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Algorithms to Compute The Demonic Transitive Closure of Fuzzy Relations Using Demonic Operators

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Abstract. In this paper, we investigate the demonic transitive closure of fuzzy relations in the sense of demonic operators. To address this, we focus on the demonic order refinement of fuzzy relations, which has a special treatment associated with the membership function of fuzzy relations. We look closely at the transitive closure of fuzzy relations with the use of angelic operators (the usual operators $\cup$, $\cap$ and $\circ$) and reform it by applying the demonic operators ($\tilde{\sqcup}$, $\tilde{\sqcap}$ and $\tilde{\Box}$). In order to carry out this task, we adopt existing algorithms and reformulate them in the means of demonic operators.

Keywords: Transitive Closure · Demonic Transitive Closure · Demonic Fuzzy Operators

1 Introduction

Fuzzy set theory was initiated by L. A. Zadeh in 1965 [20]. Fuzziness can be found in many areas of daily life, such as in medicine [17], economics [4], and others.
Transitivity property is a fundamental concept in equivalence and order relations which ensure a relationship between two elements by an intermediate element, that is if aRb and bRc then, aRc. In computer science the concept of transitive closure appears in field of the relational database in terms of capacity of the database to answer queries by means of transitive closure help [11].
The demonic calculus of relations has the advantage that the demonic operations are defined on top of the conventional relation algebraic operations.
In this paper, we consider needed definitions for introducing the concept of *refinement ordering* for fuzzy relations [8] in Section 2. Then, in Section 3 we investigate the concept of demonic transitive closure in case of fuzzy relations. In Section 4, we propose algorithms to compute the demonic transitive closure of fuzzy relations. Finally, we give our conclusion in Section 5.

2 Demonic Fuzzy Order and Fuzzy Demonic Operators

2.1 Demonic Fuzzy Order Refinement

Definition 21. *We say that a fuzzy relation $\tilde{Q}$ fuzzy refines a fuzzy relation $\tilde{R}$, denoted by $\tilde{Q} \tilde{\sqsubseteq} \tilde{R}$, iff*

$$\vee_{y\in B}\{\mu_{\tilde{R}}(x,y)\} \subseteq \vee_{y\in B}\{\mu_{\tilde{Q}}(x,y)\} \text{ and } \wedge\{\mu_{\tilde{Q}}(x,y), \vee_{y\in B}\{\mu_{\tilde{R}}(x,y)\}\} \subseteq \mu_{\tilde{R}}(x,y) \tag{1}$$

Where $\mu_{\tilde{R}}$ and $\mu_{\tilde{Q}}$ are respectively the membership functions of $\tilde{R}$ and $\tilde{Q}$.

Theorem 21. *The fuzzy relation $\tilde{\sqsubseteq}$ is a partial order.*

Definition 22. *The fuzzy demonic composition of relations $\tilde{Q}$ and $\tilde{R}$ is ($\tilde{Q} \,\tilde{\Box}\, \tilde{R}$), and its membership function is given by:*

$$\mu_{\tilde{Q} \,\tilde{\Box}\, \tilde{R}}(x,y) = \wedge[\vee_{y\in B}\{\wedge\{\mu_{\tilde{Q}}(x,y), \mu_{\tilde{R}}(y,z)\}\}, 1-\vee_{y\in B}\{\wedge\{\mu_{\tilde{Q}}(x,y), 1-\vee_{y\in B}(\mu_{\tilde{R}}(x,y))\}\}] \tag{2}$$

This definition is equivalent to definition in [7], that is

$$\tilde{Q} \,\Box\, \tilde{R} = \tilde{Q}\tilde{R} \cap \tilde{Q}\tilde{R}\tilde{L} \tag{3}$$

3 Demonic Transitive Closure of Fuzzy Relations

3.1 $\tilde{T}$-transitive Closure of Demonic Fuzzy Relations

Definition 31. *A binary operator $\tilde{T} : [0,1] \times [0,1] \longrightarrow [0,1]$ is* t-norm *if it satisfies the following axioms:*

(a) $\tilde{T}(1,x) = x$
(b) $\tilde{T}(x,y) = \tilde{T}(y,x)$
(c) $\tilde{T}(x,\tilde{T}(y,z)) = \tilde{T}(\tilde{T}(x,y),z)$

(d) $If x \leq x'$ *and* $y \leq y'$, *then* $\tilde{T}(x,y) \leq \tilde{T}(x',y')$

Definition 32. *Let* $E = \{e_1, ..., e_n\}$ *be a finite set. A fuzzy relation* $\tilde{R}$ *on* E *is a map* $\tilde{R} : E \times E \longrightarrow [0,1]$. *The relation degree value for elements* e_i and e_j in E is called e_{ij}, That is e_{ij} =(e$_i$, e_j).

Definition 33. *Let* $\tilde{T}$ *be a triangular norm. A fuzzy relation* $\tilde{R} : E \times E \longrightarrow [0,1]$ *is*

$$\tilde{T} - transitivite\ iff$$

(e_{ik},e_{kj}*)* $\leq e_{ij}$, $\forall$ *i, j, k* $\leq n$.

Definition 1. *A fuzzy similarity is a reflexive, symmetric, and min-transitive fuzzy relation.*

Definition 34. *Let* $\tilde{R} \subseteq X \times X$. *The* $\tilde{T}$*-transitive closure of* $\tilde{R}$ *is the relation* $\tilde{R}^{\tilde{T}}$ *and satisfies the following:*

(a) $\tilde{R}^{\tilde{T}}$ *is transitive,*
(b) $\tilde{R} \subseteq \tilde{R}^{\tilde{T}}$,
(c) $(\tilde{R}^{\tilde{T}})^{\tilde{T}} = \tilde{R}^{\tilde{T}}$,
(d) If $\tilde{S}$ *is any transitive relation such that* $\tilde{R} \tilde{\sqsubseteq} \tilde{S}$, *then* $\tilde{R}^{\tilde{T}} \tilde{\sqsubseteq} \tilde{S}$.

Theorem 31. *Consider an arbitrary universe* E *and an arbitrary t-norm* $\tilde{T}$. *Then any fuzzy relation* $\tilde{R}$ *on* E *has a demonic* $\tilde{T}$*-transitive Closure.*

4 Algorithms to Compute The Demonic Transitive Closure of Fuzzy Relations

4.1 The Matrix Construction Algorithm to Compute Demonic *T*-transitive Closure

Algorithm 1 The Matrix Construction Algorithm to Compute $\tilde{T}$-transitive Closure

1: **while** $\tilde{R}^* \neq \tilde{R}$ **do**$\tilde{R}^* = \tilde{R} \,\tilde{\sqcup}\, (\tilde{R} \,\tilde{\Box}_{\tilde{T}} \tilde{R})$
2: $\quad \tilde{R}^{\tilde{T}} := \tilde{R}^*$

Example 41. *Let* $\tilde{Z}$ *be a fuzzy relation. We will compute* $\tilde{Z}^{\tilde{T}}$ *as follow:*

$$\tilde{Z} = \begin{pmatrix} 0.3 & 0.5 & 0.6 & 0.1 \\ 0.7 & 0.9 & 0.1 & 0.3 \\ 0.8 & 0.5 & 0.5 & 1 \\ 0.1 & 0.5 & 0.2 & 0.7 \end{pmatrix} \quad Then, \quad \tilde{Z}^{\tilde{T}} = \begin{pmatrix} 0.6 & 0.5 & 0.6 & 0.6 \\ 0.6 & 0.6 & 0.6 & 0.6 \\ 0.6 & 0.5 & 0.6 & 0.6 \\ 0.5 & 0.5 & 0.5 & 0.6 \end{pmatrix}$$

4.2 Floyd-Warshall Method to Compute The Demonic $\tilde{T}$-transitive Closure

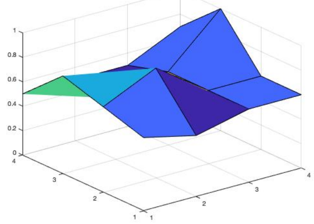

Fig. 1: The angelic $\tilde{T}$-transitive closure $\tilde{Z}^{\tilde{T}}$ in example 41

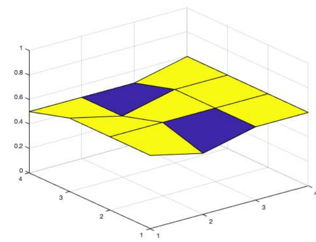

Fig. 2: The demonic $\tilde{T}$-transitive closure $\tilde{Z}^{\tilde{T}}$ in example 41

Algorithm 2 Floyd-Warshall Algorithm to Compute the $\tilde{T}$-transitive Closure

proximity $\tilde{R}$, $E = \{e_1, ..., e_n\}$, Dimension n, t-norm $\tilde{T}$

for $(i := 1; i < n; i++)$ **do**
  for $(j := 1; j < n; j++)$ **do**
    for $(k := 1; k < n; k++)$ **do**
      $e_{jk} = \min(\max(e_{jk}, \tilde{T}(e_{ji}, e_{ik}), \min_k(\max_k\ e_{jk}, \max_k\ \tilde{T}(e_{ji}, e_{ik}))))$

4.3 Fast Algorithm to Compute the Demonic $\tilde{T}$-transitive Closure of a Fuzzy Proximity

In their paper [10], the authors present a faster algorithm for computing the transitive closure of a given fuzzy proximity $\tilde{R}$. We will adopt the algorithm in [10] with a reformation that respects the fuzzy demonic operators. In Algorithm 3, we will consider N to be a set of nodes (where node is an element of $\mathcal{P}(E)$).

Algorithm 3 Fast Algorithm to Compute the Demonic $\tilde{T}$-transitive Closure

Input : a proximity $\tilde{R}$
Output : the transitive closure $\tilde{A} = [a_{ij}]$

Create a set of nodes N initially with a set of singletons $N_i = \{e_i\}$ for each element e_i in E.

Set $a_{ii} = 1$ for all i from 1 to n.

n-1 times (While N is not the universe E) compute:

$$m(N_i, N_j) = \min_{i \in N_i, j \in N_j} \{ \max_{i \in N_i, j \in N_j} e_{ij}, \max_{i \in N_i, j \in N_j} \{ \min_{i \in N_i, j \in N_j} e_{ij} \} \}$$

for all pair of nodes $N \times N$ with $i \neq j$. Record (i, j) where $m(N_i, N_j)$ is maximal. Assign $a_{rs} = a_{sr} = \max_{i \in N_i, j \in N_j} e_{ij}$ for all $r \in N_i$ and $s \in N_j$ =0

Example 42. *Let $\tilde{R}$ be a fuzzy proximity on $E = \{e_1, ..., e_6\}$ given by the following matrix:*

$$\tilde{R} = \begin{pmatrix} 1 & 0.9 & 0.5 & 0.8 & 0.1 & 0.6 \\ 0.9 & 1 & 0.3 & 0.7 & 0.2 & 0.4 \\ 0.5 & 0.3 & 1 & 0.9 & 0.9 & 0.3 \\ 0.8 & 0.7 & 0.9 & 1 & 0.2 & 0.2 \\ 0.1 & 0.2 & 0.9 & 0.2 & 1 & 0.7 \\ 0.6 & 0.4 & 0.3 & 0.2 & 0.7 & 1 \end{pmatrix}$$

The first two loops of part 3 of the algorithm record $m(N_1, N_2) = 0.9$ and $m(N_3, N_4) = 0.9$. In the third loop a maximal value record $m(N_3 \cup N_4, N_5) = 0.9$. For loop 4, the maximal value record $m(N_1 \cup N_2, N_3 \cup N_4 \cup N_5) = 0.8$. The last loop record $m(N_1 \cup N_2 \cup N_3 \cup N_4 \cup N_5, N_6) = 0.7$. Note that there is are only five loops (n-1) because $|E| = 6$. The transitive closure is

$$\tilde{A} = \begin{pmatrix} 1 & 0.9 & 0.8 & 0.8 & 0.8 & 0.7 \\ 0.9 & 1 & 0.8 & 0.8 & 0.8 & 0.7 \\ 0.8 & 0.8 & 1 & 0.9 & 0.9 & 0.7 \\ 0.8 & 0.8 & 0.9 & 1 & 0.9 & 0.7 \\ 0.8 & 0.8 & 0.9 & 0.9 & 1 & 0.7 \\ 0.7 & 0.7 & 0.7 & 0.7 & 0.7 & 1 \end{pmatrix}$$

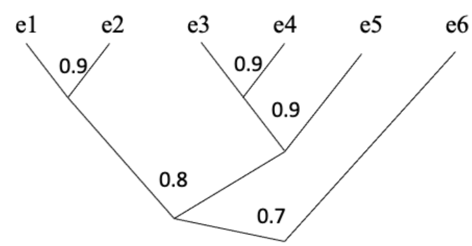

Fig. 3: Binary weighted tree for the transitive closure $\tilde{A}$ of $\tilde{R}$

Theorem 41. *The matrix $\tilde{A} = [a_{ij}]$ in algorithm 3 is fuzzy similarity.*

5 Conclusion

In this paper, we have considered needed definitions for introducing the concept of *refinement ordering* for fuzzy relations in Section 2. Then, in Section 3 and 4 we have defined the demonic transitive closure of fuzzy relations with the demonic operators and we have adopted existing algorithms and we reformulated them in the means of demonic operators.

The given algorithms in this research have their limitations due to the demonic operators and how they are defined. Many suggested approaches can be addressed for the case of transitive closure with respect to the demonic operators. First, we need more study on the demonic fuzzy operators and investigate their properties. Second, we need to investigate more algorithms for demonic transitive closure that carry on no assumptions on the properties of the fuzzy relation, in particular, in the case of reflexive fuzzy relation we have seen that the angelic transitive closure equals the demonic transitive closure of such relation.

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