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A Robust Portfolio Construction Using the Bootstrap Method to Extract Multidimensional Uncertainty Sets: An Application on BIST100 Stocks

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ABSTRACT

Asset allocation is a crucial aspect of portfolio management. The primary objective is to maximize the expected return of the portfolio while minimizing investment risk through optimal asset allocation. However, it is impossible to eliminate all investment risks due to factors such as prediction errors, flawed model construction, and uncertainties in parameters. Traditional portfolio theory models address model-based risks but fail to consider parameter uncertainties, resulting in impractical solutions. In this context, robust optimization methods, as opposed to traditional methods, incorporate parameter uncertainties into the mathematical model and construct portfolios by considering worst-case scenarios within uncertainty sets. Therefore, a robust approach ensures that the model solution remains optimal with a high probability, providing protection against model-based risks for investors. In this paper, we present a robust optimization formulation based on Bertsimas and Sim (2004) and combine it with the bootstrap technique to generate optimal portfolios. Our findings demonstrate that as the uncertainty of the models increases, the expected return of the portfolios decreases. However, for moderate levels of uncertainty, the expected return of the robust portfolio is comparable to that of the classical portfolio. Furthermore, the out-of-sample analysis reveals that the robust portfolios outperform the equally weighted portfolio.

Keywords: Robust Optimization, Bootstrap Method, Asset Allocation, Uncertainity Sets.

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INTRODUCTION

Individuals save some of their income and accumulate funds to consume more in the future. The funds accumulated by consumers erode over time due to inflation, which reduces consumers' purchasing power. Consumers, in turn, invest these funds in financial instruments in order to maintain or, if possible, increase their purchasing power. Among many other alternatives, investors invest in the stock market in the hope that their funds will increase in value. However, investing in securities involves a degree of risk due to the nature of the financial markets. These risks may come from investors, sectors or economic cycles. An investment can be exposed to two types of risk: systematic and unsystematic (Marshall 2015). The former is inherent in the market and cannot be eliminated by diversification. The latter is company or security (stock) specific and can be reduced through diversification (Lhabitant 2017; Pilbeam, 2018; Koumou 2020; Zaimovic, Omanovic, and Arnaut-Berilo 2021). In addition to systematic and unsystematic risks, an investor may also face prediction risks based on parameter uncertainties or parameter biases, i.e., a difference between the predicted parameter and its realization (Lauprete, Samarov, and Welsch 2003). Therefore, utilizing classical optimization methods, such as Markowitz's mean-variance model, may result in optimizing an incorrect model with biased parameters, leading to non-optimal solution. In this context, classical methods may not be able to construct the best portfolio that minimizes both model-based and uncertainty-based risks.

Dynamic programming, stochastic programming, and duality analysis are the methods taking parameter uncertainty into account (Gero, and Dudnik 1978; Shapiro, and Philpott 2007; Sheng, Zhu, and Wang 2020; Zakaria et al. 2020; Diwekar, and Diwekar 2020). The difficulty with these methods is that they require detailed information about the distribution of the parameters (Birge, and Louveaux 2011). In practice, however, detailed information about the distribution of the parameters is almost rarely known. Moreover, solving dynamic programming and stochastic programming problems

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becomes increasingly difficult as the number of possible scenarios with uncertain parameters increases. Although they have a solid theoretical background, the application of these models is quite limited in the literature. Robust optimization is a new technique compared to the models mentioned above. However, it is widely used in studies as it makes general assumptions about the distributions of the uncertain parameters. Moreover, the mathematical formulation of any robust problem has a linear conjugate and its solution is simple compared to stochastic programming and dynamic programming, even for large problems (Bertsimas, Brown, and Caramanis 2011; Yanıkoğlu, Gorissen, and Den Hertog 2019). Apart from post-solution methods such as duality, dynamic programming, and stochastic programming, robust optimization incorporates the uncertainty of parameters before optimizing the mathematical model (Beck and Ben-Tal, 2009; Gabrel, Murat, and Thiele, 2014). Parameter uncertainty often arises from estimation bias, changes in information flow, and shareholders' future expectations. By accounting for parameter uncertainty and incorporating it into the mathematical model, robust optimization offers several advantages for portfolio management. These include the ability to absorb errors in the mathematical model and within the uncertain sets, as well as providing a solution that remains optimal with a high probability even under the worst possible parameter realizations.

Robust optimization is one of the most widely used methods in portfolio theory (Goldfarb, and Iyengar 2003; Huang et al. 2010; Xidonas, Steuer, and Hassapis 2020). The uncertain parameters are included in the robust portfolio formulation within predetermined convex uncertainty sets. Owing to robust optimization, all possible realizations of the parameters are included in the portfolio optimization, so that the solution remains feasible with high probability. Although several robust models have been utilized to solve optimization problems with uncertain parameters, we have developed a new formulation of robust optimization based on the model proposed by Bertsimas and Sim (2004). While Bertsimas and Sim's formulation accounts for uncertainty in the constraints, the objective function does not consider uncertain parameters. In this paper, we have reorganized their robust formulation and proposed a new one that incorporates uncertain parameters in the objective function. Additionally, we have combined the bootstrap method with the model. It should be noted that there are various techniques available for determining the uncertainty sets of uncertain parameters, but in this study, we have chosen to use the bootstrap method to generate uncertainty sets for the assets analyzed. The bootstrap method is a resampling technique used to make inferences about a population based on an existing sample. In our case, it is used to create convex and symmetric uncertainty sets for the objective function parameters. The extreme values, i.e. the maximum and minimum values of the uncertainty set, were obtained for each stock using the distribution function created by the bootstrap technique.

LITERATURE REVIEW

Although the influential work of Markowitz (1952) laid the foundation for modern portfolio construction theory, the practical application of portfolio management has been disappointing due to difficulties in constructing model inputs. The inputs (expected returns and covariance between assets) for mean-variance optimization must be estimated, either statistically from historical data or pricing model (Tütüncü, and Fabozzi 2014). The uncertainty in the expected returns has a much greater influence on the optimal solution than the covariance matrix (Chopra and Ziemba 1993; and Kallberg and Ziemba 1984; Yam et al., 2016). Therefore, we focus on the uncertainty in the expected returns, assuming that the covariance matrix is known. The emphasis here is not on the risk of returns. Risk, as used in Markowitz's mean-variance model, and uncertainty, which is the difference between the estimated value and the realized value of a parameter, are different concepts. The mean-variance model assumes that asset returns are normally distributed and will continue to be normally distributed in the future. However, returns typically have a fat-tailed distribution with infinite variance (Fama 1965; Mandelbrot 1997; Campbell et al. 2008; Fabozzi et al. 2007; Bhansali 2008; Sheikh, and Qiao 2009; Haas, and Pigorsch, 2009; Stoyanov et al. 2011; Eom, Kaizoji, and Scalas, 2019; Eom 2020). Under the assumption of normality, the ordinary mean estimator is the best linear unbiased estimator (BLUE) and its use in the optimization model is unproblematic. However, in the case of non-normality, robust statistics or models must be used to construct efficient portfolios (Reyna et al. 2005; Kaszuba, 2012; Yang, Couillet, and McKay 2015; Li, Hong, and Wang 2015; Hubert, Debruyne, and Rousseeuw 2018; Bakar, and Rosbi, 2019).

The expected returns and the variance-covariance matrix estimated from historical data can be a good representation of the past. However, their ability to predict the future is not always perfect. At this point, the reliability of the solution obtained from the robust model increases, because the robust optimization solves the mathematical model with uncertain parameters (Ben-Tal, and Nemirovski 2002; Fabozzi et al. 2007; Gülpınar, and Hu 2016). Although robust optimization dates back to the study of Sosyter (1973), it received the most attention in the early 2000s (Ghaoui, Oks, and Oustry 2003; Zymler, Rustem, and Kuhn 2011; Qiu et al. 2015; Lee et al. 2020; Xidonas, Steuer, and Hassapis 2020). The logic of the Soyster model is to assume the worst-case realization within uncertainty sets for all assets in the portfolio. This makes it the most conservative of the robust optimization models and therefore the most sensitive to uncertainty. Even if the financial markets exhibit a high degree of uncertainty, it is unlikely that all assets in the portfolio will perform at their worst. Over an investment horizon, some securities will provide lower than expected returns, while others will provide higher than expected returns. The main drawback of the Soyster model is its excessive conservatism with respect to parameter uncertainty. To overcome the problem of conservatism, Ben-Tal and Nemirovski (1998, 1999, 2000) have proposed a new robust model that is less conservative to parameter uncertainties. Compared to Soyster's model, the robust model proposed by Ben-Tal and Nemirovski is less likely to remain feasible due to its lower conservatism. The robust formulation of Ben-Tal and Nemirovski is theoretically convincing, but could not be used by the researchers because of the complications in solving the model. Finally, Bertsimas and Sim (2004) proposed a robust model allowing a trade-off between the value of the objective function and the robustness of the solution (Bertsimas, Pachamanova, and Sim 2004). In addition to the financial studies, the robust optimization formulations have been used in many academic studies, such as production planning and inventory management (Alem, and Morabito 2012; Agra et al. 2018; Rodrigues et al. 2019; Golsefidi, and Jokar 2020), energy storage and planning (Zhang et al. 2018; Zhao et al. 2019; Shen et al. 2020; Moret et al. 2020), supply chain and planning (Bertsimas, and Thiele 2004; Pishvaee, Rabbani, and Torabi 2011; Hahn, and Kuhn 2012), water management and planning (Zeferino, Cunha, and Antunes 2012); finance and portfolio theory (Tütüncü, and Koening 2004; Fabozzi et al. 2007; Quaranta, and Zaffaroni 2008; Gregory, Darby-Dowman, and Mitra 2011; Scutella, and Recchia 2013; Deng et al. 2013; Kapsos, Christofides, and Rustem 2014; Wang, and Cheng 2016; Sengupta, and Kumar 2017; Solares et al. 2019; Dai, and Wang 2019; Dai, and Kang 2021; Georgantas, Doumpos, and Zopounidis 2021).

Carefully defining uncertainty sets is crucial in order to achieve feasible outcomes, although robust optimization effectively reduces the impact of parameter biases. An uncertainty set is a region that encompasses all potential realizations of an uncertain parameter with a specified likelihood. Convexity, symmetry, and closed clusters are required for uncertainty sets. Here, a closed cluster refers to an interval with a finite number of parameter realizations. There are several ways to construct uncertainty sets, including methods proposed by Ben-Tal and Nemirovski (2000), Bertsimas and Brown (2009), Bandi and Bertsimas (2012), Guan and Wang (2013), Bertsimas, Gupta, and Kallus (2018), Zhu et al. (2020), and Daneshvari and Shafaei (2021). However, we chose to use the bootstrap method to create our uncertainty sets due to its statistical advantages.

METHODOLOGY

Robust Optimization

Robust optimization is a technique that takes into account uncertain parameters during the pre-solution phase of mathematical formulation. These parameters can have a range of values within an uncertainty set. The main concept is to define an uncertainty set for potential realizations of the uncertain parameters and then optimize the mathematical model against the worst-case scenarios within the uncertain set. In many optimization problems, the values of the parameters are either unknown or inaccurately predicted at the time of solution. This is a critical factor in the optimization process, as the solution heavily relies on these parameters. It is worth noting that the optimal solution of a linear programming problem occurs at a corner point of the feasible region. However, a potential bias in the parameters can significantly alter the optimization problem and result in an infeasible solution (Ben-Tal and Nemirovski 2000). By incorporating parameter uncertainties into the optimization model, the solution becomes more resistant to prediction bias. It is common for there to be a discrepancy between the actual and predicted values of a parameter in financial data, which is typically based on past information. Discrepancy between prediction and realization of a parameter can lead to uncertainty and risk in portfolio management. To address this issue, robust optimization techniques have been developed. The optimization model of Soyster (1973), Ben-Tal and Nemirovski (2000), and Bertsimas and Sim (2004) are widely cited in the literature. However, the robust model proposed by Bertsimas and Sim (2004) offers distinct advantages, such as the ability to control the level of conservatism through a control parameter and ensuring computational feasibility in both theory and practice. In this study, we aim to enhance the robust model of Bertsimas and Sim by incorporating uncertainty into the objective function. Our proposed model for portfolio optimization with general constraints is outlined below.

$$
\begin{aligned}
&Z_{max} = \boldsymbol{\mu}'\boldsymbol{w} - \boldsymbol{\lambda}.\boldsymbol{w}^{t}\boldsymbol{\Sigma}\boldsymbol{w} \\
&\boldsymbol{A}\boldsymbol{w} \leq \boldsymbol{b} \\
&\boldsymbol{l} \leq \boldsymbol{w} \leq \boldsymbol{u}
\end{aligned} \tag{1}
$$

where $\boldsymbol{\mu}$ is the coefficient vector of the objective function, $\boldsymbol{w}$ is the weight vector, $\boldsymbol{A}$ is a matrix of technology coefficients, $\boldsymbol{b}$ is a vector of right-hand side coefficients, $\boldsymbol{\Sigma}$ is the variance-covariance matrix, $\boldsymbol{\lambda}$ is the risk aversion constant, and $\boldsymbol{l}$ and $\boldsymbol{u}$ are the lower and upper bounds of the weights, respectively. It is assumed that all coefficients in the model are certain or predetermined. However, Bertsimas and Sim (2004) proposed a robust model including uncertain parameters.

$$
\begin{aligned}
&Z_{max} = \boldsymbol{\mu}'\boldsymbol{w} \\
&\sum_j a_{ij}w_j + \max_{\{S_i\cup\{t_i\}|S_i\subseteq J_i,|S_i|=\lfloor\Gamma_i\rfloor,t_i\in J_i\setminus S_i\}} \left\{\sum_{j\in S_i} \hat{a}_{ij}y_j + (\Gamma_i - \lfloor\Gamma_i\rfloor)\hat{a}_{it_i}y_t\right\} \leq b_i \quad ;\forall i \\
&-y_j \leq x_j \leq y_j \\
&\boldsymbol{l} \leq \boldsymbol{x} \leq \boldsymbol{u} \\
&\boldsymbol{y} \geq \boldsymbol{0}
\end{aligned} \tag{2}
$$

where w_j is the j^{th} weight of the jth parameter, a_{ij} is the coefficient of the j^{th} certain parameter in the i^{th} constraint, $\hat{a}_{ij}$ is the coefficient of uncertain parameter j in the i^{th} constraint, and b_i is the constant of the right-hand side. The i^{th} constraint contains a sub-optimization model by itself. If w_j^* is the optimal weight of the j^{th} parameter, it is obvious that y_j will be equal to $|w_j^*|$ at the optimal point. If y_j takes a non-zero value, it will be equal to either $-w_j^*$ or w_j^*, because the optimal solution occurs at one of the extreme values. Hence, the constraint in equation (2) can be expressed as:

$$
\sum_j a_{ij}w_j + \max_{\{S_i\cup\{t_i\}|S_i\subseteq J_i,|S_i|=\lfloor\Gamma_i\rfloor,t_i\in J_i\setminus S_i\}} \left\{\sum_{j\in S_i} \hat{a}_{ij}|w_j|\right\} \leq b_i \quad ; \forall i \tag{3}
$$

where Γ_i represents the number of uncertain parameters included in the constraint with a possible range of 0 and $\lfloor\Gamma_i\rfloor$. This value corresponds to the maximum number of parameters in the model. The chosen integer in the analysis reflects a trade-off between the risk of uncertainty and the value of the objective function. When Γ_i is equal to 0, the model (2) will be equivalent to the Markowitz mean-variance model, which only includes certain parameters. On the other hand, if Γ_i is equal to $\lfloor\Gamma_i\rfloor$, the objective function will take on a more conservative value. In order to reach the optimal solution for equation (2), the constraint in equation (2) should be expressed as a sub-optimization model:

$$
\begin{aligned}
&\beta_i(\boldsymbol{w}^*, \Gamma_i) = max \sum_{j\in J_i} \hat{a}_{ij}|w_j^*|z_{ij} \\
&\textstyle\sum_{j\in J_i} z_{ij} \leq \Gamma_i \\
&0 \leq z_{ij} \leq 1 \quad ; \quad \forall j \in J_i
\end{aligned} \tag{3}
$$

The solution to model (2) is achieved in two steps: the first step is to solve the sub-optimization model (3), and the second step is to use the solution of model (3) to solve model (2). However, the uncertainty in returns have much more impact on feasible solution than uncertainty in the constraints. By integrating uncertainty into objective function, we develop a robust formulation based on the model of Bertsimas and Sim (2004). The objective of the proposed model is the Sharpe ratio1 .

$$
\begin{aligned}
&Z_{max} = \frac{\max\left(\sum_{i=1}^{K}\mu_i w_i + \min_{\{S_i\cup\{t_i\}|S_i\subseteq J_i,|S_i|=\lfloor\Gamma_i\rfloor,t_i\in J_i\setminus S_i\}}\left\{\sum_{j\in S_i}\tilde{\mu}_j|w_j|\right\}\right) - R_f}{\sqrt{\mathrm{Var}[R_p]}} \\
&\boldsymbol{A}\boldsymbol{w} \leq \boldsymbol{b} \\
&\boldsymbol{l} \leq \boldsymbol{w} \leq \boldsymbol{u}
\end{aligned} \tag{4}
$$

where $\tilde{\mu}_j$ is return vector of the uncertain parameters, Γ_i is the number of uncertain parameter in model, and R_f is the risk-free rate. To optimize the objective, the numerator must be maximized while the denominator must be minimized. The numerator of the objective function contains two nested optimization problems: the inner one is a minimization problem, while the outer one is a maximization problem. The denominator is the portfolio's variance, which is a measure of the portfolio's risk. The model can incorporate conventional constraints, such as transaction cost constraints, minimum and maximum limit constraints, and others. These constraints do not involve uncertain parameters.

[1] The Sharpe ratio is proposed by Sharpe (1966) and formulated as $\frac{\mathrm{E}[R_p]-R_f}{\sqrt{\mathrm{Var}[R_p]}}$; where $\mathrm{E}[R_p]$ is the expected return of portfolio, R_f is the risk free rate, and $\mathrm{Var}[R_p]$ is the variance of portfolio. $\mathrm{E}[R_p] = \mu^t w$ and $\mathrm{Var}[R_p] = w^t\Sigma w$, where Σ is the covariance matrix of returns.

Bootstrap Method

Determining the appropriate uncertainty set or interval is crucial for the success of robust optimization. Poorly determined uncertainty sets can lead to unreliable solutions for optimization problems. Therefore, well-determined uncertainty sets result in reliable and feasible worst-case solutions. In this context, we use the bootstrap technique to determine appropriate intervals for the assets used in the analysis. The bootstrap is a procedure for repeating samples in order to derive statistics on population parameters. This method uses the resampling procedure to create new samples from the existing sample, with the aim of obtaining a good representation of the population parameters. Assuming that a series consists of a random sample from an unknown probability distribution F, bootstrapping can be used to predict a representative probability distribution of the series, represented as $\hat{F}$. To obtain $\hat{F}$, multiple samples are taken from the realized sample through resampling. There may be a bias between the predicted distribution $\hat{F}$ and the population distribution F, as well as between θ and $\hat{\theta}$, which are unknown population parameters and estimated parameters derived from the resampling procedure, respectively. However, in practice, the bias between θ and $\hat{\theta}$ is usually negligible due to the superior statistical properties of the method.

The distribution function of a random variable X with observed values $X_1, X_2, \ldots\ldots, X_n$ is denoted by F. However, the distribution of X is usually unknown. Fortunately, the empirical distribution $\hat{F}$ can be obtained from random samples $\boldsymbol{x_i} = (x_{i1}, \ldots x_{ik})$ from X, where $k<n$. The estimated parameter $\hat{\theta}$ derived from $\hat{F}$ can be used as a representation of the population parameter θ. By resampling $\boldsymbol{x_i} = (x_{i1}, \ldots x_{ik})$ for $\mathrm{i} = 1,2,\ldots,\xi$, we can obtain ξ new samples from X. This allows us to create $\hat{\theta}$ based on ξ subsamples drawn from X using the resampling process. The confidence interval for θ can then be calculated using the estimated parameter $\hat{\theta}$, by taking into account the probability distribution of $\hat{\theta} - \theta$. Let s_α represent the α-percentile of the distribution of $\hat{\theta} - \theta$. A confidence interval for can be calculated using the following statement:

$$P(s_{\alpha/2} \leq \hat{\theta} - \theta \leq s_{1-\alpha/2}) = 1 - \alpha$$

or

$$\hat{\theta} - s_{1-\alpha/2} \leq \theta \leq \hat{\theta} - s_{\alpha/2}$$

The equation above states that the probability of containing the unknown parameter $\hat{\theta}$ is equal to 1-a. However, in order to use this interval, the distribution of $\hat{\theta} - \theta$ must be known. It is more common to use the distribution of the studentized estimator $(\hat{\theta} - \theta)/\widehat{\mathrm{se}}$, where $\widehat{\mathrm{se}}$ is the standard error of the estimator $\hat{\theta}$. This random variable often follows an approximate t-distribution with $df = n - p$ degrees of freedom, where p is the total number of unknown parameters to be estimated from the data. If $t_{df,\ \alpha}$ represents the α-percentile of the t-distribution with df degrees of freedom, the following confidence interval can be derived:

$$\hat{\theta} - t_{df;\alpha/2}\widehat{\mathrm{se}} \leq \theta \leq \hat{\theta} + t_{df;\alpha/2}\widehat{\mathrm{se}}$$

According to statistical theory, this interval will contain a population parameter with a probability of . In the context of robust optimization, this confidence interval represents the uncertainty associated with a risky asset used in portfolio optimization (Efron, and Tibshirani 1985; Wehrens, Putter, and Buydens, 2000).

DATA and ANALYSIS

We utilized the developed model to analyze BIST100 shares, with the exception of financial firms. This exclusion was due to the unique calculations involved in their balance sheets and their market-to-book ratio. Typically, investors are drawn to assets with low market-to-book ratios, as this is widely considered a key indicator of whether an asset is undervalued or overvalued in the market. However, financial institutions and banks tend to have very low equity, resulting in consistently low market-to-book ratios compared to manufacturing companies. As a result, the model may give disproportionate weight to the shares of financial institutions and banks. To avoid biased results, financial institutions and banks are typically excluded from financial studies (Fama and French 1992; Fama and French 1993; Azimli 2020). Therefore, our analysis was based on data from 56 assets. We used monthly data from January 2014 to March 2021 for the analysis, with closing prices on the last trading day of each month. Returns were calculated as the percentage change in prices from period t-1 to period t. In terms of the total asset space, 46.42% of shares were from the manufacturing sector, 10.71% from electricity, gas, and water, 10.71% from oil, gas, and chemicals, 8.92% from transportation, storage, and communications, 8.92% from technology, 7.14% from wholesale and retail trade, restaurants and hotels, 5.35% from mining and quarrying, and one share from the construction industry. The mathematical model used for portfolio construction and its constraints are outlined below.

$$Z_{max} = \frac{\mathrm{E}[R_p] - R_f}{\sqrt{\mathrm{Var}[R_p]}}$$

where $\mathrm{E}[R_p] = \max\left(\sum_{i=1}^{\kappa} \mu_i w_i + \min_{\{S_i \cup \{t_i\} | S_i \subseteq J_i, |S_i| = \lfloor \Gamma_i \rfloor, t_i \in J_i \setminus S_i\}} \left\{ \sum_{j \in S_i} \tilde{\mu}_j \, |w_j| \right\}\right)$ while R_f is the average interest rate of government debt securities and $\mathrm{Var}[R_p]$ is the variance of the portfolio. Here, in order to optimize the expected return, we need to minimize the sub-problem inside the parentheses. This is because the expression inside the parentheses is optimized based on worst-case scenarios, resulting in a negative value for the objective function. Consequently, to maximize the expected return of portfolio, $\left\{ \sum_{j \in S_i} \tilde{\mu}_j \, |w_j| \right\}$ part of $\mathrm{E}[R_p]$ which have zero or negative value must be minimized. Therefore, the objective can be reorganized as follows:

$$Z_{max} = \frac{\max\left(\sum_{i=1}^{\kappa} \mu_i w_i + \min_{\{S_i \cup \{t_i\} | S_i \subseteq J_i, |S_i| = \lfloor \Gamma_i \rfloor, t_i \in J_i \setminus S_i\}} \left\{ \sum_{j \in S_i} \tilde{\mu}_j \, |w_j| \right\}\right) - R_f}{\sqrt{\mathrm{Var}[R_p]}}$$

Since R_f is constant in the objective function, maximizing the objective is essentially maximizing the ratio of expected return to portfolio risk within the constraints of the model shown in Table below.

The first constraint was implemented to ensure that the total weight of the portfolio equaled 1. Constraints 2-7 were utilized to promote portfolio diversification and restrict the upper and lower limits of stocks within each sector. The final constraint was put in place to prevent the weighting of any individual stock from exceeding 5% and to prohibit short selling. These optimization constraints allow for potential losses in one sector to be offset by gains in others, thanks to measures such as sector-specific investment ratios, maximum investment limits, and restrictions on investing in certain sectors.

Above are the descriptive statistics for the assets used in the analysis. The data shows that SASA has the highest monthly return of 6.1285%, while BIZIM has the lowest monthly return of 0.2831%. When conducting mean-variance analysis, it is important to consider the risk and expected return of each asset in order to create a portfolio with minimum risk or maximum expected return. The standard deviation, which represents risk, is a crucial factor in selecting assets for the portfolio. In this case, IPEKE has the highest risk of 17.7770, while

Table 1. The Constraints and Their Definition

#	The Constraint	Definition
1)	$\sum_{i=1}^{N} w_i = 1$	It ensures that the sum of the weights is equal to 1
2)	$\frac{\sum_{i=1}^{N} a_i w_i}{\sum_{i=1}^{N} w_i} \leq 7.00$	It limits the maximum weight of market-to-book ratio of an assets. Here, a_i is the market-to-book ratio of asset i.
3)	$\frac{\sum w_{jm}}{\sum_{i=1}^{N} w_i} \leq 0.35$	It ensures that the total weight of manufacturing stocks in the portfolio does not exceed 35%. Here, $\sum w_{jm}$ is the total weight of assets from manufacturing sector.
4)	$\frac{\sum w_{tp}}{\sum_{i=1}^{N} w_i} \geq 0.05$	It ensures that the total weight of the shares of wholesale, retail, restaurants and hotels is at least 5% of the portfolio. Here, $\sum w_{tp}$ is the total weight of assets from wholesale, retail, restaurants and hotels sector.
5)	$\frac{\sum w_{tk}}{\sum_{i=1}^{N} w_i} \geq 0.20$	It ensures that the total weight of shares in the technology sector is at least 20% of the portfolio. Here, $\sum w_{tk}$ is the total weight of assets from technology sector.
6)	$\frac{\sum w_{km}}{\sum_{i=1}^{N} w_i} \geq 0.10$	It ensures that the total weight of oil, gas and chemical sector stocks is at least 10% of the portfolio. Here, $\sum w_{km}$ is the total weight of assets from oil, gas and chemical sector.
7)	$\frac{\sum w_{ul}}{\sum_{i=1}^{N} w_i} \geq 0.08$	It ensures that the weights of transportation, storage, and communications shares in the portfolio are a maximum of 8%. Here, $\sum w_{ul}$ is the total weight of assets from transportation, storage, and communications sector.
8)	$0 \leq w_i \leq 0.05$	It limits the lower and upper bounds of the weights.

Table 2. Descriptive Statistics

Asset	Mean	Median	Max.	Min.	Std. D.	Skewness	Kurtosis	Jarque-Bera
AEFES	0.315	0.609	22.035	-24.166	8.465	-0.124	3.129	0.280
AKSA	2.427	1.888	31.103	-21.998	9.874	0.199	3.389	1.109
AKSEN	1.968	1.652	32.668	-22.233	10.022	0.376	3.568	3.181
ALKIM	3.007	2.029	23.182	-11.596	7.817	0.342	2.424	2.861
ARCLK	1.571	0.897	23.261	-31.644	9.216	-0.485	4.256	**9.019***
ASELS	2.586	1.641	25.503	-20.456	8.465	-0.074	2.979	0.079
AYGAZ	1.622	1.096	21.423	-24.981	8.158	-0.018	3.651	1.525
BIMAS	1.709	1.171	17.196	-11.749	5.982	0.311	2.760	1.597
BIZIM	0.283	-0.267	35.888	-25.167	10.509	0.256	4.246	**6.506***
BRISA	2.349	1.032	30.124	-20.143	9.987	0.307	3.069	1.366
BRSAN	2.585	1.815	54.002	-28.465	12.266	0.646	5.833	**34.758***
BTCIM	2.291	2.233	82.149	-41.958	15.260	1.444	10.943	**255.988***
BUCIM	1.836	0.756	33.156	-22.400	10.304	0.890	4.351	**17.882***
CCOLA	0.646	0.586	22.181	-25.353	8.797	0.062	3.504	0.966
CEMTS	3.229	2.750	37.764	-25.568	12.050	0.443	3.371	3.304
CIMSA	1.246	0.967	25.073	-35.905	9.375	-0.270	5.306	**20.101***
CLEBI	4.281	2.749	69.586	-32.158	14.624	1.714	9.068	**174.041***
DEVA	3.411	3.150	44.727	-24.525	11.028	0.707	5.772	**34.705***
DOAS	2.387	1.790	45.735	-33.896	14.020	0.705	4.678	**17.225***
EGEEN	4.244	1.958	49.790	-29.515	11.431	0.570	5.043	**19.607***
ENKAI	1.221	0.515	18.704	-10.942	6.006	0.429	3.269	2.893
ERBOS	3.166	2.341	35.783	-23.603	11.662	0.249	2.978	0.894
EREGL	2.773	1.645	30.548	-21.256	10.118	0.027	2.670	0.401
FROTO	2.990	3.705	25.037	-39.001	9.494	-0.774	6.086	**42.720***
GUBRF	3.837	2.271	50.947	-25.983	14.259	1.041	5.117	**31.6035***
HEKTS	4.963	4.006	41.521	-22.704	10.497	0.654	4.822	**18.021***
INDES	3.047	3.259	33.749	-30.772	12.915	-0.185	3.602	1.791
IPEKE	2.089	-1.812	55.667	-57.982	17.777	0.319	4.226	6.847*
KAREL	3.826	2.709	32.030	-37.243	13.810	-0.071	2.818	0.190
KARSN	2.036	1.978	48.168	-40.306	14.769	0.512	5.150	**20.333***
KARTN	2.627	0.147	93.849	-22.629	14.752	3.181	18.956	**1057.307***
KLMSN	2.978	2.427	35.140	-40.042	15.550	-0.274	3.280	1.358
KORDS	2.897	2.236	32.371	-35.452	10.845	0.097	4.524	**8.462***
KOZAA	2.336	-0.129	50.067	-39.256	17.504	0.425	3.269	2.851
KOZAL	2.373	2.971	45.562	-32.918	14.177	-0.014	3.474	0.809
KRDMD	2.692	0.569	45.783	-31.366	13.195	0.342	3.912	**4.653***
LOGO	3.823	2.364	63.517	-23.877	14.398	1.194	5.849	**49.507***
MGROS	1.148	1.503	25.835	-35.742	9.849	-0.426	4.280	**8.475***
NETAS	2.176	0.426	41.985	-25.229	13.531	0.799	4.043	**13.045***
ODAS	2.174	0.879	55.500	-36.546	16.030	0.165	4.224	**5.762***
OTKAR	2.810	2.531	38.295	-35.094	11.682	-0.081	4.676	**10.161***
PETKM	2.408	2.701	21.591	-37.046	9.513	-0.802	5.474	**31.157***
PGSUS	1.404	-0.899	43.807	-45.676	15.560	0.147	3.716	2.147
SASA	6.129	4.796	47.717	-22.159	13.278	0.771	4.148	**13.254***
TATGD	1.932	2.247	27.895	-22.108	11.290	0.080	2.352	1.595
TCELL	0.914	1.632	17.586	-19.947	7.272	-0.404	3.248	2.558

THYAO	0.919	-0.277	24.668	-30.654	11.374	-0.040	2.886	0.069
TMSN	1.743	1.119	48.980	-33.384	12.531	0.449	5.122	19.014*
TOASO	2.034	0.586	30.605	-33.700	8.746	-0.233	6.025	**33.565***
TTKOM	0.629	0.250	25.643	-27.115	8.611	-0.152	4.241	**5.854***
TTRAK	2.186	2.177	36.282	-35.377	11.059	0.117	5.273	**18.702***
TUPRS	1.728	2.397	19.671	-29.395	8.623	-0.353	3.954	**5.045***
ULKER	0.688	0.263	22.655	-19.651	9.180	0.088	2.671	0.500
VESTL	3.593	3.846	40.945	-36.335	15.383	-0.039	3.074	0.041
YATAS	5.007	4.913	38.349	-37.445	14.374	-0.440	3.277	3.049
ZOREN	1.563	1.123	62.390	-22.691	13.076	1.206	7.310	**87.421***

Table 3. The Uncertainty Sets of the Assets

Asset				**Asset**			
AKSA	[-1.25, 6.11]	[-0.24, 5.09]	[0.34, 4.51]	***KRDMD***	[-2.38, 7.76]	[-0.83, 6.21]	[-0.08, 5.47]
AKSEN	[-1.73, 5.67]	[-0.70, 4.63]	[-0.15, 4.08]	***KAREL***	[-1.38, 9.03]	[0.07, 7.58]	[0.87, 6.78]
ALKIM	[0.07, 5.94]	[0.88, 5.13]	[1.35, 4.66]	***KARSN***	[-3.60, 7 68]	[-2.01, 6.08]	[-1.07, 5.14]
AEFES	[-2.97, 3.60]	[-2.01, 2.64]	[-1.49, 2.12]	***KARTN***	[-1.01, 6. 27]	[-0.05, 5.30]	[0.48, 4.77]
ARCLK	[-2.09, 5.24]	[-0.99, 4.13]	[-0.42, 3.56]	***KLMSN***	[-3.07, 9.03]	[-1.30, 7.26]	[-0.27, 6.23]
ASELS	[-0.74, 5.92]	[0.26, 4.91]	[0.79, 4.39]	***KORDS***	[-1.28, 7.07]	[-0.06, 5.86]	[0.60, 5.19]
AYGAZ	[-1.54, 4.78]	[-0.58, 3.83]	[-0.09, 3.33]	***KOZAL***	[-3.04, 7.79]	[-1.44, 6.19]	[-0.64, 5.38]
BTCIM	[-3.24, 7.83]	[-1.67, 6.26]	[-0.84, 5.43]	***KOZAA***	[-4.29, 8.96]	[-2.42, 7.09]	[-1.34, 6.01]
BIMAS	[-0.53, 3.95]	[0.11, 3.31]	[0.44, 2.97]	***LOGO***	[-0.22, 7.86]	[0.91, 6.74]	[1.52, 6.13]
BIZIM	[-3.74, 4.30]	[-2.55, 3.11]	[-1.93, 2.50]	***MGROS***	[-2.68, 4.98]	[-1.57, 3.87]	[-1.01, 3.31]
BRSAN	[-2.01, 7.18]	[-0.78, 5.95]	[-0.01, 5.18]	***NETAS***	[-2.81, 7.16]	[-1.39, 5.74]	[-0.64, 5.00]
BRISA	[-1.44, 6.14]	[-0.35, 5.05]	[0.24, 4.46]	***ODAS***	[-3.81, 8.16]	[-2.17, 6.52]	[-1.23, 5.58]
BUCIM	[-2.00, 5.67]	[-0.89, 4.57]	[-0.32, 3.99]	***OTKAR***	[-1.72, 7.35]	[-0.36, 5.98]	[0.34, 5.28]
CCOLA	[-2.84, 4.13]	[-1.74, 3.03]	[-1.21, 2.50]	***PGSUS***	[-4.48, 7.29]	[-2.79, 5.59]	[-1.80, 4.61]
CLEBI	[0.23, 8.34]	[1.32, 7.24]	[1.93, 6.63]	***PETKM***	[-1.48, 6.29]	[-0.27, 5.08]	[0.36, 4.45]
CEMTS	[-1.38, 7.83]	[0.06, 6.40]	[0.72, 5.73]	***SASA***	[1.25, 11.00]	[2.63, 9.63]	[3.39, 8.87]
CIMSA	[-2.39, 4.88]	[-1.30, 3.79]	[-0.74, 3.23]	***TATGD***	[-2.34, 6.20]	[-1.13, 4.99]	[-0.46, 4.33]
DEVA	[-0.70, 7.52]	[0.51, 6.31]	[1.13, 5.69]	***TOASO***	[-1.44, 5.51]	[-0.36, 4.43]	[0.16, 3.90]
DOAS	[-2.67, 7.45]	[-1.27, 6.04]	[-0.52, 5.29]	***TCELL***	[-2.00, 3.83]	[-1.12, 2.95]	[-0.66, 2.48]
EGEEN	[-0.05, 8.54]	[1.19, 7.30]	[1.84, 6.64]	***TMSN***	[-3.03, 6.51]	[-1.63, 5.12]	[-0.89, 4.37]
ENKAI	[-1.04, 3.49]	[-0.42, 2.86]	[-0.06, 2.50]	***TUPRS***	[-1.68, 5.14]	[-0.63, 4.09]	[-0.10, 3.55]
ERBOS	[-1.21, 7.55]	[0.06, 6.27]	[0.75, 5.58]	***THYAO***	[-3.47, 5.31]	[-2.19, 4.03]	[-1.53, 3.37]
EREGL	[-1.19, 6.74]	[-0.01, 5.56]	[0.62, 4.92]	***TTKOM***	[-2.73, 3.99]	[-1.70, 2.96]	[-1.19, 2.45]
FROTO	[-0.83, 6.81]	[0.35, 5.63]	[0.94, 5.04]	***TTRAK***	[-2.09, 6.47]	[-0.79, 5.17]	[-0.18, 4.55]
GUBRF	[0.09, 7.58]	[1.13, 6.55]	[1.70, 5.97]	***ULKER***	[-2.91, 4.29]	[-1.84, 3.22]	[-1.29, 2.66]
HEKTS	[1.14, 8.78]	[2.20, 7.73]	[2.76, 7.16]	***VESTL***	[-2.36, 9.54]	[-0.61, 7.80]	[0.33, 6.85]
INDES	[-1.96, 8.05]	[-0.54, 6.64]	[0.27, 5.83]	***YATAS***	[-0.70,10.72]	[1.07, 8.95]	[1.98, 8.03]
IPEKE	[-4.56, 8.73]	[-2.66, 6.84]	[-1.61, 5.79]	***ZOREN***	[-3.08, 6.21]	[-1.81, 4.94]	[-1.10, 4.23]

BIMAS has the lowest risk of 5.9819. Additionally, the distribution of the series is also important in portfolio selection. The coefficients of skewness and kurtosis provide valuable information about the shape of the distribution. It is worth noting that a large proportion of assets do not have a symmetric distribution, as indicated by the coefficients of skewness and kurtosis. The Jarque-Bera test is another indicator of normality, with the null hypothesis being "the series is normally distributed." The results in Table 2 show that 31 out of 56 series are not normally distributed, which is more than half of the total assets used in the analysis. This proportion increases even further as the frequency of the data increases. Therefore, assuming normality would render the model solutions invalid.

The bootstrap method was used to determine uncertainty intervals, which are presented in Table 3. These intervals were constructed using three alpha values, representing the significance level of the uncertainty sets. The significance levels used were 1%, 5%, and 10%, which are commonly used in statistics for confidence intervals. As the investment in a portfolio was planned for three years, subsamples of 36 observations were created using the bootstrapping procedure. For each asset, 15000 subsamples were drawn to obtain a representative distribution of returns. Once the asset distributions were determined through bootstrapping, the lower and upper bounds of the uncertainty sets were calculated using the first, fifth, and tenth quantiles for each asset. In the case of an alpha is equal to 0.01, corresponding to a 99% confidence level, the lower bounds of the uncertainty sets were generally negative and all upper bounds were positive. This is due to the wide confidence interval. As the confidence level increases, it is expected for the bounds of the confidence intervals to expand. For example, the difference between the boundary values of AKSA at 99%, 95%, and 90% confidence levels were 7.36, 5.48, and 4.85, respectively. A confidence interval of 99% is quite high and means that 99 out of 100 realizations would fall within the interval. Therefore, compared to intervals at lower confidence levels, a larger interval can be expected at a higher confidence level. Asset returns can randomly take on any value within the uncertainty set, including the lower and upper bounds. As robust optimization seeks a feasible solution to the worst-case scenario of assets, it is likely that one of the extreme values will be assumed during model optimization.

Table 4 summarizes the Sharpe ratio, expected returns (%), and portfolio risks in terms of the number of uncertain parameters in the models. The gamma value ranges from zero to 56. The gamma value of zero indicating that the model contains no uncertain variables. The gamma value of 56 corresponds to the most conservative robust formulation, the robust optimization model of Soyster. The confidence levels of the intervals or uncertainty sets are represented by 99%, 95%, and 90%, which are determined by the number of uncertain parameters included in the model. For example, a gamma value of zero would correspond to the classical mean-variance model of Markowitz, while a gamma value of 56 would correspond to the robust optimization model of Soyster.

As the number of uncertain parameters included in the model increased, the expected return and Sharpe ratio decreased. This relationship was observed at different levels of uncertainty, with the Sharpe ratio decreasing by more than 20% at a 99% confidence level when the number of uncertain parameters increased from 1 to 2. However, the decrease in Sharpe ratio was minimal when the number of uncertain parameters was between 2 and 10. As the number of uncertain parameters continued to increase from 15 to 56, the Sharpe ratio decreased significantly, and for models with 45 or more uncertain parameters, it even took on a negative value. This suggests that the risk-free interest rate was higher than the expected return of the portfolio. The expected returns of the portfolios also followed a similar trend as the Sharpe ratio. The model without any uncertain parameters had an expected return of 52%, which remained almost unchanged when one uncertain parameter was included. At different confidence levels (99%, 95%, and 90%), the expected returns were 52.095%, 51.59%, and 51.59%, respectively.

In terms of risk, the variance of the portfolio, which represents investment risk, tended to decrease as the number of uncertain parameters included in the model increased. This was in line with expectations, as robust optimization takes into account parameter uncertainties before the solution, thereby reducing some of the investment risk. However, this reduction in risk came at the cost of sacrificing some of the expected return. As the level of uncertainty increased, the solution space of the optimization model became smaller. This inverse relationship between expected return and uncertainty is referred to as the "price of robustness" by Bertsimas and Sim (2004). It is important to note that a gamma value of zero (indicating no uncertain parameters) or the maximum value of 56 (indicating all uncertain parameters) is not expected in practice. In reality, the assets in a portfolio fall somewhere between these two extremes. Some sectors may perform below the expected return, while others may perform above it. This highlights the importance of diversification in a portfolio, as investing in assets from different sectors and with different characteristics can help mitigate the impact of underperforming assets.

Figure 1 illustrates the relationship between the Sharpe ratio and portfolio risk in relation to the uncertain parameters included in the models. The dashed line represents portfolio risk, while the straight line represents the Sharpe ratio. The figure suggests that as the number of uncertain parameters increases, the Sharpe ratio

Table 4. The Expected Return Of Portfolio, The Risk Of Portfolio, and The Sharpe Ratio

	Sharpe Ratio				**Sharpe Ratio**		
Gama=0	0.01812	%52.090	25.66250				
Gama=1				***Gama=4***			
99%	0.01810	%52.095	25.69165	***99%***	0.01490	%41.232	23.93431
95%	0.01682	%51.590	27.34654	***95%***	0.01490	%41.233	23.93460
90%	0.01683	%51.590	27.33443	***90%***	0.01503	%40.878	23.48085
Gama=2				***Gama=5***			
99%	0.01447	%41.352	24.72033	***99%***	0.01500	%40.877	23.52785
95%	0.01448	%41.352	24.70964	***95%***	0.01491	%40.910	23.69429
90%	0.01460	%41.336	24.48440	***90%***	0.01494	%40.907	23.64958
Gama=3				***Gama=6***			
99%	0.01448	%41.342	24.69349	***99%***	0.01497	%40.888	23.58361
95%	0.01496	%41.207	23.81680	***95%***	0.01489	%40.912	23.73513
90%	0.01506	%41.159	23.61990	***90%***	0.01494	%40.914	23.65005
Gama=7				***Gama =10***			
99%	0.01451	%39.445	23.33683	***99%***	0.01389	%37.804	23.19995
95%	0.01482	%39.752	23.04991	***95%***	0.01455	%39.869	23.55889
90%	0.01445	%39.595	23.53577	***90%***	0.01451	%39.430	23.32053
Gama =8				***Gama =15***			
99%	0.01459	%39.606	23.31754	***99%***	0.01258	%31.950	20.95607
95%	0.01623	%39.690	21.01095	***95%***	0.01382	%32.230	19.28916
90%	0.01623	%39.693	21.01858	***90%***	0.01425	%32.031	18.55739
Gama =9				***Gama =20***			
99%	0.01556	%39.800	21.99614	***99%***	0.01116	%29.720	21.62301
95%	0.01515	%40.000	22.71826	***95%***	0.01272	%29.790	19.03467
90%	0.01478	%39.674	23.07072	***90%***	0.01323	%31.510	19.60297
Gama=25				***Gama=45***			
99%	0.01094	%24.713	17.48019	***99%***	0.00156	%08.758	20.37770
95%	0.01221	%29.008	19.19222	***95%***	-0.00100	%3.330	22.50142
90%	0.01308	%29.662	18.40892	***90%***	0.00703	%20.133	20.70634
Gama=30				***Gama=50***			
99%	0.00827	%22.543	20.49859	***99%***	-0.00462	-%4.218	21.19801
95%	0.00984	%24.842	19.56484	***95%***	0.00343	%12.725	20.82131
90%	0.01081	%27.116	19.92363	***90%***	0.00099	%7.779	22.17366
Gama=35				***Gama=56***			
99%	0.00768	%21.013	20.10443	***99%***	-0.00720	-%9.667	21.17044
95%	0.00864	%23.427	20.64725	***95%***	-0.00577	-%10.000	27.01822
90%	0.00981	%25.708	20.51236	***90%***	-0.00175	%1.618	22.60142
Gama=40							
99%	0.00141	%8.471	20.45621				
95%	0.00492	%15.811	20.78953				
90%	0.00671	%19.545	20.80928				

decreases. For instance, when the model contains no uncertainty, the ratio is 0.017, but it drops to almost -0.003 when the optimization model includes 56 uncertain parameters. Similarly, the trend of portfolio risk follows that of the Sharpe ratio, until the point where 25 uncertain parameters are present in the models. Beyond this point, the trend of risk slightly increases with the number of uncertain parameters. Interestingly, both portfolio

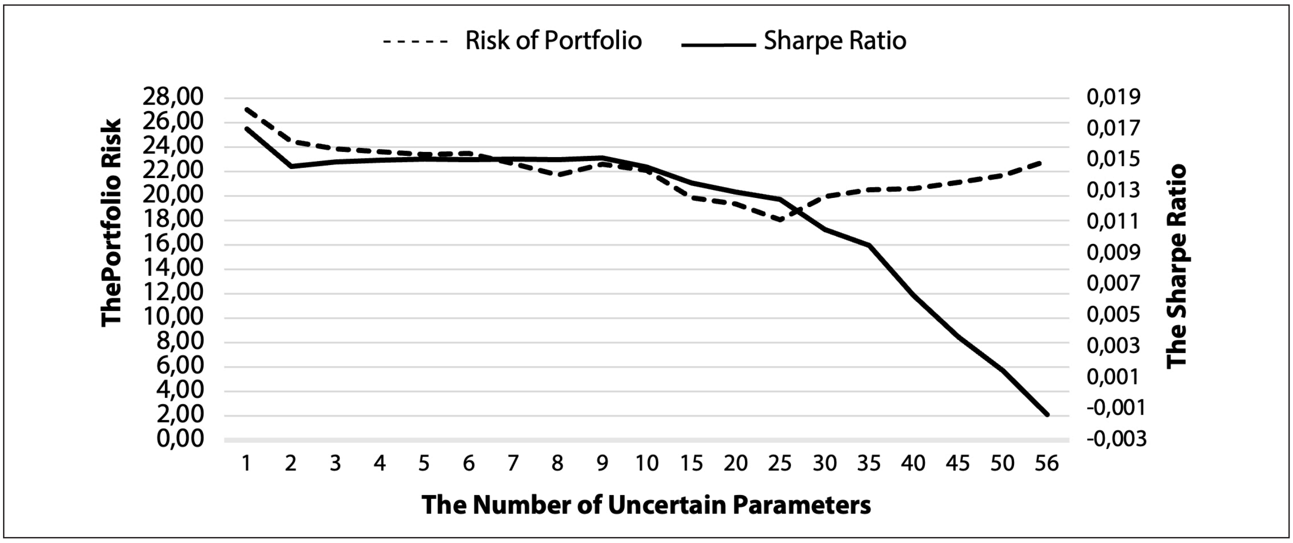

Figure 1: The Sharpe Ratio and the Risk of Portfolios Concerning Uncertain Parameters

risk and Sharpe ratio remain relatively stable when the number of uncertain parameters ranges from 2 to 10, indicating the robustness of the optimization solution. This suggests that solving the portfolio optimization problem with 2 uncertain parameters may be feasible for models with up to 10 uncertain parameters. However, for models with more than 10 uncertain parameters, a decrease in Sharpe ratio and an increase in risk can be expected due to the increased uncertainty in the models.

Figure 2 illustrates the relationship between the Sharpe ratio and expected returns of different models, based on the number of uncertain parameters in each model. As the number of uncertain parameters increases, the expected returns of the models decrease. This is reflected in the numerator of the Sharpe ratio, which includes the expected return of the portfolio. Interestingly, the decrease in expected return is more significant than the decrease in the Sharpe ratio when going from one uncertain parameter to two. This suggests that the impact of risk on the Sharpe ratio is greater than the impact of return when additional uncertain parameters are added to the model. As the number of uncertain parameters increases from two to ten, there is a gradual decline in expected returns. However, when there are more than ten uncertain parameters, the expected returns of the portfolios decrease even further and eventually become negative.

Figure 3 shows the portfolio efficient frontiers obtained by robust optimization. The efficient frontiers were computed from the constructed efficient portfolios with 95% confidence intervals for each gamma representing the uncertain parameters in the models. The efficient frontiers of , , , , , and almost overlapped, and the efficient frontier of was slightly below the first frontier. The efficient frontier of the portfolios gradually decreases as model uncertainty increases. This means that the efficient frontier of models with more uncertain parameters is lower than that of models with less uncertain parameters. The figure illustrates this by showing a decrease in expected return from almost 62% to almost 18%, depending on the level of uncertainty in the models. As investors' risk appetite increases, the expected return of the portfolios also increases, but at a decreasing rate. However, there comes a point where it is impossible to further increase the expected return for a given level of risk. At this point, there is no reason for the investor to take on more risk. The efficient frontier represents a combination of all the portfolios in which investors have invested, and it is technically impossible to achieve a higher return than this frontier. Therefore, investors should choose a portfolio on the efficient frontier based on their risk tolerance. Expected returns of a portfolio reflect the performance of the portfolio during a specific analysis period. However, investors interest in portfolio performance in a real investment process, because the expected return of a portfolio is not necessarily the same as the return achieved at the end of an investment period. Therefore, the average appreciation of an investment is often more important than the expected return of a portfolio. As a result, theoretically constructed portfolios are expected to generate a higher return than the average market returns in the investment process.

Table 5 displays the average annual returns of two portfolios: one constructed using robust optimization and the other using an equally weighted approach with 56 assets. These figures cover a seven-year period from November 2014 to February 2021, deliberately chosen to include the Covid-19 pandemic and demonstrate the robustness of the robust optimization solution. The results align with the expected return

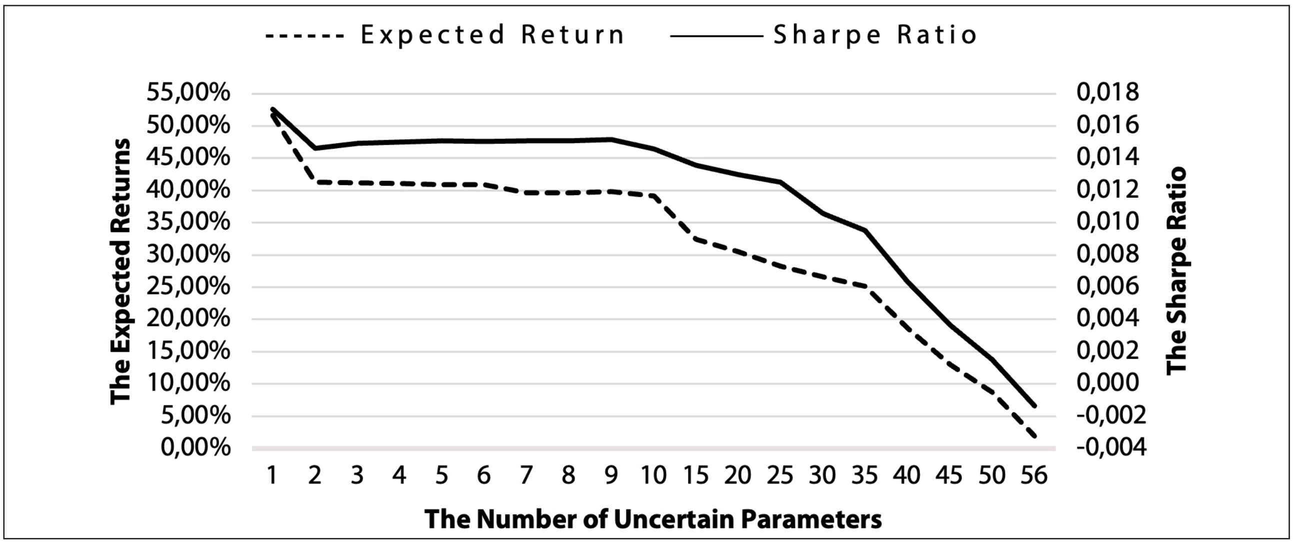

Figure 2: The Sharpe Ratio and the Expected Returns

and Sharpe ratio, as illustrated in Figures 1 and 2. As the number of uncertain parameters in the model increases, the average annual returns of the invested portfolio naturally decrease. For instance, the model without any uncertain parameters (corresponding to Markowitz's classical mean-variance model) yielded an average annual return of 51.64%. However, when an additional uncertain parameter (Gamma) was introduced and increased by two, the average annual return of the invested portfolio significantly decreased. In comparison, the average annual return of the market index during the same period was 44.60%. This suggests that portfolios without uncertainty provided approximately 7% more return than the market or equally weighted portfolio. However, if an investor aims to minimize risk, they may have to sacrifice some potential return. As the number of uncertain parameters in the model increases, the average annual returns of the portfolios decrease. For instance, when the number of uncertain parameters was 1, 2, 15, and 56, the average annual returns of the portfolios were 53.80%, 41.74%, 27.19%, and 24.82%, respectively. Notably, the average annual return of the model with two uncertain parameters was lower than that of the model without uncertainty, while the portfolios with one uncertain parameter provided approximately 9% more return than the equally weighted portfolio. Finally, the portfolios with the maximum number

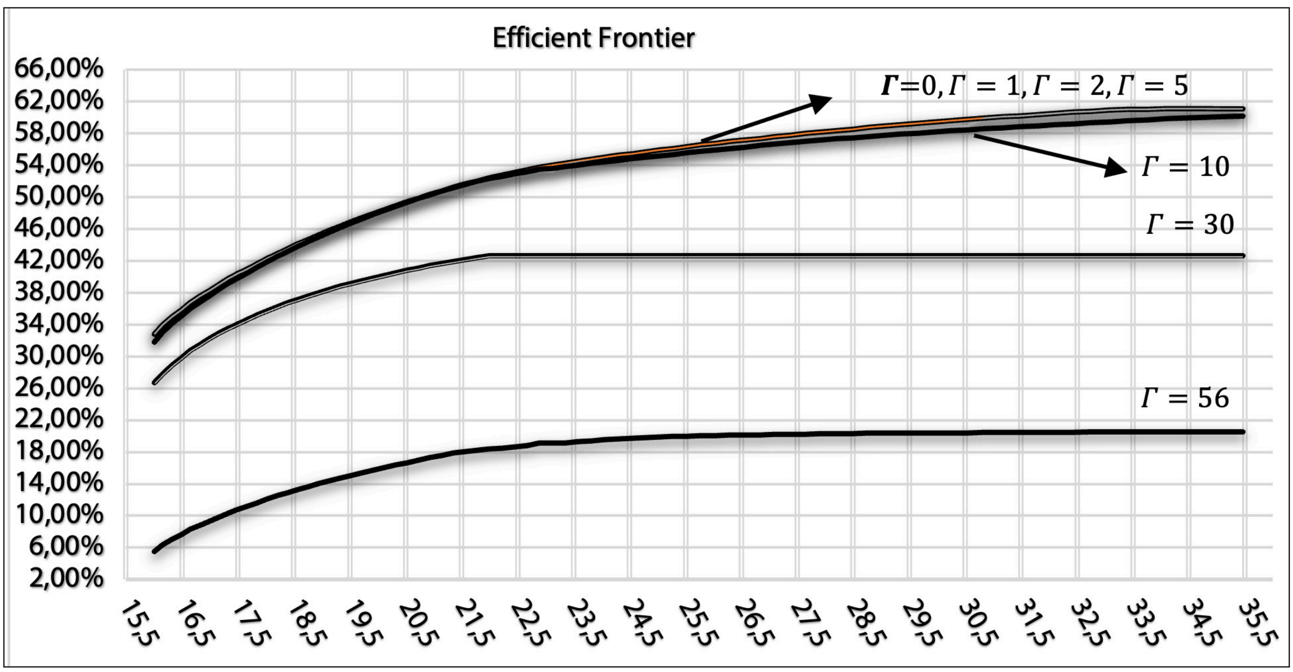

Figure 3: The Efficient Frontier of Portfolios Based on Uncertain Parameters

Table 5. The Average Annual Return of Portfolios and The Equally Weighted Index

#	The Constraint	Definition
1)	$\sum_{i=1}^{N} w_i = 1$	It ensures that the sum of the weights is equal to 1
2)	$\frac{\sum_{i=1}^{N} a_i w_i}{\sum_{i=1}^{N} w_i} \leq 7.00$	It limits the maximum weight of market-to-book ratio of an assets. Here, a_i is the market-to-book ratio of asset i.
3)	$\frac{\sum w_{jm}}{\sum_{i=1}^{N} w_i} \leq 0.35$	It ensures that the total weight of manufacturing stocks in the portfolio does not exceed 35%. Here, $\sum w_{jm}$ is the total weight of assets from manufacturing sector.
4)	$\frac{\sum w_{tp}}{\sum_{i=1}^{N} w_i} \geq 0.05$	It ensures that the total weight of the shares of wholesale, retail, restaurants and hotels is at least 5% of the portfolio. Here, $\sum w_{tp}$ is the total weight of assets from wholesale, retail, restaurants and hotels sector.
5)	$\frac{\sum w_{tk}}{\sum_{i=1}^{N} w_i} \geq 0.20$	It ensures that the total weight of shares in the technology sector is at least 20% of the portfolio. Here, $\sum w_{tk}$ is the total weight of assets from technology sector.
6)	$\frac{\sum w_{km}}{\sum_{i=1}^{N} w_i} \geq 0.10$	It ensures that the total weight of oil, gas and chemical sector stocks is at least 10% of the portfolio. Here, $\sum w_{km}$ is the total weight of assets from oil, gas and chemical sector.
7)	$\frac{\sum w_{ul}}{\sum_{i=1}^{N} w_i} \geq 0.08$	It ensures that the weights of transportation, storage, and communications shares in the portfolio are a maximum of 8%. Here, $\sum w_{ul}$ is the total weight of assets from transportation, storage, and communications sector.
8)	$0 \leq w_i \leq 0.05$	It limits the lower and upper bounds of the weights.

Note: AARP is the abbreviation for Average Annual Return of Portfolios.

of uncertain parameters had a 19.78% lower return than the equally weighted portfolio. It is worth mentioning that both portfolios with zero and 56 uncertain parameters are theoretically possible, but their realization is rare. In practice, portfolios with a number of uncertain parameters between these two extremes are more common. Therefore, calculating the number of assets with lower returns than the target index and incorporating this information into the investment process can assist investors in determining the appropriate number of uncertain parameters to include in their portfolio.

CONCLUSION

Modern portfolio theory aims to maximize returns and minimize risk in line with investor expectations. However, achieving both objectives in one model can be challenging. This is due to the volatility of securities and the difficulty in accurately predicting expected returns. Assuming that predicted values are precise and certain can invalidate the solution, as these values may contain errors. Instead, it is more realistic to consider a range of possible values for the predicted returns. This approach eliminates computational errors and biases within certain limits, while still ensuring a high probability of a successful optimization. The robust optimization model used in this paper combines the advantages of the bootstrap method, which allows for inferences to be made about the population, increasing the reliability of the model solution. The use of well-established uncertainty sets is crucial in ensuring the reliability of the model solution. By incorporating the bootstrap method, we are able to control both the uncertainty levels and the confidence levels of the uncertainty sets. This is the main advantage of the model proposed in this paper. The analysis includes uncertainty sets for three different confidence levels: 99%, 95%, and 90%. Additionally, the number of uncertain parameters is gradually increased from zero to 56, resulting in 58 different portfolio optimizations. This allows investors or researchers to select a model that meets their expectations and use the weights of the model for their investments. For example, a risk-sensitive investor may choose a portfolio with a high probability level for the uncertainty quantities and a high number of uncertain parameters, while a less risk-sensitive investor may prefer a portfolio with a lower probability level and fewer uncertain parameters. The portfolios constructed in this paper offer a flexible range of options for investors.

The results indicate that investors must make a trade-off between the stability of their portfolio and the expected return. If an investor wants to ensure that their portfolio remains feasible under all possible market conditions, they may have to sacrifice some of their expected return. However, the portfolio is still expected to outperform the market or target index. The study covers the period from January 2014 to March 2021, and during this time, the recommended portfolios had an average return higher than the market return. From April 1, 2021 to September 31, 2021, the average return of the model portfolios was 173.54%, 176.01%, and 173.16% at confidence levels of 99%, 95%, and 90% respectively. In comparison, the BIST100 increased by 69.7871%, BIST50 increased by 59.67%, BIST30 increased by 41.19%, and the equally weighted portfolio increased by 167.10%. These results demonstrate that the model portfolios consistently achieved higher returns than the index returns at all confidence levels. This shows that portfolios created using robust optimization not only provided high returns but also remained feasible under all possible market conditions. It is worth noting that extreme cases, where there are either zero or 56 uncertain parameters, are not expected in practical situations due to the diversification of the portfolio. Additionally, the robust formulation of the portfolios resulted in resistance to extreme fluctuations during the pandemic period and maintained high out-of-sample valuation rates. The robust models used in the analysis, which account for parameter uncertainty, produced optimal solutions that remained feasible with a high probability. In other words, portfolios created using robust optimization are not significantly affected by potential market fluctuations.

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