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Research Article

# A new goodness-of-fit test for the inverse Gaussian distribution

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## Abstract

The Inverse Gaussian (IG) distribution is widely used in practice and therefore an important issue is to develop a powerful goodness-of-fit test (GOF) for this distribution. In this article, we propose and examine a new GOF test for the IG distribution based on a new estimate of Kullback-Leibler (KL) information. The properties of the test statistic are presented. In order to compute the proposed test statistic, parameters of the IG distribution are estimated by maximum likelihood estimators, which are simple explicit estimators. Critical values and the actual sizes of the proposed test are obtained. Through a simulation study, power values of the proposed test are compared with some prominent existing tests. Finally, two illustrative examples are presented and analyzed.

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**Keywords.** Inverse Gaussian distribution, Kullback-Leibler information, Monte Carlo simulation, goodness-of-fit tests, critical points, test power

## 1. Introduction

The IG distribution is an important statistical model for analyzing right skewed data with positive support. Its density function is

$$f(x;\mu,\lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left\{-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\right\}, \qquad x > 0,$$

where $\mu$ and $\lambda$ are parameters. The mean and variance of this distribution are $\mu$ and $\mu^3/\lambda$, respectively.

Various applications based on IG distribution assumption are widely addressed by the literature in different fields of science as electrical networks, cardiology, hydrology, meteorology, ecology, physiology, demography, employment service, and etc., (e.g., [5, 6, 9, 18, 19, 24, 42]). Therefore, constructing powerful GOF tests for the IG distribution is an important issue. In this article, we develop a distribution-free test for the IG distribution using an estimate of KL information.

Assuming that $X_1, \ldots, X_n$ is the sample from a distribution $F$, we wish to assess whether the unknown $F(x)$ can be satisfactorily approximated by a IG model $G(x)$. GOF tests are

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designed to measure how well the observed sample data fits some proposed model. One class of GOF tests that can be used consists of tests based on the distance between the empirical and hypothesized distribution functions. Five of the known tests in this class are Cramer-von Mises ($W^2$), Kolmogorov-Smirnov ($D$), Kuiper ($V$), Watson ($U^2$), and Anderson-Darling ($A^2$). For more details about these tests (see, [13]).

Many researchers have been interested in GOF tests for different distributions and then different tests are developed in the literature. For example, see [7, 11, 13, 23, 32, 33]. Moreover, GOF tests based on censored samples are developed by some authors including [3, 4, 28, 35–39, 41].

Suppose a random variable $X$ has a distribution function $F(x)$ with a continuous density function $f(x)$. Then, the entropy $H(f)$ of $X$ was defined by [43] to be

$$H(f) = -\int_{-\infty}^{\infty} f(x)\, \log f(x)\, dx.$$

The problem of estimation of $H(f)$ based on a random sample from $F(x)$ has been considered by many authors including [12, 17, 22, 30, 44, 45, 50].

Vasicek [45] expressed a useful representation of entropy for the univariate $X$ in terms of the quantile function as

$$H(f) = \int_0^1 \log\left\{\frac{d}{du}Q(u)\right\} du,$$

where $Q(u) = F^{-1}(u) = \inf\{x: \ F(x) \geq u\}$ is the quantile function. Then, he constructed an estimate by replacing the distribution function $F$ by the empirical distribution function $F_n$, and using a difference operator instead of the differential operator. The derivative of $F^{-1}(p)$ is then estimated by a function of the order statistics obtained from the sample. With $X_1, \ldots, X_n$ being the sample, the estimator is given by

$$HV_{mn} = \frac{1}{n}\sum_{i=1}^{n} \log\left\{\frac{n}{2m}(X_{(i+m)} - X_{(i-m)})\right\},$$

where $m$ is positive integer, $m \leq \frac{n}{2}$, and $X_{(1)} \leq X_{(2)} \leq ... \leq X_{(n)}$ are the order statistics and $X_{(i)} = X_{(1)}$ if $i < 1$, $X_{(i)} = X_{(n)}$ if $i > n$. Vasicek (1976) established the consistency of $HV_{mn}$ for the population entropy $H(f)$.

Vasicek's sample entropy has been most widely used extensively for developing entropy-based statistical procedures, see [2, 15, 16, 20, 34].

In many practical problems, it is very important to test whether the underlying distribution has a specific form since most parametric statistical methods assume an underlying distribution in the development of methods.

Assuming that $X_1, \ldots, X_n$ is the sample from a distribution $F$, we wish to assess whether the unknown $F(x)$ can be satisfactorily approximated by a parametric model $G(x|\theta)$, where $\theta$ is a model parameter which is usually unknown.

The KL discrimination has been widely studied in the literature as a central index for measuring quantitative similarity between two probability distributions. The KL discrimination of $f$ from $g$ is defined by

$$D(f, g|\theta) = \int f(x)\, \log\frac{f(x)}{g(x|\theta)}\, dx.$$

Note that $D(f, g) = 0$ if and only if $f(x) = g(x|\theta)$ with probability 1. The KL discrimination is used for developing entropy models and tests, for example, Arizono and Ohta [2] and Ebrahimi et al. [16] proposed tests for normality and exponentiality, respectively, based on $D(f, g|\theta)$ that are the same as the tests introduced by [20] and [45] based on maximum entropy. Mudholkar and Tian [29] and Choi and Kim [10] proposed tests for inverse Gaussian and Laplace distributions, respectively, based on maximum entropy models. Recently, Alizadeh and Arghami [34] investigated general treatment of GOF tests based on KL information.

Recently, Alizadeh [31] proposed a new estimate of the KL discrimination and then constructed a test statistic for testing the validity of a model. His test statistic is

$$DA_{mn} = -\frac{1}{n}\sum_{i=1}^{n}\log\left\{\frac{n}{2m}\left(G(X_{(i+m)};\hat{\theta}) - G(X_{(i-m)};\hat{\theta})\right)\right\},$$

where $G$ is the distribution function of $g$. Here, $\theta$ is a model parameter which is usually unknown, and $\hat{\theta}$ is a reasonable equivariant estimate of $\theta$.

Noughabi [31] showed that the test statistic is non-negative just like the KL divergence, i.e., $DA_{mn} \geq 0$. Also, the test based on $DA_{mn}$ is consistent. Then, he proposed tests for normal, exponential, Laplace and Weibull distributions and compared the power of these tests with the other existing tests and showed that his test has a good power against different alternatives. In this paper, we apply the Alizadehs test statistic and suggest a GOF test for the IG distribution.

In Section 2, we propose a new GOF test statistic for the IG distribution based on an estimate of KL divergence. In Section 3, the critical points, the actual sizes and the power values of the proposed test are computed by Monte Carlo simulations and then compared with some known competing tests. Section 4 contains a real example for illustrative purpose. The following section contains a brief conclusion.

## 2. The proposed test statistic

In information theory, the KL divergence is a non-symmetric measure of the difference between two probability distributions $f$ and $g$. Typically, $f$ represents the true distribution of the observations and $g$ represents a theoretical model or approximation of $f$.

Suppose $X_1, \ldots, X_n$ is a random sample from an unknown continuous distribution $F$ with a probability density function $f(x)$. Let $G(x;\theta)$ be a parametric family of distributions with probability density function $g(x;\theta)$. Then, the hypothesis of interest is

$$H_0: \ f(x) = g(x;\theta), \qquad for\ some\ \theta \in \Omega.$$

The alternative to $H_0$ is a two-sided alternative of the form

$$H_1: \ f(x) \neq g(x;\theta), \qquad for\ any\ \theta \in \Omega.$$

To discriminate between the two hypotheses $H_0$ and $H_1$, Noughabi [31] used the KL divergence.

Let $f$ denote the true density function and $G = \{g(.,\theta) : \theta \in \Omega\}$ be a selected statistical model for the data distribution $f$, where $\Omega$ is a subset of $R^p$. When $f$ actually belongs to $G$, the minimal value, $\min_{\theta\in\Omega} D(f,\, g(.,\theta))$, of the KL divergence is zero. On the other hand, when $f$ does not belong to $G$, the minimal KL divergence is strictly positive. Therefore, a GOF test can be constructed which would reject $H_0: \ f \in G$ for large value of $D(f,g)$, where

$$D(f,g;\theta) = \int_{-\infty}^{\infty} f(x)\log f(x)\,dx - \int_{-\infty}^{\infty} f(x)\log\{g(x;\theta)\}\,dx$$

$$= -H(f) - E_f\{\log(g(X;\theta))\},$$

where $H(f)$ is the entropy of $f$. Noughabi [31] proposed to estimate $D(f,g)$ by

$$DA_{mn} = -HV_{mn} - HA_{mn},$$

where $HV_{mn}$ is Vasiceks estimate of $H(f)$ and $HA_{mn}$ is the semi-parametric estimate of $E_f\{\log(g(X;\theta))\}$, given by

$$HA_{mn} = \frac{1}{n}\sum_{i=1}^{n}\log\left\{\frac{G(X_{(i+m)};\hat{\theta}) - G(X_{(i-m)};\hat{\theta})}{X_{(i+m)} - X_{(i-m)}}\right\}.$$

Here, $\hat{\theta}$ can be any reasonable equivariant estimate of $\theta$. Consequently, the general test statistic proposed by [31] is as follows:

$$DA_{mn} = -\frac{1}{n}\sum_{i=1}^{n}\log\left\{\frac{n}{2m}\left(G(X_{(i+m)};\hat{\theta}) - G(X_{(i-m)};\hat{\theta})\right)\right\},$$

where $DA_{mn}$ denote the estimate of $D(f, g; \theta)$ and $G$ is the distribution function of $g$.

Given a random sample $X_1, \ldots, X_n$ from a continuous probability distribution $F$ with a density $f(x)$ over a non-negative support, the hypothesis of interest is

$$H_0: \ f(x) = f_0(x; \mu, \lambda) = \left(\frac{\lambda}{2\pi x^3}\right)^{1/2} \exp\left\{-\frac{\lambda}{2\mu^2 x}(x-\mu)^2\right\}, \ x > 0, \ \textit{for some } (\mu, \lambda) \in \Theta,$$

where $\mu$ and $\lambda$ are unspecified and $\Theta = R^+ \times R^+$. The alternative to $H_0$ is

$$H_1: \ f(x) \neq f_0(x; \mu, \lambda), \qquad \textit{for any } (\mu, \theta) \in \Theta.$$

We proposed the following test statistic for test of the IG distribution.

$$DA_{mn} = -\frac{1}{n}\sum_{i=1}^{n}\log\left\{\frac{n}{2m}\left(G(X_{(i+m)};\hat{\theta}) - G(X_{(i-m)};\hat{\theta})\right)\right\},$$

where $G$ is the IG distribution function and $\hat{\theta} = (\hat{\mu}, \hat{\lambda})$, where

$$\hat{\mu} = \bar{X} \quad \textit{and} \quad \hat{\lambda} = \frac{n}{\sum_{i=1}^{n}\left(1\big/X_i - 1\big/\bar{X}\right)}.$$

Clearly, we reject the null hypothesis for large values of the test statistic.

According to [31], the test statistic is non-negative, i.e., $DA_{mn} \geq 0$, and also the test based on $DA_{mn}$ is consistent. The mentioned properties of test statistic are presented in the following theorems. The proof of these theorems can be found in [31].

**Theorem 2.1.** *Let $X_1, ..., X_n$ be a random sample from an unknown continuous distribution $F$ with probability density function $f(x)$ and distribution function $G$ be known. Then, we have*

$$DA_{mn} \geq 0\,.$$

**Theorem 2.2.** *Let $F$ be a completely unknown continuous distribution and $G$ be the null distribution with unspecified parameters. Then under $H_1$, the test based on $DA_{mn}$ is consistent.*

## 3. Simulation study

### 3.1. Critical values and type-I error control

Distribution of the proposed test statistic $DA_{mn}$ under the null hypothesis cannot be evaluated analytically. Therefore, the critical values of the test statistic $DA_{mn}$ is obtained by the Monte Carlo method.

For our test statistic $DA_{mn}$, its sample value is calculated for 100,000 simulated random samples of size $n$ from the IG with parameters 1 and 1. Then, the critical values are determined for different significance levels $\alpha = 0.01, 0.05, 0.10$. For $\alpha = 0.05$, since $1-\alpha = 0.95 = 95000/100000$, the 95000-th order statistic is evaluated and the critical value is specified. Also, for $\alpha = 0.01$ and $\alpha = 0.10$, the 99000-th and 90000-th order statistics are evaluated and the critical values are determined. The critical values obtained for the proposed test statistic and sample sizes $10 \leq n \leq 100$ are given in Table 1.

**Table 1.** Critical values of the test statistic.

| | $\alpha$ | | |
|---|---|---|---|
| $n$ | 0.01 | 0.05 | 0.10 |
| 10 | 0.5546 | 0.4523 | 0.4130 |
| 15 | 0.4509 | 0.3868 | 0.3613 |
| 20 | 0.3809 | 0.3308 | 0.3107 |
| 25 | 0.3511 | 0.3129 | 0.2979 |
| 30 | 0.3336 | 0.3025 | 0.2899 |
| 40 | 0.2991 | 0.2755 | 0.2663 |
| 50 | 0.2776 | 0.2589 | 0.2515 |
| 75 | 0.2599 | 0.2480 | 0.2431 |
| 100 | 0.2456 | 0.2366 | 0.2329 |

Clearly, the proposed test statistic depends on the window size $m$. The value of $m$ can be obtained from heuristic formula $m = [n/3 + 1]$ for a given $n$, where $[x]$ means the integer part of $x$. For example, we recommend $m = 4$ for $n = 10$, $m = 7$ for $n = 20$, $m = 11$ for $n = 30$, and $m = 17$ for $n = 50$. We observe that the optimal $m$ increases as $n$ increases while $m/n \to 0$.

Figure 1 shows the empirical densities of the proposed test statistic for sample sizes $n = 10, 20, 30, 50$. Moreover, for different values of $\lambda$ ($\lambda = 0.5, 1, 2, 4$), the empirical density of the test statistic is derived with 100,000 samples of sizes $n = 10, 20, 30, 50$ from the IG distribution with parameter $\lambda$. The empirical densities of the proposed test are displayed in Figure 2.

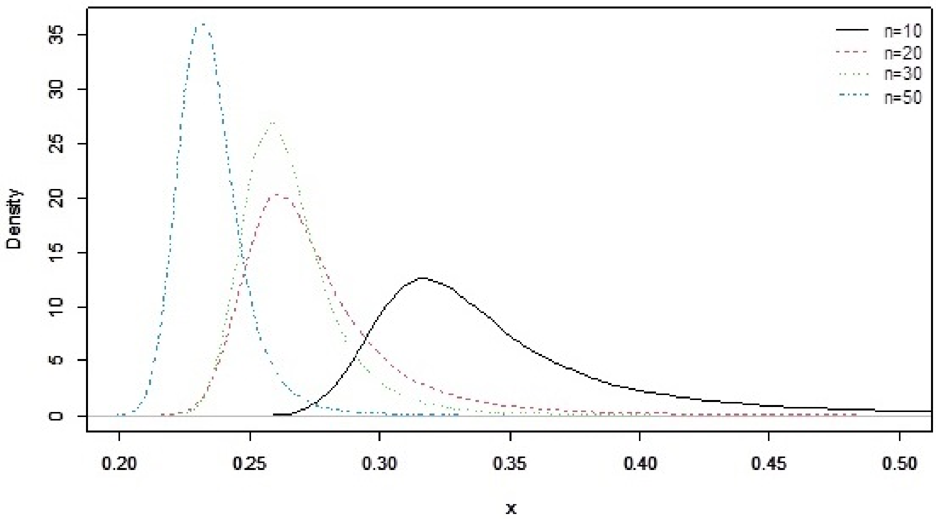


**Figure 1.** Estimated empirical densities of the test statistic generated with 100,000 samples of size $n = 10, 20, 30, 50$ from the IG distribution with parameters $\mu = 1$ and $\lambda = 1$.

We evaluate the estimated type I error control using the critical values of the proposed test. We generated random samples from different IG populations and then obtained the actual sizes of the proposed test. The results are displayed in Table 2. It is evident that the empirical percentiles given in Table 1 provides an excellent type I error control. Also, we can see that the actual sizes of the proposed test are acceptable but for the other tests the actual sizes are different with the nominal size $\alpha = 0.05$. Therefore, we can use our test in practice confidently.

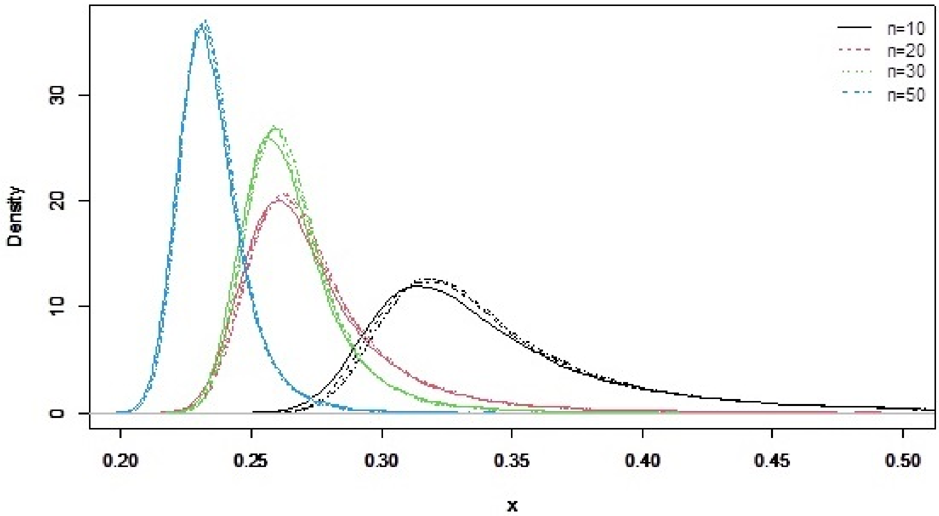


**Figure 2.** Estimated empirical densities of the test statistic generated with 100,000 samples of size $n = 10$, 20, 30, 50 from the IG distribution with parameters $\mu = 1$ and $\lambda = 0.5, 1, 2, 4$.

**Table 2.** Type I error control of the tests for the nominal significance level $alpha = 0.05$.

| | $n$ | $W^2$ | $D$ | $V$ | $U^2$ | $A^2$ | $T_1$ | $T_2$ | $DA_{mn}$ |
|---|---|---|---|---|---|---|---|---|---|
| $IG(1,0.5)$ | 10 | 0.0669 | 0.0641 | 0.0567 | 0.0573 | 0.0644 | 0.0581 | 0.0411 | 0.0493 |
| | 20 | 0.0699 | 0.0664 | 0.0572 | 0.0585 | 0.0671 | 0.0558 | 0.0462 | 0.0530 |
| | 30 | 0.0708 | 0.0664 | 0.0571 | 0.0587 | 0.0679 | 0.0506 | 0.0476 | 0.0545 |
| | 50 | 0.0716 | 0.0686 | 0.0574 | 0.0598 | 0.0678 | 0.0488 | 0.0522 | 0.0532 |
| $IG(1,2)$ | 10 | 0.0380 | 0.0384 | 0.0459 | 0.0449 | 0.0408 | 0.0482 | 0.0415 | 0.0485 |
| | 20 | 0.0362 | 0.0381 | 0.0464 | 0.0442 | 0.0393 | 0.0513 | 0.0488 | 0.0509 |
| | 30 | 0.0375 | 0.0385 | 0.0461 | 0.0453 | 0.0407 | 0.0489 | 0.0478 | 0.0488 |
| | 50 | 0.0366 | 0.0396 | 0.0456 | 0.0453 | 0.0394 | 0.0503 | 0.0498 | 0.0469 |
| $IG(1,4)$ | 10 | 0.0312 | 0.0316 | 0.0445 | 0.0421 | 0.0361 | 0.0488 | 0.0439 | 0.0486 |
| | 20 | 0.0297 | 0.0309 | 0.0434 | 0.0411 | 0.0337 | 0.0473 | 0.0455 | 0.0507 |
| | 30 | 0.0291 | 0.0306 | 0.0427 | 0.0404 | 0.0336 | 0.0502 | 0.0464 | 0.0482 |
| | 50 | 0.0288 | 0.0319 | 0.0434 | 0.0417 | 0.0335 | 0.0484 | 0.0516 | 0.0469 |
| $IG(1,8)$ | 10 | 0.0269 | 0.0278 | 0.0426 | 0.0397 | 0.0329 | 0.0489 | 0.0405 | 0.0486 |
| | 20 | 0.0253 | 0.0280 | 0.0433 | 0.0398 | 0.0309 | 0.0451 | 0.0492 | 0.0474 |
| | 30 | 0.0249 | 0.0271 | 0.0425 | 0.0389 | 0.0306 | 0.0497 | 0.0462 | 0.0476 |
| | 50 | 0.0240 | 0.0268 | 0.0412 | 0.0379 | 0.0292 | 0.0486 | 0.0471 | 0.0448 |

## 3.2. Power comparison

In this section, we compare the power values of our test with the power values of the competing tests against various alternatives by Monte Carlo simulation. We consider the popular and common tests which are used in practice and statistical software as competitor tests. The test statistics of these tests are briefly described as follows. For more details about these tests, see [13].

Let $X_{(1)} \leq X_{(2)} \leq \ldots \leq X_{(n)}$ are the order statistics based on the random sample $X_1, \ldots, X_n$.

(1) The Cramer-von Mises statistic [48]:

$$W^2 = \frac{1}{12n} + \sum_{i=1}^{n} \left( \frac{2i-1}{2n} - F_0(X_{(i)}; \hat{\mu}, \hat{\lambda}) \right)^2.$$

(2) The Watson statistic [49]:

$$U^2 = W^2 - n\Big(\bar{P} - 0.5\Big)^2,$$

where $\bar{P}$ is the mean of $F_0(X_{(i)}; \hat{\mu}, \hat{\lambda}),\;\; i = 1, \ldots, n$.

(3) The Kolmogorov-Smirnov statistic [25]:

$$D = \max(D^+, D^-)$$

where

$$D^+ = \max_{1\le i\le n}\left\{\frac{i}{n} - F_0(X_{(i)}; \hat{\mu}, \hat{\lambda})\right\};\quad D^- = \max_{1\le i\le n}\left\{F_0(X_{(i)}; \hat{\mu}, \hat{\lambda}) - \frac{i-1}{n}\right\}.$$

(4) The Kuiper statistic [26]:

$$V = D^+ + D^-$$

(5) The Anderson-Darling statistic [1]:

$$A^2 = -n - \frac{1}{n}\sum_{i=1}^{n}(2i-1)\left\{\log F_0(X_{(i)}; \hat{\mu}, \hat{\lambda}) + \log\left[1 - F_0(X_{(n-i+1)}; \hat{\mu}, \hat{\lambda})\right]\right\}.$$

In the above test statistics, $F_0(x)$ is the cumulative distribution function of the IG distribution and $(\hat{\mu}, \hat{\lambda})$ are the maximum likelihood estimates of the parameter $(\mu, \lambda)$.

Moreover, we consider two recent tests suggested by [46] and [47] in our power comparison. One test transforms the observations to approximately normally distributed observations and then uses Shapiro-Wilk test for assessing univariate normality ($T_1$). The other test is based on a transformation of data to gamma variables with shape parameter equal to 1/2 and uses Anderson-Darling test for testing the gamma distribution ($T_2$). For more details about these tests, see [46,47]. Also, an R package for these tests is provided by [21].

In power comparison, we considered the following alternatives.

- the exponential distribution $Exp(\theta)$ with density $\theta\exp(-\theta x)$,
- the Weibull distribution with density $\theta x^{\theta-1}\exp\left(-x^{\theta}\right)$, denoted by $W(\theta)$,
- the gamma distribution with density $\Gamma(\theta)^{-1}x^{\theta-1}\exp(-x)$, denoted by $\Gamma(\theta)$,
- the lognormal law $LN(\theta)$ with density $(\theta x)^{-1}(2\pi)^{-1/2}\exp\left(-(\log x)^2\big/(2\theta^2)\right)$,
- the Pareto distribution $Pa(\theta)$ with density $\theta\big/x^{\theta+1}$,
- the half-normal $HN$ distribution with density $\Gamma(2/\pi)^{1/2}\exp(-x^2/2)$,
- the uniform distribution $U$ with density 1, $\;0 \le x \le 1$,
- the Beta distribution $Beta(\alpha,\beta)$ with density $x^{\alpha-1}(1-x)^{\beta-1}\big/Beta(\alpha,\beta),\;\; 0 \le x \le 1$,
- the modified extreme value $EW(\theta)$, with distribution function $1-\exp\left(\theta^{-1}(1-e^{x})\right)$;
- the linear increasing failure rate law $LF(\theta)$ with density $(1+\theta x)\exp\left(-x-\theta x^2/2\right)$,
- Dhillons [14] law $DL(\theta)$ with distribution function $1-\exp\left(-(\log(x+1))^{\theta+1}\right)$,
- Chens [8] distribution $CH(\theta)$, with distribution function $1-\exp\left(2\left(1-e^{x^{\theta}}\right)\right)$.

The powers of the considered tests are computed by Monte Carlo simulation. Under each alternative, 100,000 samples of size 10, 20, 30 and 50 are generated. Then, the power of the corresponding test was estimated by the frequency of the event the test statistic is smaller than the critical point. The power estimates are presented in Tables 3–6. For each alternative, the bold type in these tables indicates the test achieving the maximal power.

**Table 3.** Monte Carlo power estimates of the tests for $n = 10$ and at level $\alpha = 0.05$.

| *Alternative* | $W^2$ | $D$ | $V$ | $U^2$ | $A^2$ | $T_1$ | $T_2$ | $DA_{mn}$ |
|---|---|---|---|---|---|---|---|---|
| $Exp(1)$ | 0.4182 | 0.3847 | 0.2977 | 0.3320 | **0.4223** | 0.3466 | 0.3184 | 0.2800 |
| $W(0.05)$ | 0.7950 | 0.7628 | 0.6818 | 0.6972 | **0.8079** | 0.6443 | 0.6877 | 0.5327 |
| $W(2)$ | 0.1811 | 0.1609 | 0.1391 | 0.1597 | **0.1916** | 0.1930 | 0.1443 | 0.1907 |
| $\Gamma(0.5)$ | 0.7527 | 0.7189 | 0.6341 | 0.6594 | **0.7606** | 0.6202 | 0.6417 | 0.5398 |
| $\Gamma(2)$ | 0.1731 | 0.1563 | 0.1220 | 0.1401 | **0.1783** | 0.1743 | 0.1351 | 0.1446 |
| $HN$ | 0.4271 | 0.3917 | 0.3146 | 0.3557 | **0.4330** | 0.3565 | 0.3243 | 0.3452 |
| $LN(0, 0.5)$ | 0.0380 | 0.0376 | 0.0468 | 0.0454 | 0.0430 | 0.0490 | 0.0526 | **0.0550** |
| $LN(0, 1)$ | **0.1045** | 0.0966 | 0.0711 | 0.0773 | 0.1041 | 0.0923 | 0.0668 | 0.0671 |
| $LN(0, 2)$ | 0.4827 | 0.4337 | 0.3221 | 0.3367 | **0.5037** | 0.3199 | 0.3265 | 0.1757 |
| $Pa(0.5)$ | 0.2353 | 0.1932 | 0.1865 | 0.1999 | **0.2596** | 0.0612 | 0.0810 | 0.2391 |
| $Pa(1)$ | 0.3581 | 0.2962 | 0.3025 | 0.3229 | 0.3607 | 0.1355 | 0.2005 | **0.4646** |
| $Pa(2)$ | 0.3338 | 0.2679 | 0.3251 | 0.3418 | 0.3673 | 0.0942 | 0.1149 | **0.4923** |
| $U$ | 0.5341 | 0.4795 | 0.4445 | 0.4849 | **0.5559** | 0.4338 | 0.3778 | 0.5414 |
| $Beta(2, 2)$ | 0.2535 | 0.2188 | 0.2098 | 0.2351 | 0.2739 | 0.2388 | 0.1733 | **0.3072** |
| $Beta(2, 0.5)$ | 0.6005 | 0.4831 | 0.6214 | 0.6238 | 0.6557 | 0.4053 | 0.2915 | **0.7679** |
| $Beta(0.5, 2)$ | 0.7666 | 0.7343 | 0.6496 | 0.6819 | **0.7755** | 0.6387 | 0.6598 | 0.5936 |
| $Beta(2, 5)$ | 0.1984 | 0.1764 | 0.1482 | 0.1713 | **0.2100** | 0.1995 | 0.1510 | 0.1989 |
| $CH(0.5)$ | 0.7736 | 0.7397 | 0.6559 | 0.6784 | **0.7833** | 0.6312 | 0.6625 | 0.5462 |
| $CH(1)$ | 0.4386 | 0.4029 | 0.3216 | 0.3631 | **0.4451** | 0.3710 | 0.3383 | 0.3359 |
| $CH(1.5)$ | 0.2908 | 0.2593 | 0.2208 | 0.2526 | **0.3025** | 0.2684 | 0.2157 | 0.2831 |
| $LF(2)$ | 0.4093 | 0.3758 | 0.3047 | 0.3443 | **0.4158** | 0.3664 | 0.3237 | 0.3315 |
| $LF(4)$ | 0.3877 | 0.3561 | 0.2927 | 0.3296 | **0.3958** | 0.3458 | 0.3006 | 0.3339 |
| $EV(0.5)$ | 0.4398 | 0.4024 | 0.3207 | 0.3625 | **0.4459** | 0.3679 | 0.3380 | 0.3354 |
| $EV(1.5)$ | 0.4444 | 0.4064 | 0.3421 | 0.3846 | **0.4539** | 0.3782 | 0.3345 | 0.3937 |
| $DL(1)$ | 0.1599 | 0.1460 | 0.1091 | 0.1250 | **0.1628** | 0.1527 | 0.1237 | 0.1120 |
| $DL(1.5)$ | 0.1203 | 0.1101 | 0.0900 | 0.1017 | 0.1264 | **0.1411** | 0.1066 | 0.1080 |

**Table 4.** Monte Carlo power estimates of the tests for $n = 20$ and at level $\alpha = 0.05$.

| *Alternative* | $W^2$ | $D$ | $V$ | $U^2$ | $A^2$ | $T_1$ | $T_2$ | $DA_{mn}$ |
|---|---|---|---|---|---|---|---|---|
| $Exp(1)$ | 0.6787 | 0.6309 | 0.5197 | 0.5686 | **0.6804** | 0.5927 | 0.5749 | 0.5532 |
| $W(0.05)$ | 0.9637 | 0.9508 | 0.9106 | 0.9224 | **0.9659** | 0.9040 | 0.9288 | 0.8578 |
| $W(2)$ | 0.3415 | 0.2955 | 0.2573 | 0.2957 | 0.3613 | 0.3232 | 0.2574 | **0.3846** |
| $\Gamma(0.5)$ | 0.9521 | 0.9363 | 0.8887 | 0.9074 | **0.9535** | 0.8815 | 0.9054 | 0.8653 |
| $\Gamma(2)$ | 0.3034 | 0.2676 | 0.2067 | 0.2416 | **0.3141** | 0.3106 | 0.2480 | 0.2786 |
| $HN$ | 0.7046 | 0.6538 | 0.5689 | 0.6178 | **0.7112** | 0.6230 | 0.5956 | 0.6614 |
| $LN(0, 0.5)$ | 0.0362 | 0.0365 | 0.0465 | 0.0458 | 0.0417 | **0.0620** | 0.0513 | 0.0566 |
| $LN(0, 1)$ | **0.1423** | 0.1247 | 0.0867 | 0.0995 | 0.1418 | 0.1374 | 0.0997 | 0.0846 |
| $LN(0, 2)$ | 0.7217 | 0.6672 | 0.5238 | 0.5531 | **0.7307** | 0.5434 | 0.5650 | 0.3694 |
| $Pa(0.5)$ | 0.4541 | 0.3582 | 0.3473 | 0.3584 | 0.4808 | 0.1480 | 0.2134 | **0.5582** |
| $Pa(1)$ | 0.7189 | 0.6121 | 0.6635 | 0.6604 | 0.7428 | 0.3493 | 0.4557 | **0.8717** |
| $Pa(2)$ | 0.6754 | 0.5456 | 0.6733 | 0.6686 | 0.7338 | 0.2492 | 0.2266 | **0.8865** |
| $U$ | 0.8481 | 0.7826 | 0.7883 | 0.8033 | 0.8702 | 0.6977 | 0.6592 | **0.9100** |
| $Beta(2, 2)$ | 0.5009 | 0.4216 | 0.4257 | 0.4602 | 0.5415 | 0.4200 | 0.3343 | **0.6449** |
| $Beta(2, 0.5)$ | 0.9248 | 0.8252 | 0.9439 | 0.9304 | 0.9556 | 0.6858 | 0.6121 | **0.9916** |
| $Beta(0.5, 2)$ | 0.9606 | 0.9453 | 0.9066 | 0.9249 | **0.9633** | 0.8949 | 0.9105 | 0.9120 |
| $Beta(2, 5)$ | 0.3770 | 0.3252 | 0.2793 | 0.3182 | 0.3967 | 0.3472 | 0.2781 | **0.4172** |
| $CH(0.5)$ | 0.9606 | 0.9463 | 0.9030 | 0.9201 | **0.9621** | 0.8900 | 0.9077 | 0.8739 |
| $CH(1)$ | 0.7197 | 0.6679 | 0.5791 | 0.6276 | **0.7262** | 0.6167 | 0.5939 | 0.6644 |
| $CH(1.5)$ | 0.5413 | 0.4761 | 0.4278 | 0.4749 | 0.5619 | 0.4730 | 0.4037 | **0.5846** |
| $LF(2)$ | 0.6855 | 0.6355 | 0.5549 | 0.6034 | **0.6934** | 0.6153 | 0.5791 | 0.6542 |
| $LF(4)$ | 0.6630 | 0.6094 | 0.5367 | 0.5843 | **0.6727** | 0.6038 | 0.5658 | 0.6415 |
| $EV(0.5)$ | 0.7216 | 0.6685 | 0.5810 | 0.6297 | **0.7281** | 0.6148 | 0.5939 | 0.6654 |
| $EV(1.5)$ | 0.7407 | 0.6856 | 0.6242 | 0.6693 | **0.7522** | 0.6397 | 0.6065 | 0.7444 |
| $DL(1)$ | 0.2639 | 0.2328 | 0.1711 | 0.1994 | **0.2685** | 0.2641 | 0.2134 | 0.2009 |
| $DL(1.5)$ | 0.2050 | 0.1784 | 0.1417 | 0.1638 | 0.2150 | **0.2256** | 0.1741 | 0.1918 |

**Table 5.** Monte Carlo power estimates of the tests for $n = 30$ and at level $\alpha = 0.05$.

| *Alternative* | $W^2$ | $D$ | $V$ | $U^2$ | $A^2$ | $T_1$ | $T_2$ | $DA_{mn}$ |
|---|---|---|---|---|---|---|---|---|
| $Exp(1)$ | 0.8294 | 0.7864 | 0.6874 | 0.7323 | **0.8316** | 0.7440 | 0.7361 | 0.7343 |
| $W(0.05)$ | 0.9945 | 0.9912 | 0.9784 | 0.9825 | **0.9948** | 0.9733 | 0.9835 | 0.9554 |
| $W(2)$ | 0.4821 | 0.4159 | 0.3698 | 0.4166 | 0.5093 | 0.4488 | 0.3630 | **0.5520** |
| $\Gamma(0.5)$ | 0.9902 | 0.9853 | 0.9684 | 0.9754 | **0.9904** | 0.9683 | 0.9769 | 0.9591 |
| $\Gamma(2)$ | 0.4259 | 0.3742 | 0.2948 | 0.3398 | **0.4393** | 0.4098 | 0.3441 | 0.4096 |
| $HN$ | 0.8588 | 0.8151 | 0.7481 | 0.7884 | **0.8651** | 0.7669 | 0.7480 | 0.8356 |
| $LN(0, 0.5)$ | 0.0360 | 0.0369 | 0.0470 | 0.0462 | 0.0426 | **0.0681** | 0.0558 | 0.0579 |
| $LN(0, 1)$ | 0.1721 | 0.1497 | 0.1052 | 0.1202 | **0.1740** | 0.1712 | 0.1299 | 0.1106 |
| $LN(0, 2)$ | 0.8503 | 0.8077 | 0.6785 | 0.7078 | **0.8553** | 0.7001 | 0.7274 | 0.5227 |
| $Pa(0.5)$ | 0.6399 | 0.5167 | 0.5173 | 0.5076 | 0.6780 | 0.3295 | 0.3703 | **0.7628** |
| $Pa(1)$ | 0.8970 | 0.8137 | 0.8710 | 0.8563 | 0.9193 | 0.5374 | 0.6370 | **0.9754** |
| $Pa(2)$ | 0.8669 | 0.7469 | 0.8743 | 0.8563 | 0.9136 | 0.4228 | 0.3540 | **0.9789** |
| $U$ | 0.9572 | 0.9201 | 0.9361 | 0.9347 | 0.9692 | 0.8481 | 0.8237 | **0.9846** |
| $Beta(2, 2)$ | 0.6929 | 0.5933 | 0.6137 | 0.6404 | 0.7402 | 0.5506 | 0.4471 | **0.8412** |
| $Beta(2, 0.5)$ | 0.9904 | 0.9623 | 0.9953 | 0.9910 | 0.9968 | 0.8511 | 0.8196 | **0.9997** |
| $Beta(0.5, 2)$ | 0.9937 | 0.9894 | 0.9782 | 0.9834 | **0.9943** | 0.9703 | 0.9777 | 0.9807 |
| $Beta(2, 5)$ | 0.5279 | 0.4575 | 0.4017 | 0.4498 | 0.5562 | 0.4691 | 0.3831 | **0.5990** |
| $CH(0.5)$ | 0.9929 | 0.9887 | 0.9741 | 0.9797 | **0.9933** | 0.9700 | 0.9799 | 0.9650 |
| $CH(1)$ | 0.8706 | 0.8265 | 0.7564 | 0.7966 | **0.8767** | 0.7678 | 0.7507 | 0.8374 |
| $CH(1.5)$ | 0.7183 | 0.6476 | 0.6028 | 0.6486 | 0.7422 | 0.6143 | 0.5429 | **0.7721** |
| $LF(2)$ | 0.8424 | 0.7979 | 0.7321 | 0.7726 | **0.8498** | 0.7591 | 0.7368 | 0.8254 |
| $LF(4)$ | 0.8236 | 0.7772 | 0.7159 | 0.7562 | **0.8320** | 0.7416 | 0.7133 | 0.8191 |
| $EV(0.5)$ | 0.8706 | 0.8270 | 0.7565 | 0.7959 | **0.8769** | 0.7720 | 0.7499 | 0.8395 |
| $EV(1.5)$ | 0.8891 | 0.8458 | 0.8045 | 0.8369 | 0.8987 | 0.7894 | 0.7666 | **0.9013** |
| $DL(1)$ | 0.3544 | 0.3118 | 0.2323 | 0.2705 | **0.3626** | 0.3515 | 0.2934 | 0.2973 |
| $DL(1.5)$ | 0.2844 | 0.2474 | 0.1945 | 0.2261 | 0.2999 | **0.3071** | 0.2429 | 0.2808 |

**Table 6.** Monte Carlo power estimates of the tests for $n = 50$ and at level $\alpha = 0.05$.

| *Alternative* | $W^2$ | $D$ | $V$ | $U^2$ | $A^2$ | $T_1$ | $T_2$ | $DA_{mn}$ |
|---|---|---|---|---|---|---|---|---|
| $Exp(1)$ | 0.9554 | 0.9324 | 0.8777 | 0.9052 | **0.9558** | 0.9040 | 0.9015 | 0.9067 |
| $W(0.05)$ | **0.9998** | 0.9997 | 0.9988 | 0.9992 | **0.9998** | 0.9981 | 0.9993 | 0.9965 |
| $W(2)$ | 0.7032 | 0.6209 | 0.5682 | 0.6205 | 0.7306 | 0.6180 | 0.5357 | **0.7562** |
| $\Gamma(0.5)$ | **0.9997** | 0.9993 | 0.9977 | 0.9985 | **0.9997** | 0.9980 | 0.9987 | 0.9971 |
| $\Gamma(2)$ | 0.6186 | 0.5477 | 0.4489 | 0.5063 | **0.6319** | 0.5643 | 0.4944 | 0.5886 |
| $HN$ | 0.9717 | 0.9523 | 0.9237 | 0.9416 | **0.9740** | 0.9173 | 0.9117 | 0.9626 |
| $LN(0, 0.5)$ | 0.0379 | 0.0379 | 0.0483 | 0.0475 | 0.0446 | **0.0764** | 0.0589 | 0.0565 |
| $LN(0, 1)$ | 0.2295 | 0.1937 | 0.1360 | 0.1584 | **0.2333** | 0.2309 | 0.1772 | 0.1418 |
| $LN(0, 2)$ | 0.9572 | 0.9358 | 0.8620 | 0.8815 | **0.9580** | 0.8778 | 0.9040 | 0.7385 |
| $Pa(0.5)$ | 0.8684 | 0.7541 | 0.7878 | 0.7384 | 0.9024 | 0.7076 | 0.6600 | **0.9535** |
| $Pa(1)$ | 0.9906 | 0.9655 | 0.9874 | 0.9804 | 0.9950 | 0.8270 | 0.8630 | **0.9996** |
| $Pa(2)$ | 0.9857 | 0.9422 | 0.9884 | 0.9806 | 0.9946 | 0.6486 | 0.5927 | **0.9997** |
| $U$ | 0.9980 | 0.9918 | 0.9964 | 0.9950 | 0.9990 | 0.9620 | 0.9568 | **0.9998** |
| $Beta(2, 2)$ | 0.9039 | 0.8222 | 0.8527 | 0.8627 | 0.9331 | 0.7394 | 0.6427 | **0.9773** |
| $Beta(2, 0.5)$ | 0.9999 | 0.9994 | **1.0000** | 0.9999 | **1.0000** | 0.9704 | 0.9720 | **1.0000** |
| $Beta(0.5, 2)$ | **0.9999** | 0.9997 | 0.9993 | 0.9995 | **0.9999** | 0.9978 | 0.9984 | 0.9995 |
| $Beta(2, 5)$ | 0.7565 | 0.6718 | 0.6159 | 0.6655 | 0.7820 | 0.6391 | 0.5456 | **0.8159** |
| $CH(0.5)$ | **0.9998** | 0.9997 | 0.9988 | 0.9993 | **0.9998** | 0.9978 | 0.9991 | 0.9979 |
| $CH(1)$ | 0.9750 | 0.9561 | 0.9274 | 0.9448 | **0.9774** | 0.9204 | 0.9148 | 0.9663 |
| $CH(1.5)$ | 0.9091 | 0.8536 | 0.8265 | 0.8572 | 0.9229 | 0.8008 | 0.7458 | **0.9390** |
| $LF(2)$ | 0.9653 | 0.9439 | 0.9130 | 0.9338 | **0.9683** | 0.9137 | 0.9042 | 0.9580 |
| $LF(4)$ | 0.9574 | 0.9321 | 0.9013 | 0.9238 | **0.9614** | 0.9026 | 0.8852 | 0.9542 |
| $EV(0.5)$ | 0.9754 | 0.9564 | 0.9271 | 0.9448 | **0.9771** | 0.9203 | 0.9139 | 0.9659 |
| $EV(1.5)$ | 0.9834 | 0.9673 | 0.9544 | 0.9653 | 0.9861 | 0.9365 | 0.9240 | **0.9864** |
| $DL(1)$ | 0.5117 | 0.4537 | 0.3520 | 0.4016 | **0.5198** | 0.5080 | 0.4455 | 0.4284 |
| $DL(1.5)$ | 0.4260 | 0.3664 | 0.2901 | 0.3374 | **0.4426** | 0.4327 | 0.3514 | 0.4110 |

The power of the aforementioned test statistic depends on the alternative distribution and the window size. It is not possible to have the best value of $m$ which attains the maximum powers for all alternatives. Therefore, based on a broad Monte Carlo analysis, we determine the optimal $m$ to be the values of $m$ which attain good (not best) powers for all alternative distributions. The value of $m$ can be obtained from heuristic formula $m = [n/3+1]$ for a given $n$, where $[x]$ means the integer part of $x$. For example, we recommend $m = 4$ for $n = 10$, $m = 7$ for $n = 20$, $m = 11$ for $n = 30$, and $m = 17$ for $n = 50$ as the optimal values which the proposed test attains good (not best) power values against all alternatives. We observe that the optimal $m$ increases as $n$ increases.

It is evident from Tables 3-6 that there is no one uniformly most powerful test against all alternatives. Under various alternatives, sometimes the proposed test has a higher power, and sometimes other tests do. However, for almost alternatives the tests based on $A^2$ and $DA_{mn}$ statistics have the most power.

Power study reveals the tests $A^2$ and $DA_{mn}$ have a high power and generally they outperform the other tests under the different alternatives. The power differences between these tests and the other tests are substantial. In other hand, from Table 2, we found that the actual size of the proposed test based on $DA_{mn}$ was acceptable. Consequently, the proposed test based on $DA_{mn}$ statistic should be recommended in practice. Finally, we can generally conclude that the test $DA_{mn}$ has a good performance against almost alternatives and this test can be confidently recommended in practice.

## 4. Illustrative examples

In this section, we illustrate how the tests can be applied to test the GOF for the IG distribution when the observations are available.

**Example 4.1.** Folks and Chhikara [18] considered the following dataset, consisting of 19 fracture toughness of MIG (metal inert gas) welds.

54.4, 62.6, 63.2, 67.0, 70.2, 70.5, 70.6, 71.4, 71.8, 74.1, 74.1, 74.3, 78.8, 81.8, 83.0, 84.4, 85.3, 86.9, 87.3.

They concluded by using the KS statistic that the IG distribution is a reasonable fit. Histogram of the considered data set is presented in Figure 3.

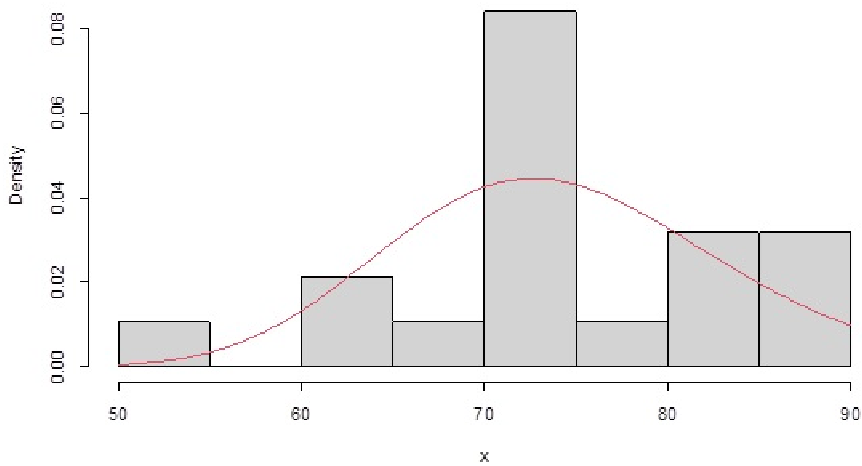


**Figure 3.** Histogram of data in Example 1 and a fitted IG density function.

Here, we apply the considered tests to these data. First, the ML estimates of $\mu$ and $\lambda$ are computed as:

$$\hat{\mu} = 74.3 \quad and \quad \hat{\lambda} = 4923.952\,.$$

Then, the value of each test statistic is computed and also the critical value of each test at the significance level 0.05 is obtained by Monte Carlo simulation. Results are summarized in Table 7.

**Table 7.** The value of the test statistics and critical values at 5% level.

| Test | Value of the test statistic | Critical value | Decision |
|---|---|---|---|
| $W^2$ | 0.05379 | 0.83473 | Not reject $H_0$ |
| $D$ | 0.13339 | 0.21478 | Not reject $H_0$ |
| $V$ | 0.24056 | 0.33114 | Not reject $H_0$ |
| $U^2$ | 0.05030 | 0.12381 | Not reject $H_0$ |
| $A^2$ | 0.37997 | 0.83192 | Not reject $H_0$ |
| $T_1$ | 0.96487 | 0.90046 | Not reject $H_0$ |
| $T_2$ | 0.02417 | 1.44810 | Not reject $H_0$ |
| $DA_{mn}$ | 0.29149 | 0.34384 | Not reject $H_0$ |

Because the value of each test statistic is smaller than the corresponding critical value, the IG hypothesis is not rejected for these data at the significance level of 0.05. Based on the considered tests, the data did not provide evidence against the null hypothesis. Also, two tests $T_1$ and $T_2$ do not reject the null hypothesis. Therefore, we can conclude that the underlying distribution of these data is an IG distribution.

**Example 4.2.** The following data represent active repair times (in hours) for an airborne communication transceiver.

0.2, 0.3, 0.5, 0.5, 0.5, 0.5, 0.6, 0.6, 0.7, 0.7, 0.7, 0.8, 0.8, 1.0, 1.0, 1.0, 1.0, 1.1, 1.3, 1.5, 1.5, 1.5, 1.5, 2.0, 2.0, 2.2, 2.5, 3.0, 3.0, 3.3, 3.3, 4.0, 4.0, 4.5, 4.7, 5.0, 5.4, 5.4, 7.0, 7.5, 8.8, 9.0, 10.3, 22.0, 24.5.

These data are used by [27] and [40]. They fitted the IG distribution and finally concluded that the fit is good. Histogram of the considered data set is presented in Figure 4.

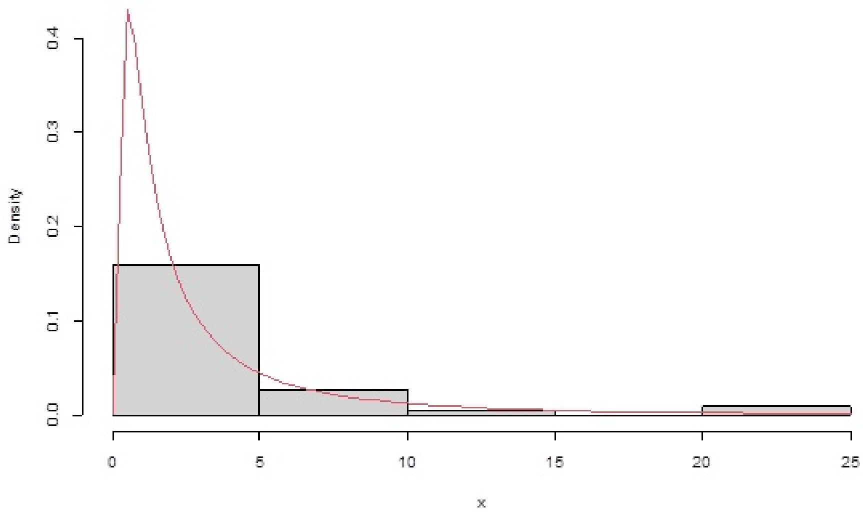


**Figure 4.** Histogram of data in Example 2 and a fitted IG density function.

The ML estimates of $\mu$ and $\lambda$ for the considered data are computed as:

$$\hat{\mu} = 3.6267 \quad and \quad \hat{\lambda} = 1.6242\,.$$

In Table 8, the values of test statistics and also the critical values of tests at the significance level 0.05 are presented.

**Table 8.** The value of the test statistics and critical values at 5% level.

| Test | Value of the test statistic | Critical value | Decision |
|---|---|---|---|
| $W^2$ | 0.03677 | 0.15440 | Not reject $H_0$ |
| $D$ | 0.07245 | 0.14281 | Not reject $H_0$ |
| $V$ | 0.14081 | 0.22023 | Not reject $H_0$ |
| $U^2$ | 0.03676 | 0.12505 | Not reject $H_0$ |
| $A^2$ | 0.23926 | 0.85266 | Not reject $H_0$ |
| $T_1$ | 0.98273 | 0.95020 | Not reject $H_0$ |
| $T_2$ | 0.32354 | 1.47697 | Not reject $H_0$ |
| $DA_{mn}$ | 0.23126 | 0.27266 | Not reject $H_0$ |

Since the value of each test statistic is smaller than the corresponding critical value, the IG hypothesis is not rejected for these data at the significance level of 0.05. Also, two tests $T_1$ and $T_2$ do not reject the null hypothesis. Consequently, we conclude that these data follow an IG distribution.

## 5. Conclusions

In this paper, we have proposed a new GOF test for the IG distribution based on an estimate of KL information. We have stated the properties of the proposed test and computed the critical values, the actual sizes and powers of the test. Through a Monte Carlo simulation, we have shown that the proposed test is powerful against some alternatives. Also, we have shown that the actual sizes of the proposed test are acceptable. Therefore, the proposed test can be confidently used in practice. Finally, we have considered two real data sets and have illustrated how the proposed test can be employed to test the GOF for the IG distribution when a sample is available.

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