of Metatheorem, we apply it to preordered sets with $G(x, y)$ means $x \not\preceq y$ (resp. $y \not\preceq x$) as the following prototype of Maximal (resp. Minim) Element Principles which is an extended form of our previous maximal element principle in [15]:

Theorem 3.1. *Let* $(X, \preceq)$ *be a preordered set and* A *a nonempty subset of* X*. Then the following statements are equivalent:*

(α) *There exists a maximal (resp. minimal) element* $v \in A$*, that is,* $v \not\preceq w$ *(resp.* $w \not\preceq v$*) for any* $w \in X\backslash\{v\}$*.*

(β) *If* $\mathfrak{F}$ *is a family of maps* $f : A \to X$ *such that, for any* $x \in A$ *with* $x \neq f(x)$*, there exists a* $y \in X\backslash\{x\}$ *satisfying* $x \preceq y$ *(resp.* $y \preceq x$*), then* $\mathfrak{F}$ *has a common fixed element* $v \in A$*, that is,* $v = f(v)$ *for all* $f \in \mathfrak{F}$*.*

(γ) *If* $\mathfrak{F}$ *is a family of maps* $f : A \to X$ *satisfying* $x \preceq f(x)$ *(resp.* $f(x) \preceq x$*) for all* $x \in A$ *with* $x \neq f(x)$*, then* $\mathfrak{F}$ *has a common fixed element* $v \in A$*, that is,* $v = f(v)$ *for all* $f \in \mathfrak{F}$*.*

(δ) *Let* $\mathfrak{F}$ *be a family of multimaps* $F : A \multimap X$ *such that, for any* $x \in A\backslash F(x)$*, there exists* $y \in X\backslash\{x\}$ *satisfying* $x \preceq y$ *(resp.* $y \preceq x$*). Then* $\mathfrak{F}$ *has a common fixed element* $v \in A$*, that is,* $v \in F(v)$ *for all* $F \in \mathfrak{F}$*.*

(ϵ) *If* $\mathfrak{F}$ *is a family of multimaps* $F : A \multimap X$ *such that* $x \preceq y$ *(resp.* $y \preceq x$*) holds for any* $x \in A$ *and any* $y \in F(x)\backslash\{x\}$*, then* $\mathfrak{F}$ *has a common stationary element* $v \in A$*, that is,* $\{v\} = F(v)$ *for all* $F \in \mathfrak{F}$*.*

(ζ) *Let* $\mathfrak{F}$ *be a family of multimaps* $F : A \multimap X$ *such that, for all* $x \in A$ *with* $F(x) \neq \emptyset$*, there exists* $y \in X\backslash\{x\}$ *satisfying* $x \preceq y$ *(resp.* $y \preceq x$*). Then there exists* $v \in A$ *such that* $F(v) = \emptyset$ *for all* $F \in \mathfrak{F}$*.*

(η) *If* Y *is a subset of* X *such that, for each* $x \in A\backslash Y$*, there exists a* $z \in X\backslash\{x\}$ *satisfying* $x \preceq z$ *(resp.* $z \preceq x$*), then there exists a* $v \in A \cap Y$*.*

Remark. (1) Note that we claimed that $(\alpha) - (\eta)$ are equivalent in Theorem 3.1 and did not say that they are true. For a counter-example, consider the real line $\mathbb{R}$ with the usual order. However, we gave many examples that they are true based on their original sources; see the articles mentioned in our [22].

(2) All of the elements v's in Theorem 3.1 are same as we have seen in the proof of Metatheorem in [22].

Proof. In Metatheorem, let $G(x, y)$ be the statement $x \not\preccurlyeq y$ (resp. $y \not\preccurlyeq x$) for all $x, y \in A$. Then the equivalency is a consequence of our 2023 Metatheorem. □

Remark. Some part of Theorem 3.1 coincides to Theorem 3.1 of [15] with adding its dual.

4. The Brøndsted-Jachymski Principle

From Theorem 3.1$(\alpha) \Longleftrightarrow (\gamma)$, we have the following:

Brøndsted Principle. *Let $(E, \preccurlyeq)$ be a preordered set and $f : E \to E$ be a map such that $x \preccurlyeq f(x)$ (resp. $f(x) \preccurlyeq x$) for all $x \in E$. Then a maximal (resp. minimal) element $v \in E$ is a fixed point of f.*

We adopted the maximal case in [16] instead of the main result of [2].

For a preordered set $(X, \preccurlyeq)$ and a map $f : X \to X$, we denote

Max$(\preccurlyeq)$: the set of maximal elements;
Min$(\preccurlyeq)$: the set of minimal elements;
Fix(f) : the set of fixed points of f;
Per(f) : the set of periodic points $x \in X$; that is, $x = f^n(x)$ for some $n \in \mathbb{N}$.

In our previous works [18, 19], we established the following based on Brøndsted [2] in 1976 and Jachymski [7] in 2003:

Brøndsted-Jachymski Principle. *Let $(X, \preccurlyeq)$ be a poset and $f : X \to X$ be a progressive map (that is, $x \preccurlyeq f(x)$ for all $x \in X$). Then we have*

$$\mathrm{Max}(\preccurlyeq) \subset \mathrm{Fix}(f) = \mathrm{Per}(f).$$

If $f : X \to X$ is a anti-progressive (that is, $f(x) \preccurlyeq x$ for all $x \in X$), then we have

$$\mathrm{Min}(\preccurlyeq) \subset \mathrm{Fix}(f) = \mathrm{Per}(f).$$

,

This is not claiming the non-emptiness of those three sets. We noticed that, in most applications of this principle, the existence of a maximal element or a fixed point is achieved by an upper bound of a chain in $(X, \preccurlyeq)$ as we showed many examples in our previous works.

From now on, we give examples holding Theorem 3.1.

5. Example 1: Zorn and Kasahara

Motivated our recent work on generalized Zorn's Lemma in [20], we obtain the following:

Theorem 5.1. *Let $(X, \preccurlyeq)$ be a partially ordered set having a chain A with a lower bound $v \in A$.*
Then the seven equivalent statements of the minimum case of Theorem 3.1 hold including

(α) *$v \in A$ is a minimal element, that is, $w \not\preccurlyeq v$ for any $w \in X \backslash \{v\}$.*

(γ) *If $\mathfrak{F}$ is a family of maps $f : A \to X$ satisfying $f(x) \preccurlyeq x$ for all $x \in A$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in A$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

Proof. (α) Since A has a lower bound $v \in A$, v is minimal by our new Zorn's Lemma in [20]. Therefore $(\alpha) - (\eta)$ holds by Theorem 3.1. □

The following was due to Kasahara [8] in 1976:

Corollary 5.2 [8] *Let $\mathcal{F}$ be a family of selfmaps of a poset $(X, \preccurlyeq)$ such that $\forall f \in \mathcal{F}, \forall x \in X, f(x) \preccurlyeq x$. If for some element $e \in X$ each chain in X containing e has a lower bound, then the family $\mathcal{F}$ has a common fixed point.*

This follows from Theorem 5.1(γ) and has equivalent forms $(\alpha) - (\eta)$. Moreover, we have $\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Min}(\preccurlyeq) \neq \emptyset$ for all $f \in \mathcal{F}$.

We take the following from Cobzaş [3]:

Let $(Z, \preccurlyeq)$ be a partially ordered set. For $x \in Z$ put $S_+(x) = \{z \in Z : x \preccurlyeq z\}$ and $S_-(x) = \{z \in Z : z \preccurlyeq x\}$. We shall use the notation $x \prec y$ to designate the situation $x \preccurlyeq y$ and $x \neq y$. Note that any assertion concerning maximal elements has a dual formulation in terms of minimal elements, which can be obtained by reversing the order: $x \preccurlyeq_1 y \Longleftrightarrow y \preccurlyeq x$, so we have to prove only one of the assertions.

In our recent work [20], we obtained the dual of the following:

Theorem 5.3. *Let $(X, \preccurlyeq)$ be a partially ordered set, $x_0 \in X$, let $A = S_-(x_0) = \{y \in X : y \preccurlyeq x_0\}$ have a lower bound $v \in A$.*

Then the seven equivalent statements of Theorem 3.1 hold including

(α) $v \in A$ is a minimal element, that is, $w \not\preccurlyeq v$ for any $w \in X \backslash \{v\}$.

Proof. (α) Since A has a lower bound $v \in A$, for each $y \in A$, we have $v \preccurlyeq y \preccurlyeq x_0$. If $w \preccurlyeq v$ for some $w \in X$, then $w \in S_-(x_0) = A$ and $v \preccurlyeq w$. Since $(X, \preccurlyeq)$ is partially ordered, we have $w = v$. Hence v is minimal. Therefore (α) holds.

Let $G(x, y)$ be $y \not\preccurlyeq x$. Then $(\alpha) - (\eta)$ are equivalent by Metatheorem or Theorem 3.1. □

We need the following:

Definition 5.4. Let X be a set and $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be a function. Define a preorder $\preccurlyeq$ on X by $x \preccurlyeq y$ iff $\phi(x) \leq \phi(y)$ for all $x, y \in X$.

6. Example 2: Brezis-Browder

We take the following from Cobzaş [4]:

Theorem 6.1. (Brézis-Browder [1]) *Let $(Z, \preccurlyeq)$ be a partially ordered set.*

[I] *Suppose that $\psi : Z \to \mathbb{R}$ is a function satisfying the conditions*

(a) the function ψ is strictly increasing, i.e., $x \prec y \Longrightarrow \psi(x) < \psi(y)$;

(b) for each $x \in Z$, $\psi(S_-(x))$ is bounded below;

(c) for any decreasing sequence (x_n) in Z, there exists $y \in Z$ such that $y \preccurlyeq x_n$, $n \in \mathbb{N}$.

Then for each $x \in Z$ there exists a minimal element $z \in Z$ such that $z \preccurlyeq x$.

[II] *Dually, let $\varphi : Z \to \mathbb{R}$ be a function satisfying the conditions*

(a′) the function φ is strictly increasing, i.e., $x \prec y \Rightarrow \varphi(x) < \varphi(y)$;

(b′) for each $x \in Z$, $\varphi(S_+(x))$ is bounded above;

(c′) for any increasing sequence (x_n) in Z there exists $y \in Z$ such that $x_n \preccurlyeq y$, $n \in \mathbb{N}$.

Then for each $x \in Z$ there exists a maximal element $z \in Z$ such that $x \preccurlyeq z$.

From Theorem 6.1.[I], we have the following:

Theorem 6.2. *Let $(Z, \preccurlyeq)$ be a poset whose order is defined by a function $\phi : Z \to \mathbb{R}$ as in Definition 5.2 satisfying (a), (b), and (c) in Theorem 6.1.[I] Let $x_0 \in Z$ and $A = S_-(x_0)$.*

Then the seven statements in Theorem 5.1 hold including the following:

(α) There exists a minimal point $v \in A$ such that $w \not\preccurlyeq v$ for any $w \in X \backslash \{v\}$.

(γ) If $f : A \to X$ is a map such that $f(x) \preccurlyeq x$ for all $x \in A$, then f has a minimal and fixed element $v \in A$, that is, $v = f(v)$.

Now by the Brøndsted-Jachymski principle, we have a more stronger conclusion of Theorem 6.1.[I] as follows:

Theorem 6.3. *Under the hypothesis of Theorem 6.1.[I], if $f : Z \to Z$ is a map such that $f(x) \preccurlyeq x$ for all $x \in Z$, then we have the conclusion*

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Min}(\preccurlyeq) \neq \emptyset.$$

Similarly, the dual forms of Theorems 6.2 and 6.3 can be easily obtained.

7. Example 3: Tasković

Recall that Tasković [23] in 1989 showed that Zorn's lemma is equivalent to the following related to Theorem 3.1.(δ):

Theorem 7.1. [23] *Let $\mathcal{F}$ be a family of self-maps defined on a partially ordered set A such that $x \preccurlyeq f(x)$ [resp. $f(x) \preccurlyeq x$] for all $x \in A$ and all $f \in \mathcal{F}$. If each chain in A has an upper bound (resp. a lower bound), then the family $\mathcal{F}$ has a common fixed point.*

This can be extended as the following extension of Theorem 5.1:

Theorem 7.2. *Let $(A, \preccurlyeq)$ be a partially ordered set such that a chain in A has an upper bound (resp. lower bound). Then the equivalent statements $(\alpha) - (\eta)$ of Theorem 3.1 hold including*

(α) There exists a maximal (resp. minimal) point $v \in A$ such that $v \not\preccurlyeq w$ (resp. $w \not\preccurlyeq v$) for any $w \in A \backslash \{v\}$.

(γ) If $f : A \to A$ is a map satisfying $x \preccurlyeq f(x)$ (resp. $f(x) \preccurlyeq x$) for all $x \in A$, then f has a maximal (resp. minimal) fixed element $v \in A$, that is, $v = f(v)$.

Remark 7.3. The equivalency of maximal case of Theorem 7.2 was given as Theorem 3.1 of [15]. Note that (α) is a generalization of Zorn's Lemma as shown in [20].

From Theorem 7.2 (α), (γ) and the Brøndsted-Jachymski Principle, we have the following:

Theorem 7.4. *Let $(A, \preccurlyeq)$ be a partially ordered set.*
(i) If a chain in A has an upper bound and $f : A \to A$ is progressive, then

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Max}(\preccurlyeq) \neq \emptyset.$$

(ii) If a chain in A has a lower bound and $f : A \to A$ is anti-progressive, then

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Min}(\preccurlyeq) \neq \emptyset.$$

8. Example 4: Zhong

We follow Zhong [24] in 1997:

Throughout this section X denotes a Banach space. Recall that a functional $f : X \to \mathbb{R} \cup \{+\infty\}$ is called Gateaux differentiable if at every point x with $f(x) < +\infty$, there exists a continuous linear functional $f'(x_0)$ such that for every $y \in X$,

$$\lim_{t \to \infty} \frac{f(x_0 + ty) - f(x_0)}{t} = \langle f'(x_0), y \rangle.$$

In the following, we always assume that $h : [0, +\infty) \to [0, +\infty)$ is a nondecreasing continuous function such that $\int_0^\infty (1/(1 + h(r)))dr = +\infty$, and $f : X \to \mathbb{R} \cup \{+\infty\}$ is a lower semicontinuous function, not identically $+\infty$ and Gateaux differentiable.

Definition 8.1. f is said to satisfy the weak Palais-Smale (P.S.) condition if the existence of $\{x_n\}$ in X such that $\{f(x_n)\}$ is bounded and $||f'(x_n)||(1 + h(x_n)) \to 0$ implies that $\{x_n\}$ has a convergent subsequence.

Theorem 8.2. [24] *If f is bounded from below and satisfies the weak P.S. condition, then f has a minimal point.*

From Theorem 8.2, we have the following:

Theorem 8.3. *Let X be a Banach space and $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be bounded from below and satisfy the weak P.S. condition. Let $(X, \preccurlyeq)$ be the poset defined as in Definition 4.3.*

Then the seven statements in Theorem 3.1 hold.

Proof. In Theorem 3.1, $x \preccurlyeq y$ be the statement $\phi(x) \leq \phi(y)$ for all $x, y \in X$. Then (α) follows from Theorem 8.2 [24]. Moreover, the equivalency of $(\alpha) - (\eta)$ follows from the corresponding ones in Theorem 3.1.

This completes our proof. □

Now we have the following

Theorem 8.4. *Let $(X, \preccurlyeq)$ be a Banach space and $\phi : X \to \mathbb{R} \cup \{+\infty\}$ be bounded from below and satisfy the weak P.S. condition. If $\preccurlyeq$ is defined by ϕ and $f : X \to X$ is such that $f(x) \preccurlyeq x$ for all $x \in X$, then*

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Min}(\preccurlyeq) \neq \emptyset.$$

9. Example 5: Khamsi and Park

In 2009 Khamsi [9] stated: Let A be an abstract set partially ordered by $\preccurlyeq$. We will say that $a \in A$ is a minimal element of A if and only if $b \preccurlyeq a$ implies $b = a$. The concept of minimal element is crucial in the proofs given for Caristi's fixed point theorem.

Theorem 9.1. [9] *Let $(A, \preccurlyeq)$ be a partially ordered set. Then the following statements are equivalent.*

(1) *A contains a minimal element.*

(2) *Any multimap T defined on A, such that for any $x \in A$ there exists $y \in T(x)$ with $y \prec x$, has a fixed point, i.e., there exists a in A such that $a \in T(a)$.*

This follows from Theorem 3.1(α) and (δ) with the preordered set $X = A$. Therefore Theorem 3.1 extends Theorem 9.1. Moreover, the dual form of Theorem 9.1 is easily obtained.

In most recently, we obtained the following [21]:

Theorem 9.2. *Let $(P, \preccurlyeq)$ be a poset and $f : P \to P$ be a map such that*

(a) there exists $p_0 \in P$ such that $f(p_0) \preccurlyeq p_0$,

(b) $B = \{f^n(p_0) : n \in \mathbb{N}\}$ has an infimum, and

(c) $\inf f(B) = f(\inf B)$.

Then f has a fixed point $p^* := \inf B$ and

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Min}(\preccurlyeq) = \{p^*\}.$$

Let $(X, \preccurlyeq)$ be a poset. A map $f : X \to X$ is said to be *monotone* (or *increasing*) if $f(x) \preccurlyeq f(y)$ whenever $x \preccurlyeq y$.

Theorem 9.3. *Let $(X, \preccurlyeq)$ be a poset, $x_0 \in X$, $\varphi : X \to X$ be monotone such that $\varphi(x_0) \preccurlyeq x_0$. Let $B = \{\varphi^n(x_0) \in X : n \in \mathbb{N}\}$ have an infimum $v \in X$. Then*

$$\mathrm{Fix}(\varphi) = \mathrm{Per}(\varphi) \supset \mathrm{Min}(\preccurlyeq) = \{v\}.$$

For details on Theorems 9.2 and 9.3, see [21].

10. Example 6: Cobzaş

In 2011, Cobzaş [3, Proposition 2.2] obtained the following proposition containing a typical situation when the Brezis-Browder principle 6.1 applies:

Proposition 10.1. [3] *Let (X, ρ) be a quasi-metric space such that the topology τ_ρ is T_1 and $\psi : X \to \mathbb{R}$ a function on X. Define an order relation on X by*

$$x \preccurlyeq y \iff \rho(x, y) \leq \psi(y) - \psi(x), \quad x, y \in X.$$

(1) If the space X is right ρ-K-complete and ψ is bounded below and ρ-lsc on X, then every element of X is minored by a minmal element.

(2) If the space X is right $\overline{\rho}$-K-complete and ψ is bounded above and ρ-usc on X, then every element of X is majored by a maxmal element.

For all terminology, see [3]. By applying Theorem 3.1, we follow only the case (1) and the corresponding case (2) is similarly obtained:

Theorem 10.2. *Let (X, ρ) be a quasi-metric space such that the topology τ_ρ is T_1 and $\psi : X \to \mathbb{R}$ a function on X. Define an order relation $x \preccurlyeq y$ as in Proposition 10.1. Suppose that the space X is right ρ-K-complete and ψ is bounded below and ρ-lsc on X*

Then the following seven equivalent statements hold:

(α) *There exists a minimal point $v \in X$ such that $\rho(w, v) > \psi(w) - \psi(v)$ for any $w \in X \backslash \{v\}$.*

(β) *If $\mathfrak{F}$ is a family of maps $f : X \to X$ such that, for any $x \in X$ with $x \neq f(x)$, there exists a $y \in X \backslash \{x\}$ satisfying $\rho(y, x) \leq \psi(x) - \psi(y)$, then $\mathfrak{F}$ has a common fixed element $v \in X$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(γ) *If $\mathfrak{F}$ is a family of maps $f : X \to X$ satisfying $\rho(f(x), x) \leq \psi(x) - \psi(f(x))$ for all $x \in X$ with $x \neq f(x)$, then $\mathfrak{F}$ has a common fixed element $v \in X$, that is, $v = f(v)$ for all $f \in \mathfrak{F}$.*

(δ) *If $\mathfrak{F}$ is a family of multimaps $T : X \multimap X$ such that, for any $x \in X \backslash T(x)$, there exists $y \in X \backslash \{x\}$ satisfying $\rho(y, x) \leq \psi(x) - \psi(y)$, then $\mathfrak{F}$ has a common fixed element $v \in X$, that is, $v \in T(v)$ for all $T \in \mathfrak{F}$.*

(ϵ) *If $\mathfrak{F}$ is a family of multimaps $T : X \multimap X$ such that $\rho(y, x) \leq \psi(x) - \psi(y)$ holds for any $x \in X$ and any $y \in T(x) \backslash \{x\}$, then $\mathfrak{F}$ has a common stationary element $v \in X$, that is, $\{v\} = T(v)$ for all $T \in \mathfrak{F}$.*

(ζ) *Let $\mathfrak{F}$ be a family of multimaps $T : X \multimap X$ such that, for all $x \in X$ with $T(x) \neq \emptyset$, there exists $y \in X \backslash \{x\}$ satisfying $\rho(y, x) \leq \psi(x) - \psi(y)$. Then there exists $v \in X$ such that $T(v) = \emptyset$ for all $T \in \mathfrak{F}$.*

(η) *If Y is a subset of X such that, for each $x \in X \backslash Y$, there exists a $z \in X \backslash \{x\}$ satisfying $\rho(z, x) \leq \psi(x) - \psi(z)$, then there exists an element $v \in Y$.*

Remark 10.3. (1) Theorem 10.2(α) holds by Theorem 10.1. [3, Proposition 2.2(1)].

(2) Theorem 10.2(γ) extends [3, Theorem 2.3(1)] (The Caristi-Kirk fixed point theorem).

(3) Theorem 10.2(δ) extends [3, Theorem 2.12] (The multi-valued version of the Caristi-Kirk fixed point theorem). Others of Theorem 10.2 are their equivalent formulations.

(4) We can obtain a similar theorem to Theorem 10.2 corresponding to Proposition 10.1(2). For other theorems of [3], our method of equivalent formulation in 2023 Metatheorem can be applied.

By applying the Brøndsted-Jachymski principle to Theorem 10.2, we obtain the following:

Theorem 10.4. *Under the hypothesis of Theorem 10.2, if $f : X \to X$ is anti-progressive, then we have*

$$\mathrm{Fix}(f) = \mathrm{Per}(f) \supset \mathrm{Min}(\preccurlyeq) \neq \emptyset.$$

11. Conclusion

Our new Metatheorem in 2022 is applied to many maximal element principles implying the Brøndsted principle and the Brøndsted-Jachymski principle. Such maximal element principles can be reformulated equivalently to collectively fixed point theorems, collectively stationary point theorems for progressive maps and conversely. In our works in 2022, we showed such examples are Zorn's Lemma, Caristi fixed point theorem, and many results on progressive maps. In the end of 2022 [22], we added one more equivalent statement (ζ) to the 2022 Metatheorem. We replaced statements for a single map or multimap to the corresponding ones for families of maps or multimaps, and obtain a new version called the 2023 Metatheorem.

In the present article, we deduce certain minimal element principles which can be used to obtain the dual statements of known maximal flement results. Consequently, we obtain several applications dual to known ones and for anti-progressive maps. In fact, we found several known examples related to minimality in this article. These examples seem to be obtained eventually without any ground. Hence, we provide our minimal element principles from Metatheorem as the logical common basis of them.

As the references at the end of this article show, we obtained a large number of consequences of previous Metatheorems. The readers are encouraged to find more applications of the 2023 Metatheorem and its corresponding maximal or minimal element principles.

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