INTERNATIONAL ELECTRONIC JOURNAL OF GEOMETRY
VOLUME 16 NO. 1 PAGE 111–121 (2023)
DOI: HTTPS://DOI.ORG/10.36890/IEJG.1210442

# The Spinor Expressions of Mannheim Curves in Euclidean 3-Space

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(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

**ABSTRACT**

**In this paper, the spinor formulations of Mannheim curve pair are investigated. First of all, two spinors corresponding to Mannheim curve pair are given and considering the relationships between the Frenet frames of Mannheim curve pair the relationship between two spinors corresponding to this curve pair are obtained. Therefore, some geometric interpretations of spinors are obtained using the Mannheim curve pair and considering Mannheim curve as helix the spinor formulations of Mannheim curve pair are given. Moreover, the spinor formulations for the curvatures of the Mannheim curve pair are also obtained. Consequently, an example of these spinors is given. Therefore, it is thought that this study will make an important contribution to the mathematical analysis and geometric interpretation of spinors, which have many uses in physics.**

***Keywords:*** *Spinors, Clifford algebra, Euclidean space, Mannheim curve pair*
***AMS Subject Classification (2020):*** *Primary: 15A66 ; Secondary: 53A04: 53Z05.*

## 1. Introduction

Spinors form a vector space, usually on complex numbers, through a representation of the spin group. Spinors are needed to obtain the most basic data related to the topology of the spin group, and for each rotation the spin group has two elements that represent it. In this case, geometric vectors and other tensors produce opposite signs when they affect any spinor under the representation because they cannot distinguish between these two elements. Moreover, spinors, which are the main components in the definition of fermionic particles that encode ordinary matter in the universe, have created a comprehensive mathematical study object with various applications. Especially, in applications of quantum, mechanics in physics the spinor theory has been used frequently [8, 9, 19, 30]. Cartan, one of the first people to study on Lie groups, was also the first to study spinors geometrically in mathematics [6]. Cartan obtained the spinor formulas of the basic definitions in geometry and emphasized that isotropic vectors in $C^3$ create a surface with two-dimensional in complex space $C^2$. Cartan also showed that every isotropic vector in space $C^3$ corresponds to two vectors in space $C^2$. Later, Cartan named these complex vectors in the space $C^2$ as spinors [6]. After that, Vivarelli obtained a spinor formulation of rotations by establishing a linear relationship between spinors and real quaternions considering the representation of rotational motions with quaternions in $R^3$ [29]. In a different study, Torres del Castillo and Barrales expressed the curve theory in $E^3$ with spinors [7]. In addition, the spinor equations of relationship between Bishop frame and Frenet frame were obtained in [28]. Then, considering these studies the spinor representation of the Darboux frame in $E^3$ was obtained by Kişi and Tosun [18]. In addition to that, the spinor representations of some curve pairs selected in Minkowski space were obtained [4, 10, 17]. In addition to these studies, Erişir and Kardağ found spinor formulations of involute-evolute curves, which are a special curve pair in Euclidean space $E^3$ [11] and the spinor equations of Bertrand curves were obtained in [12].

***Received :*** *28-11-2022,* ***Accepted :*** *10-04-2023*
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Many studies have been carried out in three-dimensional Euclidean space on curve theory, which covers a large part of differential geometry, which is a sub-branch of mathematics. In particular, many curve pairs have been studied using some special connections between Frenet frames taken at mutual points of any two curves such as involute-evolute, Bertrand, and Mannheim curve pairs etc. The definition of Mannheim curve pair was first introduced by Mannheim in 1878 [21]. It has been proved that the necessary and sufficient condition for any curve to be a Mannheim curve in three-dimensional Euclidean space is $\kappa = \lambda\left(\kappa^2+\tau^2\right)$ where $\lambda = constant \neq 0$, $\kappa$, and $\tau$ are curvatures of Mannheim curve [21]. Then, in 1966, some theorems about Mannheim curves were given with the help of Riccati equations in 3-dimensional Euclidean space [5]. Recently, Mannheim curve pair has been redefined by Wang and Liu [20, 31]. According to that definition given by them, if the principal normal vector of $(\alpha)$ and the binormal vector of $(\beta)$ in three dimensional Euclidean space are linearly dependent therefore, $(\alpha)$ is called the Mannheim curve and $(\beta)$ is called the Mannheim partner curve. Moreover, $(\alpha, \beta)$ is called Mannheim curve pair. Also, the necessary and sufficient condition for the curve $(\beta)$ to be the Mannheim partner curve of $(\alpha)$ is ${\tau_1}' = \frac{\kappa_1}{\lambda}\left(1+\lambda^2\tau_1^2\right)$ where $\kappa_1$ and $\tau_1$ are the curvatures of the Mannheim partner curve [20, 31]. After these studies, many studies have been done on Mannheim curves [1, 2, 3, 5, 13, 14, 15, 16, 22, 23, 24, 27].

In this study, we study a new and interesting representation (spinor representation) of the Mannheim curve pair, which is a pair of curves in Euclidean space $E^3$. For this, we first give the spinors corresponding to the Mannheim curve pair. Then, using the relations between the Frenet frames of this pair of curves, we obtain the relations between two spinors corresponding to these curves. We also give geometric interpretations and results about the spinors corresponding to these curves and the angles between these spinors. As an application, we consider the Mannheim curve as a helix and obtain the results of the spinors corresponding to these curves. Consequently, we give an example to these spinors.

## 2. Preliminaries

In this section, Mannheim curve pairs and spinors have been briefly mentioned. Now, we assume that the curves $\alpha : I \to E^3$ and $\beta : J \to E^3$ are two curves with arc-length parameter $s$ and $s_1$ in Euclidean space $E^3$, respectively. In addition, let the Frenet vector fields of the curves $(\alpha)$ and $(\beta)$ be $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ and $\{\boldsymbol{T}_1, \boldsymbol{N}_1, \boldsymbol{B}_1\}$, respectively. Therefore, we can give the following definition and theorems.

**Definition 2.1.** Assume that $\alpha : I \to E^3$ and $\beta : J \to E^3$ are two arbitrary curves in $E^3$. Therefore, if the normal vector field of $(\alpha)$ is linearly dependent with the binormal vector field of $(\beta)$, the curve $(\alpha)$ is called Mannheim curve, the curve $(\beta)$ is called Mannheim partner curve, and the curve pair $\{\alpha, \beta\}$ is called Mannheim curve pair [20, 31].

**Theorem 2.1.** *Suppose that $(\alpha, \beta)$ is Mannheim curve pair and the parameters $s$ and $s_1$ are the arc-length parameters of the curves $(\alpha)$ and $(\beta)$, respectively. In this case, the distance between mutually points of these curves is constant [20, 31].*

Therefore, from this theorem the equation

$$\alpha(s_1) = \beta(s_1) + \lambda \boldsymbol{B}_1(s_1) \tag{2.1}$$

can be written where $\lambda = constant$ [20, 31].

**Theorem 2.2.** *Assume that $\alpha : I \to E^3$ and $\beta : J \to E^3$ are Mannheim curve and Mannheim partner curve, respectively. Moreover, the angle between the tangent vector fields $\boldsymbol{T}$ and $\boldsymbol{T}_1$ is $\theta$. In this case, the relationship between Frenet vector fields of these curves is*

$$\begin{aligned} \boldsymbol{T} &= \cos\theta \boldsymbol{T}_1 + \sin\theta \boldsymbol{N}_1 \\ \boldsymbol{N} &= -\boldsymbol{B}_1 \\ \boldsymbol{B} &= -\sin\theta \boldsymbol{T}_1 + \cos\theta \boldsymbol{N}_1 \end{aligned} \tag{2.2}$$

*[24].*

**Theorem 2.3.** *Assume that the pair $(\alpha, \beta)$ is Mannheim curve pair. Let Frenet curvatures of these curves $(\alpha)$ and $(\beta)$ be $\{\kappa, \tau\}$ and $\{\kappa_1, \tau_1\}$, respectively. In this case, the equation*

$$\tau_1 = \frac{\kappa}{\lambda\tau} \tag{2.3}$$

*is provided where $\lambda = constant \neq 0$ [24].*

**Theorem 2.4.** *There are relationships between the curvatures $\{\kappa, \tau\}$ and $\{\kappa_1, \tau_1\}$ of Mannheim curve pair $(\alpha, \beta)$ as follows*

$$\begin{array}{l} i)\ \tau_1 = (-\kappa \sin\theta + \tau\cos\theta)\,\frac{ds}{ds_1}, \\ ii)\ \kappa = -\tau_1 \sin\theta \frac{ds_1}{ds}, \\ iii)\ \tau = \tau_1\cos\theta \frac{ds_1}{ds}, \end{array} \tag{2.4}$$

*where $\theta$ is the angle between $\boldsymbol{T}$ and $\boldsymbol{T}_1$, $s$ and $s_1$ are the arc-length parameter of Mannheim curve pair $(\alpha, \beta)$, respectively [24].*

**Proposition 2.1.** *Consider that $(\alpha)$ is Mannheim curve and $(\beta)$ is Mannheim partner curve. In this case, if Mannheim curve $(\alpha)$ is a helix, Mannheim partner curve $(\beta)$ is a straight line [31].*

Spinors usually construct a vector space over complex numbers with the aid of a linear group representations of the spin group. Cartan [6] was the first to express spinors on complex numbers in the geometric sense. In addition, Cartan also obtained the vector $\xi$ with two complex components corresponding to an isotropic vector $\boldsymbol{x} = (x_1,\ x_2,\ x_3)$ in $C^3$. In this case, the set of isotropic vectors in $C^3$ generate a surface in $C^2$. Therefore, we assume the surface with the parameters $\xi_1$ and $\xi_2$, then we have $x_1 = {\xi_1}^2 - {\xi_2}^2$, $x_2 = i({\xi_1}^2 + {\xi_2}^2)$, $x_3 = -2\xi\xi_1\xi_2$ and $\xi_1 = \pm\sqrt{\frac{x_1 - ix_2}{2}}$, $\xi_2 = \pm\sqrt{\frac{-x_1 - ix_2}{2}}$ [6]. Cartan said that the complex vectors mentioned above are called spinors such that

$$\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix}$$

[6]. By means of the study [6], in [7] the isotropic vector $\boldsymbol{u} + i\boldsymbol{v} \in C^3$ corresponds to the spinor $\xi = (\xi_1, \xi_2)$ where $\boldsymbol{u},\ \boldsymbol{v} \in R^3$. Therefore, with the aid of the Pauli matrices $(P_1, P_2, P_3)$, the complex symmetric matrices $\sigma$ with $2x2$ dimensional can be obtained that

$$\sigma_1 = CP_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \sigma_2 = CP_2 = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}, \sigma_3 = CP_3 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{2.5}$$

where $C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ [7, 25, 26]. Moreover, in [7] for $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in R^3$ the spinor formulations are

$$\begin{array}{l} \boldsymbol{u} + i\boldsymbol{v} = \xi^t\sigma\xi, \\ \boldsymbol{w} = -\hat{\xi}^t\sigma\xi \end{array} \tag{2.6}$$

where $\boldsymbol{u} + i\boldsymbol{v}$ is the isotropic vector in the space $C^3$, $\boldsymbol{w} \in R^3$ and the spinor mate $\hat{\xi}$ of the spinor $\xi$ is

$$\hat{\xi} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\overline{\xi} = -\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} \overline{\xi_1} \\ \overline{\xi_2} \end{pmatrix} = \begin{pmatrix} -\overline{\xi_2} \\ \overline{\xi_1} \end{pmatrix}.$$

In addition, we can say that the lengths of the vectors $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w} \in R^3$ are $\|\boldsymbol{u}\| = \|\boldsymbol{v}\| = \|\boldsymbol{w}\| = \overline{\xi}^t\xi$. At the same time, these vectors are also mutually orthogonal and $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$, $\{\boldsymbol{v}, \boldsymbol{w}, \boldsymbol{u}\}$ and $\{\boldsymbol{w}, \boldsymbol{u}, \boldsymbol{v}\}$ correspond to different spinors [7].

**Proposition 2.2.** *Assume that the arbitrary two spinors are $\xi$ and $\delta$. In this case, we have the equations*

$$\begin{array}{ll} i) & \overline{\delta^t\sigma\xi} = -\hat{\delta}^t\sigma\hat{\xi} \\ ii) & \widehat{\lambda\delta + \mu\xi} = \overline{\lambda}\hat{\delta} + \overline{\mu}\hat{\xi} \\ iii) & \hat{\hat{\xi}} = -\xi \\ iv) & \delta^t\sigma\xi = \xi^t\sigma\delta \end{array}$$

*where " $-$ " is complex conjugate and $\lambda,\ \mu \in C$ [7].*

Consider that an arbitrary curve is $\alpha : I \subseteq R \to E^3$ with arc-length parameter $s$ and also $\|\alpha'(s)\| = 1$. Moreover, we assume that the Frenet frame corresponding to the spinor $\eta$ of $(\alpha)$ is $\{\boldsymbol{N}, \boldsymbol{B}, \boldsymbol{T}\}$. In this case, with the aid of the equation (2.6) we can write the spinor equations

$$\begin{array}{l} \boldsymbol{N} + i\boldsymbol{B} = \eta^t\sigma\eta = \left({\eta_1}^2 - {\eta_2}^2,\ i({\eta_1}^2 + {\eta_2}^2),\ -2\eta_1\eta_2\right) \\ \boldsymbol{T} = -\hat{\eta}^t\sigma\eta = \left(\eta_1\overline{\eta_2} + \overline{\eta_1}\eta_2,\ i(\eta_1\overline{\eta_2} - \overline{\eta_1}\eta_2),\ \eta_1\overline{\eta_1} - \eta_2\overline{\eta_2}\right) \end{array} \tag{2.7}$$

with $\overline{\eta}^t\eta = 1$ [7].

**Theorem 2.5.** *Assume that the spinor $\eta$ corresponds to the Frenet frame $\{\boldsymbol{N},\boldsymbol{B},\boldsymbol{T}\}$ of $(\alpha)$ with arc-length parameter $s$. Therefore, the Frenet curvatures $\kappa$ and $\tau$ can be written in terms of a single spinor equation as*

$$\frac{d\eta}{ds}=\frac{1}{2}(-i\tau\eta+\kappa\hat{\eta})$$

*where $\kappa$ and $\tau$ are the curvature and torsion of $(\alpha)$, respectively [7].*

## 3. Main Theorems and Results

In this section, two spinors corresponding to the Mannheim curve pair have been considered. Then, using the relationships between the Frenet frames of the Mannheim curve pair the relationships between the spinors corresponding to this curve pair have been given. In addition, the geometric interpretations of the angles between these spinors have been made and some main theorems and corollaries have been obtained by considering the case of the Mannheim curve being helix. Consequently, an example has been given.

### *3.1. Spinor Equations of Mannheim Curve Pair*

Now, the curve $\alpha : I \to E^3$ is Mannheim curve with the arc-length parameter $s$ and the curve $\beta : J \to E^3$ is Mannheim partner curve of $(\alpha)$ with the arc-length parameter $s_1$. Moreover, the spinors $\zeta$ and $\phi$ correspond to the Frenet vectors $\{\boldsymbol{B},\boldsymbol{T},\boldsymbol{N}\}$ of the Mannheim curve $(\alpha)$ and $\{\boldsymbol{T}_1,\boldsymbol{N}_1,\boldsymbol{B}_1\}$ of the Mannheim partner curve $(\beta)$. Firstly, we consider the Mannheim curve $(\alpha)$. In this case, for the spinor $\zeta$ corresponding to the Frenet frame $\{\boldsymbol{B},\boldsymbol{T},\boldsymbol{N}\}$ of the Mannheim curve $(\alpha)$, the spinor equations

$$\begin{aligned}&\boldsymbol{B}+i\boldsymbol{T}=\zeta^t\sigma\zeta\\&\boldsymbol{N}=-\hat{\zeta}^t\sigma\zeta\end{aligned}\tag{3.1}$$

can be written where "$t$" is the transpose, $\hat{\zeta}$ is mate of the spinor $\zeta$, and the matrices $\sigma$ obtained by Pauli matrices in (2.5). Thus, the following theorem can be given for the spinor $\zeta$.

**Theorem 3.1.** *Let the curve $\alpha : I \to E^3$ be Mannheim curve with arc-length parameter $s$. In addition, the spinor $\zeta$ corresponds to $\{\boldsymbol{B},\boldsymbol{T},\boldsymbol{N}\}$ of the Mannheim curve $(\alpha)$. Therefore, the spinor equation of the curvatures $\kappa$ and $\tau$ of the Mannheim curve $(\alpha)$ is*

$$\frac{d\zeta}{ds}=\left(\frac{\tau-i\kappa}{2}\right)\hat{\zeta}.\tag{3.2}$$

*Proof.* Consider that $\zeta$ is a spinor corresponding to $\{\boldsymbol{B},\boldsymbol{T},\boldsymbol{N}\}$ of the Mannheim curve $(\alpha)$. Thus, if the derivative of the equation (3.1) according to the arc-length parameter $s$ is considered, then the equation

$$\frac{d\boldsymbol{B}}{ds}+i\frac{d\boldsymbol{T}}{ds}=\left(\frac{d\zeta}{ds}\right)^t\sigma\zeta+\zeta^t\sigma\left(\frac{d\zeta}{ds}\right)\tag{3.3}$$

can be written. On the other hand, since the spinor pair $\left\{\zeta,\hat{\zeta}\right\}$ is formed a basis for the spinors there is the equation

$$\frac{d\zeta}{ds}=f_1\zeta+f_2\hat{\zeta}\tag{3.4}$$

for the spinor $\frac{d\zeta}{ds}$ where $f_1$ and $f_2$ are two arbitrary complex functions. In this case, from the equations (3.3) and (3.4), it is obtained

$$(-\tau+i\kappa)\boldsymbol{N}=2f_1\left(\zeta^t\sigma\zeta\right)+2f_2\left(\hat{\zeta}^t\sigma\zeta\right)$$

and using the equation (3.1) we have

$$f_1=0,\quad f_2=\frac{\tau-i\kappa}{2}.$$

Also, the spinor $\frac{d\zeta}{ds}$ is written as $\frac{d\zeta}{ds}=\left(\frac{\tau-i\kappa}{2}\right)\hat{\zeta}$. □

Now, we investigate that how the spinors corresponding to the vector fields $\boldsymbol{T}$ and $\boldsymbol{B}$ in the equation (3.1) are written separately. For this, if we use the equation $\boldsymbol{T} = \operatorname{Im}(\zeta^t \sigma \zeta)$ and $\boldsymbol{B} = \operatorname{Re}(\zeta^t \sigma \zeta)$ then, we can give the following conclusion.

**Conclusion 3.1.** *Let $\zeta$ be the spinor corresponding to $\{\boldsymbol{B}, \boldsymbol{T}, \boldsymbol{N}\}$ of the Mannheim curve $(\alpha)$. In this case, the component vectors $\boldsymbol{T}$ and $\boldsymbol{B}$ of the isotropic vector $\boldsymbol{B} + i\boldsymbol{T}$ can be obtained separately that*

$$\begin{aligned} \boldsymbol{T} &= -\tfrac{i}{2}\left(\zeta^t \sigma \zeta + \hat{\zeta}^t \sigma \hat{\zeta}\right), \\ \boldsymbol{B} &= \tfrac{1}{2}\left(\zeta^t \sigma \zeta - \hat{\zeta}^t \sigma \hat{\zeta}\right). \end{aligned} \tag{3.5}$$

In addition, we can write the spinor equations of the Frenet vectors in terms of components with some algebraic calculations as

$$\begin{aligned} \boldsymbol{T} &= -\tfrac{i}{2}\left({\zeta_1}^2 - {\zeta_2}^2 + \overline{\zeta_2}^2 - \overline{\zeta_1}^2, i\left({\zeta_1}^2 + {\zeta_2}^2 + \overline{\zeta_1}^2 + \overline{\zeta_2}^2\right), -2\zeta_1\zeta_2 + 2\overline{\zeta_1\zeta_2}\right), \\ \boldsymbol{N} &= \left(\zeta_1\overline{\zeta_2} + \overline{\zeta_1}\zeta_2, i\left(\zeta_1\overline{\zeta_2} - \overline{\zeta_1}\zeta_2\right), \zeta_1\overline{\zeta_1} - \zeta_2\overline{\zeta_2}\right), \\ \boldsymbol{B} &= \tfrac{1}{2}\left({\zeta_1}^2 - {\zeta_2}^2 - \overline{\zeta_2}^2 + \overline{\zeta_1}^2, i\left({\zeta_1}^2 + {\zeta_2}^2 - \overline{\zeta_1}^2 - \overline{\zeta_2}^2\right), -2\zeta_1\zeta_2 - 2\overline{\zeta_1\zeta_2}\right). \end{aligned} \tag{3.6}$$

Now, let Mannheim partner curve of Mannheim curve $(\alpha)$ be the curve $\beta : J \to E^3$. Moreover, consider that Frenet frame of Mannheim partner curve $(\beta)$ is $\{\boldsymbol{T}_1, \boldsymbol{N}_1, \boldsymbol{B}_1\}$ and the spinor $\phi$ corresponds to this frame of the Mannheim partner curve $(\beta)$. Therefore, similar to the equation (3.1) it can be obtained

$$\begin{aligned} \boldsymbol{T}_1 - i\boldsymbol{N}_1 &= \phi^t \sigma \phi \\ \boldsymbol{B}_1 &= -\hat{\phi}^t \sigma \phi \end{aligned} \tag{3.7}$$

where $\boldsymbol{T}_1 - i\boldsymbol{N}_1$ is isotropic vector in $C^3$. Thus, the spinor equations of the Frenet frame $\{\boldsymbol{T}_1, \boldsymbol{N}_1, \boldsymbol{B}_1\}$ in terms of spinor $\phi$ can be written by

$$\begin{aligned} \boldsymbol{T}_1 &= \tfrac{1}{2}\left(\phi^t \sigma \phi - \hat{\phi}^t \sigma \hat{\phi}\right), \\ \boldsymbol{N}_1 &= \tfrac{i}{2}\left(\phi^t \sigma \phi + \hat{\phi}^t \sigma \hat{\phi}\right), \\ \boldsymbol{B}_1 &= -\hat{\phi}^t \sigma \phi \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{T}_1 &= \tfrac{1}{2}\left(\phi_1^2 - \phi_2^2 + \overline{\phi_1}^2 - \overline{\phi_2}^2, i\left(\phi_1^2 + \phi_2^2 - \overline{\phi_1}^2 - \overline{\phi_2}^2\right), -2\left(\phi_1\phi_2 + \overline{\phi_1\phi_2}\right)\right), \\ \boldsymbol{N}_1 &= -\tfrac{i}{2}\left(\phi_1^2 - \phi_2^2 + \overline{\phi_2}^2 - \overline{\phi_1}^2, i\left(\phi_1^2 + \phi_2^2 + \overline{\phi_1}^2 + \overline{\phi_2}^2\right), -2\left(\phi_1\phi_2 - \overline{\phi_1\phi_2}\right)\right), \\ \boldsymbol{B}_1 &= \left(\phi_1\overline{\phi_2} + \overline{\phi_1}\phi_2, i\left(\phi_1\overline{\phi_2} - \overline{\phi_1}\phi_2\right), \phi_1\overline{\phi_1} - \phi_2\overline{\phi_2}\right). \end{aligned} \tag{3.8}$$

Therefore, the following theorem about Mannheim partner curve $(\beta)$ can be given.

**Theorem 3.2.** *Consider that the spinor $\phi$ corresponds to the Frenet frame $\{\boldsymbol{T}_1, \boldsymbol{N}_1, \boldsymbol{B}_1\}$ of Mannheim partner curve $(\beta)$. Therefore, the curvatures $\kappa_1$ and $\tau_1$ of $(\beta)$ can be written in terms of the single spinor equation*

$$\frac{d\phi}{ds_1} = \frac{i}{2}\left(\kappa_1 \phi + \tau_1 \hat{\phi}\right) \tag{3.9}$$

*where $s_1$ is the arc-length parameter of Mannheim partner curve $(\beta)$ and the spinor $\hat{\phi}$ is the mate of the spinor $\phi$.*

*Proof.* Let the curve $(\beta)$ with the arc-length parameter $s_1$ be Mannheim partner curve of $(\alpha)$ and $\phi$ be the spinor corresponding to the Mannheim partner curve $(\beta)$. Therefore, if the derivative of the first equation in (3.7) is taken making necessary arrangements, it can be obtained

$$\frac{d\boldsymbol{T}_1}{ds_1} - i\frac{d\boldsymbol{N}_1}{ds_1} = \left(\frac{d\phi}{ds_1}\right)^t \sigma \phi + \phi^t \sigma \left(\frac{d\phi}{ds_1}\right).$$

Here, since the spinor pair $\left\{\phi, \hat{\phi}\right\}$ represents the Frenet frame $\{\boldsymbol{T}_1, \boldsymbol{N}_1, \boldsymbol{B}_1\}$ the pair $\left\{\phi, \hat{\phi}\right\}$ forms a basis for spinors. Thus, we can write $\frac{d\phi}{ds_1} = g_1 \phi + g_2 \hat{\phi}$ where $g_1$ and $g_2$ are two arbitrary complex functions. Therefore, if we use this equation in the last equation, we have

$$i\kappa_1 \left(\boldsymbol{T}_1 - i\boldsymbol{N}_1\right) - i\tau_1 \boldsymbol{B}_1 = 2g_1\left(\phi^t \sigma \phi\right) + 2g_2\left(\hat{\phi}^t \sigma \phi\right)$$

and

$$i\kappa_1(\boldsymbol{T}_1 - i\boldsymbol{N}_1) - i\tau_1\boldsymbol{B}_1 = 2g_1(\boldsymbol{T}_1 - i\boldsymbol{N}_1) - 2g_2\boldsymbol{B}_1.$$

Consequently, we obtain

$$g_1 = \frac{i\kappa_1}{2}, \quad g_2 = \frac{i\tau_1}{2},$$

and $\frac{d\phi}{ds_1} = \frac{i}{2}\left(\kappa_1\phi + \tau_1\hat{\phi}\right)$. □

**Theorem 3.3.** *Assume that the curves $\alpha : \mathrm{I} \to E^3$ and $\beta : \mathrm{J} \to E^3$ are Mannheim curve and Mannheim partner curve, respectively, and $\zeta$ and $\phi$ are the spinors corresponding to the curves $(\alpha)$ and $(\beta)$, respectively. In this case, the relationship between spinors corresponding to Mannheim curve pair $(\alpha, \beta)$ is*

$$\zeta = \pm e^{i\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}\phi. \tag{3.10}$$

*Proof.* Let $(\alpha, \beta)$ be Mannheim curve pair and $(\zeta, \phi)$ be the spinor pair corresponding to Frenet frames $\{\boldsymbol{B}, \boldsymbol{T}, \boldsymbol{N}\}$ and $\{\boldsymbol{T}_1, \boldsymbol{N}_1, \boldsymbol{B}_1\}$ of the curves $(\alpha, \beta)$, respectively. In this case, using the equation (2.2) the relationship between the isotropic vectors including Frenet frames of the Mannheim curve pair can be written

$$\begin{aligned}\boldsymbol{B} + i\boldsymbol{T} &= -\sin\theta\boldsymbol{T}_1 + \cos\theta\boldsymbol{N}_1 + i(\cos\theta\boldsymbol{T}_1 + \sin\theta\boldsymbol{N}_1)\\ &= (-\sin\theta + i\cos\theta)\boldsymbol{T}_1 + (\cos\theta + i\sin\theta)\boldsymbol{N}_1\\ &= e^{i\theta}(i\boldsymbol{T}_1 + \boldsymbol{N}_1)\\ &= ie^{i\theta}(\boldsymbol{T}_1 - i\boldsymbol{N}_1).\end{aligned}$$

Here, if we use the spinor equation in the equations (3.1) and (3.7) then, we obtain

$$\zeta^t\sigma\zeta = ie^{i\theta}\left(\phi^t\sigma\phi\right).$$

In this case, from the equations (3.6) and (3.8) we have

$$\begin{aligned}\zeta_1^2 &= ie^{i\theta}\phi_1^2,\\ \zeta_2^2 &= ie^{i\theta}\phi_2^2\end{aligned}$$

and consequently,

$$\zeta = \pm e^{i\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}\phi.$$

□

In this case, as a result of the Theorem 3.3 the following geometric interpretation can be expressed.

**Conclusion 3.2.** *Consider that the spinors $\zeta$ and $\phi$ represent the Mannheim curve pair $(\alpha)$ and $(\beta)$, respectively. Therefore, if the angle between tangent vectors of $(\alpha)$ and $(\beta)$ is $\theta$, then the angle between spinors $\zeta$ and $\phi$ is $\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$.*

In addition, the equation (3.10) giving the relationship between spinors can be written in another way as follows.

**Conclusion 3.3.** *There is the relationship*

$$\zeta = \pm\frac{\sqrt{2}}{2}(1+i)\,e^{i\frac{\theta}{2}}\phi$$

*between the spinors $\zeta$ and $\phi$ corresponding to Mannheim curve pair $(\alpha, \beta)$.*

Moreover, with the aid of Proposition 2.2 and Theorem 3.3 the following conclusion can be expressed.

**Conclusion 3.4.** *The relationship between the spinor mates $\hat{\zeta}$ and $\hat{\phi}$ of spinors $\zeta$ and $\phi$ corresponding to Mannheim curve pair $(\alpha, \beta)$, respectively, is*

$$\hat{\zeta} = \pm e^{-i\left(\frac{\pi}{4}+\frac{\theta}{2}\right)}\hat{\phi}.$$

Therefore, we give the following corollary using the result in Conclusion 3.4.

**Corollary 3.1.** *Assume that the spinors $\zeta$ and $\phi$ represent the Mannheim curve pair $(\alpha, \beta)$, respectively. In this case, the spinor $\zeta$ returns to the spinor $\phi$, while the spinor $\widehat{\zeta}$ makes a **reverse rotation** to the spinor $\widehat{\phi}$ with the same angle $\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$.*

*3.2. Spinor Equations If Mannheim Curve is Helix*

We know that the curves $(\alpha)$ and $(\beta)$ in $E^3$ are Mannheim curve and Mannheim partner curve, and the spinor pair $(\zeta,\phi)$ corresponds to this Mannheim curve pair $(\alpha,\beta)$, respectively. Now, especially we assume that Mannheim curve $(\alpha)$ is helix. Therefore, the following theorems and conclusions can be given.

**Theorem 3.4.** *Consider that the spinor pair $(\zeta,\phi)$ corresponds to Mannheim curve pair $(\alpha,\beta)$. If Mannheim curve $(\alpha)$ is helix, then the spinor $\frac{d\zeta}{ds}$ is*

$$\frac{d\zeta}{ds}=\frac{\kappa}{2\sin\gamma}e^{-i\gamma}\hat{\zeta} \tag{3.11}$$

*where $\langle \boldsymbol{T},\boldsymbol{U}\rangle=\cos\gamma$ and $\boldsymbol{U}$ is axis of helix $(\alpha)$.*

*Proof.* Let Mannheim curve $(\alpha)$ be helix. In this case, for the curvatures of Mannheim curve $(\alpha)$, $\frac{\tau}{\kappa}=\cot\gamma=constant$ can be written. Then, using the equation (3.2) we obtain

$$\frac{d\zeta}{ds}=\frac{\kappa}{2}\left(\frac{\cos\alpha}{\sin\alpha}-i\right)\hat{\zeta}=\frac{\kappa}{2\sin\alpha}\left(\cos\alpha-i\sin\alpha\right)\hat{\zeta}$$

and

$$\frac{d\zeta}{ds}=\frac{\kappa}{2\sin\alpha}e^{-i\alpha}\hat{\zeta}.$$

□

**Theorem 3.5.** *Let $\zeta$ and $\phi$ be the spinors corresponding to Mannheim curve pair $(\alpha)$ and $(\beta)$. In this case, Mannheim curve $(\alpha)$ is helix, then the necessary and sufficient condition is that the constant vector $\boldsymbol{U}$ is written as*

$$\boldsymbol{U}=-\frac{i}{2}\left(e^{i\gamma}\zeta^t\sigma\zeta+e^{-i\gamma}\hat{\zeta}^t\sigma\hat{\zeta}\right) \tag{3.12}$$

*where $\gamma=\arccos\left(\langle \boldsymbol{T},\boldsymbol{U}\rangle\right)$ and the vector $\boldsymbol{U}$ is axis of the helix.*

*Proof.* $(\Rightarrow)$ : Consider that the spinor $\zeta$ corresponds to Mannheim curve $(\alpha)$ especially selected as helix. Therefore, there is the constant vector $\boldsymbol{U}=\cos\gamma\boldsymbol{T}+\sin\gamma\boldsymbol{B}$ (axis of the helix) where $\langle \boldsymbol{T},\boldsymbol{U}\rangle=\cos\gamma=constant$. If we use the equations (3.1) and (3.5), we have

$$\boldsymbol{U}=-\frac{i}{2}\cos\gamma\left(\zeta^t\sigma\zeta+\hat{\zeta}^t\sigma\hat{\zeta}\right)+\frac{1}{2}\sin\gamma\left(\zeta^t\sigma\zeta-\hat{\zeta}^t\sigma\hat{\zeta}\right)$$

and

$$\boldsymbol{U}=-\frac{i}{2}\left[e^{i\gamma}\zeta^t\sigma\zeta+e^{-i\gamma}\hat{\zeta}^t\sigma\hat{\zeta}\right]$$

where $\boldsymbol{U}$ is the constant vector. Namely, if we derivate the vector $\boldsymbol{U}$ with respect to $s$, then we obtain

$$\boldsymbol{U}'=-\frac{i}{2}\left[e^{i\gamma}\left(\frac{d\zeta}{ds}^t\sigma\zeta+\zeta^t\sigma\frac{d\zeta}{ds}\right)+e^{-i\gamma}\left(\frac{d\hat{\zeta}}{ds}^t\sigma\hat{\zeta}+\hat{\zeta}^t\sigma\frac{d\hat{\zeta}}{ds}\right)\right]$$

where $\gamma=constant$. Therefore, using the equation (3.11) we have

$$\boldsymbol{U}'=-\frac{i}{2}\left[\frac{\kappa}{2\sin\gamma}\left(\hat{\zeta}^t\sigma\zeta+\zeta^t\sigma\hat{\zeta}\right)-\frac{\kappa}{2\sin\gamma}\left(\zeta^t\sigma\hat{\zeta}+\hat{\zeta}^t\sigma\zeta\right)\right].$$

Thus, $\boldsymbol{U}'=\boldsymbol{0}$ and we get $\mathbf{U}=constant$.

$(\Leftarrow)$ : Let $\zeta$ be the spinor corresponding to Mannheim curve $(\alpha)$. Moreover, we assume a constant vector

$$\boldsymbol{U}=-\frac{i}{2}\left[e^{i\gamma}\zeta^t\sigma\zeta+e^{-i\gamma}\hat{\zeta}^t\sigma\hat{\zeta}\right].$$

In this case, from the equation (3.5) we obtain

$$\langle \boldsymbol{T},\boldsymbol{U}\rangle=-\frac{1}{4}\begin{bmatrix}\left({\zeta_1}^2-{\zeta_2}^2+\overline{\zeta_2}^2-\overline{\zeta_1}^2\right)\left(e^{i\gamma}\left({\zeta_1}^2-{\zeta_2}^2\right)+e^{-i\gamma}\left(\overline{\zeta_2}^2-\overline{\zeta_1}^2\right)\right)\\-\left({\zeta_1}^2+{\zeta_2}^2+\overline{\zeta_1}^2+\overline{\zeta_2}^2\right)\left(e^{i\gamma}\left({\zeta_1}^2+{\zeta_2}^2\right)+e^{-i\gamma}\left(\overline{\zeta_1}^2+\overline{\zeta_2}^2\right)\right)\\+\left(-2\zeta_1\zeta_2+2\overline{\zeta_1\zeta_2}\right)\left(-2e^{i\gamma}\zeta_1\zeta_2+2e^{-i\gamma}\overline{\zeta_1\zeta_2}\right)\end{bmatrix}$$

and

$$\langle \boldsymbol{T}, \boldsymbol{U} \rangle = \frac{1}{2}\left[\left(\zeta_1\overline{\zeta_1}+\zeta_2\overline{\zeta_2}\right)^2\left(e^{i\gamma}+e^{-i\gamma}\right)\right] = \cos\gamma\left(\zeta_1\overline{\zeta_1}+\zeta_2\overline{\zeta_2}\right).$$

Since the Frenet vectors $\{\boldsymbol{B}, \boldsymbol{T}, \boldsymbol{N}\}$ are unit vectors, $\zeta_1\overline{\zeta_1}+\zeta_2\overline{\zeta_2}=1$ and we have $\langle \boldsymbol{T}, \boldsymbol{U} \rangle = \cos\gamma$. On the other hand, the vector $\boldsymbol{U}$ is constant and $\boldsymbol{U}' = \boldsymbol{0}$. Thus,

$$\boldsymbol{U}' = \boldsymbol{0} = -\frac{i}{2}\left[\begin{array}{l} i\gamma'\left(e^{i\gamma}\zeta^t\sigma\zeta - e^{-i\gamma}\hat{\zeta}^t\sigma\hat{\zeta}\right) + e^{i\gamma}\left(\frac{\tau-i\kappa}{2}\right)\left(\zeta^t\sigma\zeta+\zeta^t\sigma\hat{\zeta}\right) \\ -e^{-i\gamma}\left(\frac{\tau+i\kappa}{2}\right)\left(\zeta^t\sigma\zeta+\zeta^t\sigma\hat{\zeta}\right) \end{array}\right]$$

and

$$\boldsymbol{0} = -\frac{i}{2}\left[i\gamma'\left(e^{i\gamma}\zeta^t\sigma\zeta - e^{-i\gamma}\hat{\zeta}^t\sigma\hat{\zeta}\right) + 2\hat{\zeta}^t\sigma\zeta\left(\tau\sin\theta-\kappa\cos\theta\right)\right]$$

where $\theta$ is the angle between the tangent vector fields $\boldsymbol{T}$ and $\boldsymbol{T}_1$. If we use the equations (3.1) and (3.5), we have

$$i\gamma'\sin\theta\boldsymbol{T} - i\gamma'\cos\theta\boldsymbol{B} + \left(\tau\sin\theta-\kappa\cos\theta\right)\boldsymbol{N} = \boldsymbol{0}$$

consequently, $\gamma' = 0$. Therefore, we can say that $\gamma = constant$, $\langle \boldsymbol{T}, \boldsymbol{U} \rangle = \cos\gamma = constant$, and Mannheim curve $(\alpha)$ is helix. □

Now, we rewrite the axis of the helix $\boldsymbol{U}$ giving in the Theorem 3.5 in terms of the spinor $\phi$ corresponding to Mannheim partner curve $(\beta)$. Namely, since Mannheim curve $(\alpha)$ is helix, Mannheim partner curve $(\beta)$ is straight line and the curvature $\kappa_1 = 0$. In this case, we say that the tangent vector field $\mathbf{T}_1$ of Mannheim partner curve $(\beta)$ is constant, and $\langle \boldsymbol{T}, \boldsymbol{T}_1 \rangle = \cos\theta = constant$. Therefore, the following conclusion can be given.

**Conclusion 3.5.** *Let $\phi$ be the spinor corresponding to the Mannheim partner curve $(\beta)$ of Mannheim curve $(\alpha)$. In this case, if the vector $\boldsymbol{U}$ is the axis of Mannheim curve specially selected as helix, then this axis can be written in terms of the spinor $\phi$ as*

$$\boldsymbol{U} = \frac{1}{2}\left(\phi^t\sigma\phi - \hat{\phi}^t\sigma\hat{\phi}\right).$$

In addition to that, using the equation (3.9) we have the following conclusion.

**Conclusion 3.6.** *Let the curve pair $(\alpha, \beta)$ be Mannheim curve pair, the spinors corresponding to this Mannheim curve pair be $(\zeta, \phi)$ and Mannheim curve $(\alpha)$ be helix. In this case, the spinor $\frac{d\phi}{ds_1}$ can be obtained as*

$$\frac{d\phi}{ds_1} = \frac{i\tau_1}{2}\hat{\phi}$$

*where $s_1$ is the arc-length parameter and $\tau_1$ is torsion of Mannheim partner curve $(\beta)$, respectively.*

Now, we give the relationship between the spinors $d\zeta$ and $d\phi$ with the following conclusion.

**Conclusion 3.7.** *Assume that the spinor pair $(\zeta, \phi)$ corresponds to Mannheim curve pair $(\alpha, \beta)$ and the Mannheim curve $(\alpha)$ is helix. Therefore, the relationship between the derivative of the spinors $(\zeta, \phi)$ is*

$$d\zeta = \pm e^{i\left(\frac{\theta}{2}-\frac{3\pi}{4}\right)}d\phi$$

**Conclusion 3.8.** *Consider that the spinor pair $(\zeta, \phi)$ corresponds to Mannheim curve pair $(\alpha, \beta)$. Moreover, the Mannheim curve $(\alpha)$ is considered as helix. In this case, the angle between the spinors $d\zeta$ and $d\phi$ is $\left(\frac{\theta}{2}-\frac{3\pi}{4}\right)$.*

***Example:***
Consider that the curve $\alpha : I \to E^3$ with arc-length parameter $s$ is Mannheim curve

$$\alpha(s) = \left(3\cos\frac{s}{5}, 3\sin\frac{s}{5}, \frac{4}{5}s\right).$$

In this case, for the Frenet vectors $\{\boldsymbol{T}, \boldsymbol{N}, \boldsymbol{B}\}$ of the curve $(\alpha)$, we calculate as

$$\begin{aligned} \boldsymbol{T}(s) &= \left(-\tfrac{3}{5}\sin\tfrac{s}{5}, \tfrac{3}{5}\cos\tfrac{s}{5}, \tfrac{4}{5}\right), \\ \boldsymbol{N}(s) &= \left(-\cos\tfrac{s}{5}, -\sin\tfrac{s}{5}, 0\right), \\ \boldsymbol{B}(s) &= \left(\tfrac{4}{5}\sin\tfrac{s}{5}, -\tfrac{4}{5}\cos\tfrac{s}{5}, \tfrac{3}{5}\right). \end{aligned}$$

Consider that the spinor $\zeta$ corresponds to the Frenet frame $\{\boldsymbol{B}, \boldsymbol{T}, \boldsymbol{N}\}$. Therefore, the spinor $\zeta$ can be written as

$$\zeta = \pm\sqrt{\frac{3+4i}{10}}\begin{pmatrix} e^{-i\frac{s}{10}} \\ e^{i\frac{s}{10}} \end{pmatrix}$$

where $s$ is the arc-length parameter of Mannheim curve $(\alpha)$. In addition to that, the spinor $\frac{d\zeta}{ds}$ is

$$\frac{d\zeta}{ds} = \frac{(4-3i)}{50}\hat{\zeta}.$$

Now, we regard that Mannheim partner curve of Mannheim curve $(\alpha)$ is $(\beta)$ where $s$ is an arbitrary parameter for Mannheim partner curve $\beta$. In this case, from the equation

$$\beta(s_1) = \alpha(s) + \lambda \boldsymbol{N}(s),$$

the curve $(\beta)$ can be written as

$$\beta(s) = \left(\cos\frac{s}{5}, \sin\frac{s}{5}, \frac{4}{5}s\right).$$

Therefore, we obtain that Frenet vectors of Mannheim partner curve $\beta$ are

$$\begin{aligned} \boldsymbol{T}_1 &= \tfrac{1}{\sqrt{17}}\left(-\sin\tfrac{s}{5}, \cos\tfrac{s}{5}, 4\right), \\ \boldsymbol{N}_1 &= \tfrac{1}{\sqrt{17}}\left(4\sin\tfrac{s}{5}, -4\cos\tfrac{s}{5}, 1\right), \\ \boldsymbol{B}_1 &= \left(\cos\tfrac{s}{5}, \sin\tfrac{s}{5}, 0\right), \end{aligned}$$

and Frenet curvatures are

$$\kappa_1 = \frac{1}{17}, \quad \tau_1 = \frac{4}{17}.$$

Now, let $\phi$ be the spinor corresponding to Frenet frame $\{\boldsymbol{B}_1, \boldsymbol{T}_1, \boldsymbol{N}_1\}$. In this case, we get

$$\phi = \pm\sqrt{\frac{4-i}{2\sqrt{17}}}\begin{pmatrix} e^{-i\frac{s}{10}} \\ e^{i\frac{s}{10}} \end{pmatrix}$$

and

$$\frac{d\phi}{ds} = \frac{1}{10\sqrt{17}}\left(\phi + 4\hat{\phi}\right).$$

## 4. Conclusion

Spinors, expressed by Waerden in 1929, are used in quantum mechanics, relativity theory and especially electron spin applications. However, spinors are very difficult to introduce in quantum mechanics. Because even when spin 1/2 is taken, it is quite difficult to explain the fundamental aspects of spinors in some matters, such as how spin will affect spinors. On the other hand, although it is thought that spinors can be used without referring to the theory of relativity, they appear indirectly when it comes to Lorentz group discussions. From the physicists' point of view, spinors, like tensors, are multi-linear transformations, allowing for a more general consideration of the concept of invariance under rotation and Lorentz boosts. Spins in quantum theory are expressed with the help of Pauli matrices, where Pauli matrices are also a representation of Clifford spin algebra. Regardless of a particular application, the most important feature of spinors is their behavior under rotations. That is, when a vector or tensor object rotates by a certain angle, a spinor corresponding to this object rotates half that angle. Therefore, for this object to return to its original position, the spinor must rotate two full turns. In this paper, we have approached spinors from a geometric perspective and considered spinors as two-dimensional vectors in the complex plane. From this point of view, we have taken spinors corresponding to the Frenet frame of the Mannheim curve pair and have made geometric interpretations of the angle between these spinors. Consequently, this study is thought to make an important contribution to the geometric interpretation of spinors.

## Acknowledgements

We would like to thank to the editor and the anonymous reviewers for their helpful comments and suggestions.

## Funding

There is no funding for this work.

## Availability of data and materials

Not applicable.

## Competing interests

The authors declare that they have no competing interests.

## Author's contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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