



Warped product pointwise bi-slant submanifolds of locally product Riemannian manifolds

Prince Majeed*, Mehraj Ahmad Lone

Department of Mathematics, National Institute of Technology Srinagar, 190006, Jammu and Kashmir, India

Abstract

In this paper we introduce the concept of pointwise bi-slant submanifolds of locally product Riemannian manifolds and studied warped product pointwise bi-slant submanifolds of locally product Riemannian manifolds. We obtain some characterization results for warped products pointwise bi-slant submanifolds. Also, we provide some non-trivial examples of such warped product submanifolds.

Mathematics Subject Classification (2020). 53C15, 53C25, 53C40, 53C42

Keywords. locally product Riemannian manifold, pointwise bi-slant submanifolds, Warped products

1. Introduction

In [7], Chen introduced the notion of slant submanifolds. It includes totally real as well as holomorphic submanifolds. Numerous geometer groups continue to study and conduct research on this idea of submanifolds. Recently, the related literature of slant submanifolds has been compiled in the form of two books by Chen, Shahid and Solamy (see [15]). Since the introduction of slant submanifolds, many generalizations and extensions of slant submanifolds have been introduced, like: semi-slant, pointwise slant, hemi-slant, pointwise hemislant and many more. The related literature of these kind of generalizations can be found in (see, [14, 17, 21, 23, 24]). A more generic class of submanifolds in the form of bi-slant submanifolds was introduced by Cabrerizo and Carriazo [6]. This class of submanifolds acts as a natural generalization of CR, semi-slant, slant, hemi-slant submanifolds [21, 23, 25]. In connection to this generic notion of submanifolds, some of recent studies can be found in [20]. Further the extended notion of pointwise bi-slant submanifolds of Kaehler manifolds can be found in [16].

Bishop and O'Neill in 1960s introduced the concept of warped product manifolds. These manifolds find their applications both in physics as well as in mathematics. Since then the study of warped product submanifolds has been investigated by many geometers (see, [1, 10–12]). In particular, Chen started looking these warped products as submanifolds of different kinds of manifolds (see, [8, 9]). In this connection, in Kaehlerian settings, he

*Corresponding Author.

Email addresses: prince_05phd19@nitsri.net (P. Majeed), mehrajlone@nitsri.ac.in (M. A. Lone)

Received: 28.11.2022; Accepted: 22.02.2024

proved besides CR- products the non-existence of warped products of the form $N^\perp \times_f N^T$, where N^\perp, N^T is a totally real and holomorphic submanifold, respectively [32]. Now from the past two decades this area of research is an active area of research among many of the geometry groups. For the overall development of the subject we refer the reader to see Chen's book on it [13].

Now while importing the survey of warped products to slant cases, Sahin in [27] proved the non-existence of semi-slant warped products in any Kaehler manifold. Then in [29] he extended the study to pointwise semi-slant warped products of Kaehlerian manifolds. Sahin [28] and Atcken [2] investigated warped product semi-slant submanifolds of locally product Riemannian manifolds. They proved there is no warped product semi-slant submanifold of the form $M_T \times_f M_\theta$ of a locally product Riemannian manifold \bar{M} such that M_T and M_θ are invariant and proper slant submanifolds of \bar{M} , respectively. Moreover they provided non-trivial examples and proved a characterization theorem for warped product semi-slant submanifolds of the form $M_\theta \times_f M_T$.

The main motivation for our paper is a recent study of Uddin, Alghamdi and Solamy [31], in which they have studied the geometry of warped product pointwise semi-slant submanifolds of locally product Riemannian manifolds. In this paper, we try to generalize the notion to bi-slant warped products of locally product Riemannian manifolds. We proved several results on pointwise bi-slant submanifolds of locally product Riemannian manifolds, in addition, we proved some characterization results for pointwise bi-slant submanifolds of locally product Riemannian manifolds. Later, we also provide some non-trivial examples of such submanifolds.

2. Preliminaries

Let \bar{M} be an m -dimensional differential manifold with a tensor field F of type (1,1) such that $F^2 = I$ and $F \neq \pm I$. Then we say that \bar{M} is an almost product manifold with almost product structure F . If an almost product manifold \bar{M} has a Riemannian metric g such that

$$g(FX, FY) = g(X, Y), \quad (2.1)$$

for any $X, Y \in \Gamma(T\bar{M})$, then \bar{M} is called an almost product Riemannian manifold [33], where $\Gamma(T\bar{M})$ denotes the set of all vector fields of \bar{M} . Let $\bar{\nabla}$ denotes the Levi-Civita connection on \bar{M} with respect to the Riemannian metric g . if $(\bar{\nabla}_X F)Y = 0$, for any vector $X, Y \in \Gamma(T\bar{M})$, the \bar{M} is called a locally product Riemannian manifold [22].

Let M be a Riemannian manifold isometrically immersed in \bar{M} and we denote by the symbol g the Riemannian metric induced on M . Let $\Gamma(TM)$ denote the Lie algebra of vector fields in M and $\Gamma(T^\perp M)$, the set of all vector fields normal to M . If ∇ be the induced Levi-Civita connection on M , the Gauss and Weingarten formulas are respectively given by

$$\bar{\nabla}_X Y = \nabla_X Y + \sigma(X, Y), \quad (2.2)$$

and

$$\bar{\nabla}_X N = -A_N X + \nabla_X^\perp N, \quad (2.3)$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, where ∇^\perp is the normal connection on $T^\perp M$ and A the shape operator. The shape operator and the second fundamental form of M are related by

$$g(A_N X, Y) = g(\sigma(X, Y), N), \quad (2.4)$$

for any $X, Y \in \Gamma(TM)$ and $N \in \Gamma(T^\perp M)$, and g denotes the induced metric on M as well as the metric on \bar{M} .

For a tangent vector field X and a normal vector field N of M , we can write

$$FX = TX + \omega X, \quad FN = BN + CN, \quad (2.5)$$

where TX and ωX (respectively, BN and CN) are the tangential and normal components of FX (respectively, of FN).

Moreover, from (2.1) and (2.5), we have

$$g(TX, Y) = g(X, TY), \quad (2.6)$$

for any $X, Y \in \Gamma(TM)$.

We can now specify the following classes of submanifolds of locally product Riemannian manifolds:

(1) A submanifold M of a locally product Riemannian manifold \bar{M} is said to be slant (see [3, 7, 26]), if for each non-zero vector X tangent to M , the angle $\theta(X)$ between FX and $T_p M$ is a constant, i.e., it does not depend on the choice of $p \in M$ and $X \in T_p M$.

(2) A submanifold M of a locally product Riemannian manifold \bar{M} is called semi-invariant submanifold (see [4, 30]) of \bar{M} if there exists a differentiable distribution $D : p \rightarrow D_p \subset T_p M$ such that D is invariant with respect to F and the complementary distribution D^\perp is anti-invariant with respect to F .

(3) A submanifold M of a locally product Riemannian manifold \bar{M} is called semi-slant (see [22, 25]), if it is endowed with two orthogonal distributions D and D^θ , where D is invariant with respect to F and D^θ is slant, i.e., $\theta(X)$ is the angle between FX and D_p^θ is constant for any $X \in D_p^\theta$ and $p \in M$.

Definition 2.1. A submanifold M of a locally product Riemannian manifold \bar{M} is called pointwise slant [18], if at each point $p \in M$, the Wirtinger angle $\theta(X)$ between FX and $T_p M$ is independent of the choice of the non-zero vector $X \in T_p M$. In this case, the Wirtinger angle gives rise to a real valued function $\theta : TM - \{0\} \rightarrow \mathbb{R}$ which is called Wirtinger function or slant function of the pointwise slant function.

We note that a pointwise slant submanifold of a locally product Riemannian manifold is called slant, in the sense of [3, 26], if its Wirtinger function θ is globally constant. We also note that every slant submanifold is a pointwise slant submanifold.

From Chen's result (Lemma 2.1) of [14], we can easily show that M is a pointwise slant submanifold of a locally product Riemannian manifold \bar{M} if and only if

$$T^2 = (\cos^2 \theta)I, \quad (2.7)$$

for some real-valued function θ defined on M , where I denotes the identity transformation of the tangent bundle TM of M . The following relations are the consequences of (2.7) as

$$g(TX, TY) = (\cos^2 \theta)g(X, Y), \quad g(\omega X, \omega Y) = (\sin^2 \theta)g(X, Y). \quad (2.8)$$

Also, for a pointwise bi-slant submanifold of locally product Riemannian manifold, (2.5) and (2.7) yields

$$B\omega X = (\sin^2 \theta)X \quad C\omega X = -\omega TX. \quad (2.9)$$

3. Pointwise bi-slant submanifolds

In this section, we define and study pointwise bi-slant submanifolds of a locally product Riemannian manifold.

Definition 3.1. Let \bar{M} be a locally product Riemannian manifold and M a real submanifold of \bar{M} . Then we say M is a bi-slant submanifold if there exists a pair of orthogonal distributions D_1 and D_2 of M , at a point $p \in M$ such that

- (a) $TM = D_1 \oplus D_2$;
- (b) $JD_1 \perp D_2$ and $JD_2 \perp D_1$;
- (c) The distributions D_1, D_2 are pointwise slant with slant functions θ_1, θ_2 , respectively.

The pair $\{\theta_1, \theta_2\}$ of slant functions is called the bi-slant function. A pointwise bi-slant submanifold M is called proper if its bi-slant function satisfies $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$ and both θ_1, θ_2 are not constant on M .

Note that (2.5) and condition (b) in the Definition 3.1 imply that

$$T(D_i) \subset D_i, \quad i = 1, 2. \quad (3.1)$$

Given a pointwise bi-slant submanifold M of locally product Riemannian manifold \bar{M} , for any $X \in \Gamma(TM)$, we put

$$X = P_1X + P_2X \quad (3.2)$$

where P_i is the projection from $\Gamma(TM)$ onto D_i . Clearly, P_iX is the components of X in $D_i, i = 1, 2$. In particular, if $X \in D_i$, we have $X = P_iX$. If we put $T_i = P_i \circ T$, then we can find (3.2) that

$$FX = T_1X + T_2X + \omega X, \quad (3.3)$$

for any $X \in \Gamma(TM)$.

From now onwards, we assume the ambient manifold \bar{M} is locally product Riemannian manifold and M is pointwise bi-slant submanifold in \bar{M} .

Now we give the following useful lemma for later use.

Lemma 3.2. *Let M be a pointwise bi-slant submanifold of a locally product Riemannian manifold \bar{M} with pointwise slant distributions D_1 and D_2 with distinct slant functions θ_1 and θ_2 , respectively. Then*

(i) *For $X, Y \in D_1$ and $Z \in D_2$, we have*

$$\begin{aligned} (\sin^2 \theta_2 - \sin^2 \theta_1)g(\nabla_X Y, Z) &= g\{(\sigma(X, Z), \omega T_1 Y) + (\sigma(X, T_2 Z), \omega Y)\} \\ &\quad + g\{(\sigma(X, Y), \omega T_2 Z) + (\sigma(X, T_1 Y), \omega Z)\}. \end{aligned} \quad (3.4)$$

(ii) *For $Z, W \in D_2$ and $X \in D_1$, we have*

$$\begin{aligned} (\sin^2 \theta_1 - \sin^2 \theta_2)g(\nabla_Z W, X) &= g\{(\sigma(X, Z), \omega T_2 W) + (\sigma(Z, T_1 X), \omega W)\} \\ &\quad + g\{(\sigma(Z, W), \omega T_1 X) + (\sigma(Z, T_2 W), \omega X)\}. \end{aligned} \quad (3.5)$$

Proof. For $X, Y \in D_1$ and $Z \in D_2$, we have

$$g(\nabla_X Y, Z) = g(\bar{\nabla}_X Y, Z) = g(F\bar{\nabla}_X Y, FZ).$$

Using the locally product structure and (2.5), we have

$$\begin{aligned} g(\nabla_X Y, Z) &= g(\bar{\nabla}_X T_1 Y, FZ) + g(\bar{\nabla}_X \omega Y, T_2 Z) + g(\bar{\nabla}_X \omega Y, \omega Z). \\ &= g(\bar{\nabla}_X T_1^2 Y, Z) + g(\bar{\nabla}_X \omega T_1 Y, Z) - g(A_{\omega Y} X, T_2 Z) \\ &\quad - g(\bar{\nabla}_X \omega Z, \omega Y). \end{aligned}$$

Again using (2.5) and (2.7), we obtain

$$\begin{aligned} g(\nabla_X Y, Z) &= \cos^2 \theta_1 g(\bar{\nabla}_X Y, Z) - \sin 2(\theta_1) X(\theta_1) g(Y, Z) - g(A_{\omega T_1 Y} X, Z) \\ &\quad - g(A_{\omega Y} X, T_2 Z) - g(\bar{\nabla}_X \omega Z, FY) + g(\bar{\nabla}_X \omega Z, T_1 Y). \end{aligned}$$

By using the orthogonality of two distributions and the symmetry of shape operator, the above equation reduces to

$$\begin{aligned} \sin^2 \theta_1 g(\nabla_X Y, Z) &= -g(\sigma(X, Z), \omega T_1 Y) - g(\sigma(X, T_2 Z), \omega Y) \\ &\quad - g(\bar{\nabla}_X B \omega Z, Y) - g(\bar{\nabla}_X C \omega Z, Y) - g(A_{\omega Z} X, T_1 Y). \end{aligned}$$

Thus, from (2.9), we obtain

$$\begin{aligned} \sin^2 \theta_1 g(\nabla_X Y, Z) &= -g(\sigma(X, Z), \omega T_1 Y) - g(\sigma(X, T_2 Z), \omega Y) \\ &\quad - \sin^2 \theta_2 g(\bar{\nabla}_X Z, Y) - \sin 2(\theta_2) X(\theta_2) g(Y, Z) \\ &\quad + g(\bar{\nabla}_X \omega T_2 Z, Y) - g(A_{\omega Z} X, T_1 Y). \end{aligned}$$

Using (2.3) and the orthogonality of vector fields, we have

$$\begin{aligned} \sin^2 \theta_1 g(\nabla_X Y, Z) &= -g(\sigma(X, Z), \omega T_1 Y) - g(\sigma(X, T_2 Z), \omega Y) \\ &\quad + \sin^2 \theta_2 g(\bar{\nabla}_X Y, Z) - g(A_{\omega T_2 Z} X, Y) \\ &\quad - g(A_{\omega Z} T_1 Y, X). \end{aligned}$$

Now, part (i) of the lemma follows from the above equation by using (2.4). In the similar fashion, we can prove part (ii). \square

The following corollary is the immediate consequence of the Lemma 1(i)

Corollary 3.3. *Let M be a pointwise semi-slant submanifold of a locally product Riemannian manifold \bar{M} . Then,*

$$\sin^2 \theta g(\nabla_X Y, Z) = g(\sigma(X, Y), \omega T Z) + g(\sigma(X, F Y), \omega Z),$$

for any $X, Y \in D_1$ and $Z \in D_2$.

Proof. If we put $\theta_1 = 0$ and $\theta_2 = \theta$, a slant function, then the submanifold M of locally product Riemannian manifold \bar{M} becomes pointwise semi-slant submanifold. In this case, the first two terms in the right hand side of (3.4) vanish identically. Thus the relation (3.4) reduces to

$$\sin^2 \theta g(\nabla_X Y, Z) = g(\sigma(X, Y), \omega T Z) + g(\sigma(X, F Y), \omega Z).$$

The same result has been proved in [31]. \square

4. Warped product pointwise bi-slant submanifolds of locally product Riemannian manifold

Let (M_1, g_1) and (M_2, g_2) be two Riemannian manifolds and $f > 0$, be a positive differentiable function on M_1 . Consider the product manifold $M_1 \times M_2$ with its canonical projections $\pi : M_1 \times M_2 \rightarrow M_1$ and $\rho : M_1 \times M_2 \rightarrow M_2$. The warped product $M = M_1 \times_f M_2$ is the product manifold $M_1 \times M_2$ equipped with the Riemannian metric g such that

$$g(X, Y) = g_1(\pi_*(X), \pi_*(Y)) + (f \circ \pi)^2 g_2(\rho_*(X), \rho_*(Y)),$$

for any tangent vector $X, Y \in TM$, where $*$ is the symbol for the tangent maps. It was proved in [5] that for any $X \in TM_1$ and $Z \in TM_2$, the following holds

$$\nabla_X Z = \nabla_Z X = (X \ln f) Z, \quad (4.1)$$

where ∇ denotes the Levi-Civita connection of g on M . A warped product manifold $M = M_1 \times_f M_2$ is said to be trivial if the warping function f is constant. If $M = M_1 \times_f M_2$ is a warped product manifold then M_1 is totally geodesic and M_2 is a totally umbilical (see [5, 9]).

Lemma 4.1. *Let $M_T \times_f M_\theta$ be a warped product pointwise bi-slant submanifold of a locally product Riemannian manifold \bar{M} such that M_T and M_θ are pointwise slant submanifolds with slant functions θ_1 and θ_2 , respectively of \bar{M} . Then we have the following*

$$g(\sigma(X, W), \omega T_2 Z) + g(\sigma(X, T_2 Z), \omega W) = -(\sin 2\theta_2)X(\theta_2)g(Z, W), \quad (4.2)$$

for any $X \in TM_T$ and $Z, W \in TM_\theta$.

Proof. For any $X \in TM_T$ and $Z, W \in TM_\theta$, we have

$$g(\bar{\nabla}_X Z, W) = g(\nabla_X Z, W) = X(\ln f)g(Z, W). \quad (4.3)$$

On the other hand, we also have

$$g(\bar{\nabla}_X Z, W) = g(F\bar{\nabla}_X Z, FW) = g(\bar{\nabla}_X FZ, FW).$$

Now for any $X \in TM_T$ and $Z, W \in TM_\theta$. Using (2.5), we obtain

$$g(\bar{\nabla}_X Z, W) = g(\bar{\nabla}_X T_2 Z, T_2 W) + g(\bar{\nabla}_X T_2 Z, \omega W) + g(\bar{\nabla}_X \omega Z, FW).$$

Then from (2.1), (2.2), (4.1) and the locally product Riemannian structure, we derive

$$\begin{aligned} g(\bar{\nabla}_X Z, W) &= \cos^2 \theta_2 X(\ln f)g(Z, W) + g(\sigma(X, T_2 Z), \omega W) + g(\bar{\nabla}_X F\omega Z, W) \\ &= \cos^2 \theta_2 X(\ln f)g(Z, W) + g(\sigma(X, T_2 Z), \omega W) + g(\bar{\nabla}_X B\omega Z, W) \\ &\quad + g(\bar{\nabla}_X C\omega Z, W). \end{aligned}$$

Using (2.9), we find

$$\begin{aligned} g(\bar{\nabla}_X Z, W) &= \cos^2 \theta_2 X(\ln f)g(Z, W) + g(\sigma(X, T_2 Z), \omega W) \\ &\quad + \sin^2 \theta_2 g(\bar{\nabla}_X Z, W) + \sin 2\theta_2 X(\theta_2)g(Z, W) \\ &\quad - g(\bar{\nabla}_X \omega T_2 Z, W). \end{aligned} \quad (4.4)$$

Thus the lemma follows from (4.3) and (4.4) by using (2.3) and (4.1). \square

Lemma 4.2. *Let $M_T \times_f M_\theta$ be a warped product pointwise bi-slant submanifold of a locally product Riemannian manifold \bar{M} such that M_T and M_θ are pointwise slant submanifolds with slant functions θ_1 and θ_2 , respectively of \bar{M} . Then we have the following*

$$g(\sigma(X, Z), \omega W) + g(\sigma(X, W), \omega Z) = -2(\tan \theta_2)X(\theta_2)g(T_2 Z, W), \quad (4.5)$$

for any $X \in TM_T$ and $Z, W \in TM_\theta$.

Proof. The proof of this lemma follows by Interchanging Z by $T_2 Z$ for any $Z \in TM_2$ in (4.2) and then by using (2.7). \square

Lemma 4.3. *Let $M_T \times_f M_\theta$ be a warped product pointwise bi-slant submanifold of a locally product Riemannian manifold \bar{M} such that M_T and M_θ are pointwise slant submanifolds with slant functions θ_1 and θ_2 , respectively of \bar{M} . Then*

$$(i) \quad g(\sigma(X, Z), \omega W) = g(\sigma(X, W), \omega Z), \quad (4.6)$$

$$(ii) \quad g(\sigma(X, Z), \omega Y) = -g(\sigma(X, Y), \omega Z), \quad (4.7)$$

for any $X \in TM_T$ and $Z, W \in TM_\theta$.

Proof. For any $X \in TM_T$ and $Z, W \in TM_\theta$, we have

$$\begin{aligned} g(\sigma(X, Z), \omega W) &= g(\bar{\nabla}_Z X, \omega W) \\ &= g(\bar{\nabla}_Z X, FW) - g(\bar{\nabla}_Z X, T_2 W). \end{aligned}$$

Using (2.1), (2.5) and (4.1), we obtain

$$g(\sigma(X, Z), \omega W) = g(\bar{\nabla}_Z T_1 X, W) + g(\bar{\nabla}_Z \omega X, W) - X(\ln f)g(Z, T_2 W).$$

On simplification and using (2.3), (2.4) and (4.1), we derive

$$g(\sigma(X, Z), \omega W) = T_1 X(\ln f)g(Z, W) - g(\sigma(Z, W), \omega X) - X(\ln f)g(Z, T_2 W). \quad (4.8)$$

Then from polarization, we get

$$g(\sigma(X, W), \omega Z) = T_1 X(\ln f)g(Z, W) - g(\sigma(Z, W), \omega X) - X(\ln f)g(T_2 Z, W). \quad (4.9)$$

Subtracting (4.9) from (4.8) and using (2.6), we obtain

$$g(\sigma(X, Z), \omega W) - g(\sigma(X, W), \omega Z) = 0.$$

Hence, the proof follows from the above relation.

For part (ii), the proof follows same as part (i). \square

Theorem 4.4. *Let $M = M_T \times_f M_\theta$ be a warped product pointwise bi-slant submanifold of a locally product Riemannian manifold \bar{M} such that M_T and M_θ are pointwise slant submanifolds with distinct slant functions θ_1 and θ_2 , respectively of \bar{M} . Then we have*

$$\begin{aligned} g(A_{\omega T_1 X} W + A_{\omega X} T_2 W, Z) + g(A_{\omega T_2 W} X + A_{\omega W} T_1 X, Z) \\ = (\sin^2 \theta_2 - \sin^2 \theta_1) X(\ln f)g(Z, W), \end{aligned} \quad (4.10)$$

for any $X, Y \in TM_T$ and $Z, W \in TM_\theta$.

Proof. For any $X, Y \in TM_T$ and $Z, W \in TM_\theta$, we have

$$g(\bar{\nabla}_Z X, W) = g(\nabla_Z X, W) = X(\ln f)g(Z, W). \quad (4.11)$$

On the other hand, for $X, Y \in TM_T$ and $Z, W \in TM_\theta$, we have

$$g(\bar{\nabla}_Z X, W) = g(F\bar{\nabla}_Z X, FW) = g(\bar{\nabla}_Z FX, FW).$$

Therefore, by using (2.5), we get

$$g(\bar{\nabla}_Z X, W) = g(\bar{\nabla}_Z T_1 X, FW) + g(\bar{\nabla}_Z \omega X, T_2 W) + g(\bar{\nabla}_Z \omega X, \omega W).$$

Using (2.1), (2.3) and definition of locally product Riemannian manifold, we obtain

$$g(\bar{\nabla}_Z X, W) = g(\bar{\nabla}_Z FT_1 X, W) - g(A_{\omega X} Z, T_2 W) - g(\bar{\nabla}_Z \omega W, \omega X).$$

From (2.5) and symmetry of shape operator, we derive

$$\begin{aligned} g(\bar{\nabla}_Z X, W) &= g(\bar{\nabla}_Z T_1^2 X, W) + g(\bar{\nabla}_Z \omega T_1 X, W) - g(A_{\omega X} T_2 W, Z) \\ &\quad - g(F\bar{\nabla}_Z \omega W, X) + g(\bar{\nabla}_Z \omega W, T_1 X) \\ &= \cos^2 \theta_1 g(\bar{\nabla}_Z X, W) - \sin 2\theta_1 Z(\theta_1)g(X, W) - g(A_{\omega T_1 X} Z, W) \\ &\quad - g(A_{\omega X} T_2 W, Z) - g(\bar{\nabla}_Z F\omega W, X) - g(A_{\omega W} Z, T_1 X). \end{aligned}$$

Using (2.2), (2.5), (4.1), (4.11) and the orthogonality of vector fields and symmetry of shape operator, we get

$$\begin{aligned} \sin^2 \theta_1 X(\ln f)g(Z, W) &= -g(A_{\omega T_1 X} W + A_{\omega X} T_2 W, Z) - g(\bar{\nabla}_Z B\omega W, X) \\ &\quad - g(\bar{\nabla}_Z C\omega W, X) - g(A_{\omega W} T_1 X, Z). \end{aligned}$$

Using (2.9), we arrive at

$$\begin{aligned} \sin^2 \theta_1 X(\ln f)g(Z, W) &= -g(A_{\omega T_1 X} W + A_{\omega X} T_2 W, Z) - \sin^2 \theta_2 g(\bar{\nabla}_Z W, X) \\ &\quad - \sin 2\theta_2 Z(\theta_2)g(X, W) + g(\bar{\nabla}_Z \omega T_2 W, X) \\ &\quad - g(A_{\omega W} T_1 X, Z). \end{aligned}$$

Further, using orthogonality of vector fields and the relation (2.2), (2.3) and (4.1), we obtain

$$\begin{aligned} \sin^2 \theta_1 X(\ln f)g(Z, W) &= -g(A_{\omega T_1 X} W + A_{\omega X} T_2 W, Z) \\ &\quad + \sin^2 \theta_2 X(\ln f)g(Z, W) - g(A_{\omega T_2 W} Z, X) \\ &\quad - g(A_{\omega W} T_1 X, Z). \end{aligned}$$

Again using the symmetry of shape operator, we obtain (4.10) from the above relation. Hence the proof is complete. \square

5. Characterization for warped product pointwise bi-slant submanifolds of locally product Riemannian manifolds

In this section, we will prove the characterization for warped product pointwise bi-slant submanifolds of locally product Riemannian manifolds. For this, we need the following well known theorem of Hiepko's.

Theorem 5.1 ([19]). *Let D_1 and D_2 be two orthogonal distribution on a Riemannian manifold M . Suppose that D_1 and D_2 both are involutive such that D_1 is totally geodesic foliation and D_2 is a spherical foliation. Then M is locally isometric to a non-trivial warped product $M_1 \times_f M_2$, where, M_1 and M_2 are integral manifolds of D_1 and D_2 , respectively.*

The following result provides a characterization for warped product pointwise bi-slant submanifolds of locally product Riemannian manifolds.

Theorem 5.2. *Let M be a proper pointwise bi-slant submanifold of a locally product Riemannian manifold \bar{M} with pointwise slant distributions D_1 and D_2 . Then M is locally a warped product pointwise bi-slant submanifold of the form $M_T \times_f M_\theta$, where M_T and M_θ are pointwise slant submanifolds with distinct slant functions θ_1 and θ_2 , respectively of \bar{M} if and only if the shape operator of M satisfies*

$$A_{\omega T_1 X} Z + A_{\omega X} T_2 Z + A_{\omega T_2 Z} X + A_{\omega Z} T_1 X = (\sin^2 \theta_2 - \sin^2 \theta_1) X(\mu) Z, \quad (5.1)$$

for $X \in D_1$, $Z \in D_2$ and for a smooth function μ on M satisfying $W(\mu) = 0$ for any $W \in D_2$.

Proof. Let $M = M_T \times_f M_\theta$ be a warped product pointwise bi-slant submanifold of a locally product Riemannian manifold \bar{M} . Then from Lemma 4.3(ii), we have

$$g(A_{\omega Y} Z + A_{\omega Z} Y, X) = 0 \quad (5.2)$$

for any $X, Y \in TM_1$ and $Z \in TM_2$. Interchanging Y by $T_1 Y$ in (5.2), we obtain

$$g(A_{\omega T_1 Y} Z + A_{\omega Z} T_1 Y, X) = 0. \quad (5.3)$$

Again interchanging Z by $T_2 Z$ in (5.2), we obtain

$$g(A_{\omega Y} T_2 Z + A_{\omega T_2 Z} Y, X) = 0. \quad (5.4)$$

Adding equations (5.3) and (5.4), we get

$$g(A_{\omega T_1 Y} Z + A_{\omega Z} T_1 Y + A_{\omega Y} T_2 Z + A_{\omega T_2 Z} Y, X) = 0. \quad (5.5)$$

Then (5.1) follows from (4.10) by using the above fact.

Conversely, if M be a pointwise bi-slant submanifold of a locally product Riemannian manifold with pointwise slant distributions D_1 and D_2 such that (5.1) holds, then from Lemma 3.2(i), we have

$$\begin{aligned} (\sin^2 \theta_2 - \sin^2 \theta_1)g(\nabla_X Y, Z) &= g(A_{\omega T_1 Y} Z + A_{\omega Z} T_1 Y \\ &\quad + A_{\omega Y} T_2 Z + A_{\omega T_2 Z} Y, X) \end{aligned}$$

for any $X, Y \in D_1$ and $Z \in D_2$. Using the above condition (5.1), we have

$$g(\nabla_X Y, Z) = X(\mu)g(X, Z) = 0$$

which indicates that the leaves of the distributions are totally geodesic in M .

On the other hand, from Lemma 3.2(ii), we have

$$\begin{aligned} (\sin^2 \theta_1 - \sin^2 \theta_2)g(\nabla_Z W, X) &= g(A_{\omega T_2 W} X + A_{\omega W} T_1 X \\ &\quad + A_{\omega T_1 X} W + A_{\omega X} T_2 W, Z). \end{aligned}$$

From the hypothesis of the theorem i.e., (5.1), we get

$$g(\nabla_Z W, X) = -X(\mu)g(Z, W). \quad (5.6)$$

By polarization, we arrive at

$$g(\nabla_W Z, X) = -X(\mu)g(Z, W). \quad (5.7)$$

On subtracting (5.7) from (5.6) and by the definition of Lie bracket, we obtain $g([Z, W], X) = 0$, which depicts that the distribution D_2 is integrable. If we consider a leaf M_2 of D_2 and the second fundamental form σ_2 of M_2 in M , then from (5.6), we have

$$g(\sigma_2(Z, W), X) = g(\nabla_Z W, X) = -X(\mu)g(Z, W).$$

Now, by the definition of the gradient we have $\sigma_2(Z, W) = -\bar{\nabla}_\mu g(Z, W)$, such that $\bar{\nabla}_\mu$ is the gradient of μ . The above relations shows that the leaf of M_2 is totally Umbilical in M with the mean curvature vector $H_2 = -\bar{\nabla}_\mu$. Since $W(\mu) = 0$ for any $W \in D_2$, which clearly shows that the mean curvature is parallel. Thus, the spherical condition is satisfied. Then by Hiepko's Theorem M is locally a warped product pointwise bi-slant submanifold. Hence the proof is complete. \square

The following immediate consequences of the above theorem are given below:

1. In Theorem 5.2, if $\theta_1 = 0$ and $\theta_2 = \theta$, a slant function, then the submanifold M of locally product Riemannian manifold \bar{M} becomes a pointwise semi-slant submanifold which has been studied in [31]. In this case, the first two terms in the left hand side of (5.1) vanish identically. Thus, the relation (5.1) is true for pointwise semi-slant warped product and it reduces to

$$A_{\omega T Z} X + A_{\omega Z} F X = (\sin^2 \theta) X(\mu) Z,$$

for $X \in D_1$ and $Z \in D_2$, where D_1 and D_2 are complex and proper pointwise slant distributions of M . The same has been proved in [31].

2. In Theorem 5.2, if we consider $\theta_1 = \theta$ a constant slant angle and $\theta_2 = \frac{\pi}{2}$, then it is a case of hemi-slant warped products. In this case, the second and third term in the left hand side of (5.1) vanish identically. Thus the relation (5.1) is true for hemi-slant warped products and it reduces to

$$A_{\omega T X} Z + A_{F Z} T X = (\cos^2 \theta) X(\mu) Z,$$

for $X \in D_\theta$ and $Z \in D^\perp$, where D_θ and D^\perp are proper slant and totally real distributions.

3. In Theorem 5.2, if $\theta_1 = 0$ and $\theta_2 = \frac{\pi}{2}$, then it is a case of CR-warped product. In this case all the terms in the left hand side of (5.1) vanish identically. Thus the relation (5.1) is true for CR-warped products and it will be

$$A_{F Z} F X = X(\mu) Z,$$

for $X \in D$ and $Z \in D^\perp$, where D and D^\perp are complex and totally real distributions of M .

6. Some examples on warped product pointwise bi-slant submanifolds of locally product Riemannian manifold

Example 6.1. Let \mathbb{R}^4 be the Euclidean space with the cartesian coordinates given by (x_1, x_2, y_1, y_2) and the almost product structure

$$F\left(\frac{\partial}{\partial x_i}\right) = \frac{\partial}{\partial x_i}, \quad F\left(\frac{\partial}{\partial y_j}\right) = -\frac{\partial}{\partial y_j}, \quad 1 \leq i, j \leq 2.$$

A submanifold M of \mathbb{R}^4 defined by

$$\chi(u, v, w) = (wu \cos v, wu \sin v, w \cos v, w \sin v).$$

It is easy to see that the tangent bundle TM of M is spanned by the following vectors

$$\begin{aligned} v_1 &= w \cos v \frac{\partial}{\partial x_1} + w \sin v \frac{\partial}{\partial x_2}, \\ v_2 &= -wu \sin v \frac{\partial}{\partial x_1} + wu \cos v \frac{\partial}{\partial x_2} - w \sin v \frac{\partial}{\partial y_1} + w \cos v \frac{\partial}{\partial y_2}, \\ v_3 &= u \cos v \frac{\partial}{\partial x_1} + u \sin v \frac{\partial}{\partial x_2} + \cos v \frac{\partial}{\partial y_1} + \sin v \frac{\partial}{\partial y_2}. \end{aligned}$$

Then, clearly we obtain

$$\begin{aligned} Fv_1 &= w \cos v \frac{\partial}{\partial x_1} + w \sin v \frac{\partial}{\partial x_2}, \\ Fv_2 &= -wu \sin v \frac{\partial}{\partial x_1} + wu \cos v \frac{\partial}{\partial x_2} + w \sin v \frac{\partial}{\partial y_1} - w \cos v \frac{\partial}{\partial y_2}, \\ Fv_3 &= u \cos v \frac{\partial}{\partial x_1} + u \sin v \frac{\partial}{\partial x_2} - \cos v \frac{\partial}{\partial y_1} - \sin v \frac{\partial}{\partial y_2}. \end{aligned}$$

Then, we find that $D_1 = \text{span}\{v_1, v_3\}$ is a proper pointwise slant distribution with slant angle $\theta_1 = \cos^{-1}\left(\frac{u}{\sqrt{1+u^2}}\right)$ and $D_2 = \text{span}\{v_2\}$ is again a proper pointwise slant distribution with slant angle $\theta_2 = \cos^{-1}\left(\frac{u^2-1}{u^2+1}\right)$. Thus, M is a proper pointwise bi-slant submanifold of \mathbb{R}^4 .

It is easy to verify that both the distributions D_1 and D_2 are integrable. If we denote the integrable manifolds of D_1 and D_2 by M_T and M_θ , respectively. Then the metric tensor g of product manifold M is given by

$$g = g_{M_T} + w^2(1 + u^2)g_{M_\theta},$$

where,

$$g_{M_T} = w^2 du^2 + (1 + u^2)dw^2 \quad \text{and} \quad g_{M_\theta} = dv^2.$$

Hence, M is a proper non-trivial warped product pointwise bi-slant submanifold of \mathbb{R}^4 with warping function $f = \sqrt{w^2(1 + u^2)}$ and whose bi-slant angles $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$.

Example 6.2. Let $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ be a locally product Riemannian manifold with cartesian coordinates $(x_1, x_2, x_3, y_1, y_2, y_3)$. Consider a submanifold M of \mathbb{R}^6 defined by

$$\chi(u, v, w) = (v \cos u, v \sin u, -v + w, w \cos u, w \sin u, v + w),$$

with almost product structure F defined by

$$F\left(\frac{\partial}{\partial x_i}\right) = -\left(\frac{\partial}{\partial x_i}\right), \quad F\left(\frac{\partial}{\partial y_j}\right) = \left(\frac{\partial}{\partial y_j}\right), \quad 1 \leq i, j \leq 3.$$

It is easy to see that its tangent space TM of M is spanned by the following vectors

$$\begin{aligned} v_1 &= -v \sin u \frac{\partial}{\partial x_1} + v \cos u \frac{\partial}{\partial x_2} - w \sin u \frac{\partial}{\partial y_1} + w \cos u \frac{\partial}{\partial y_2}, \\ v_2 &= \cos u \frac{\partial}{\partial x_1} + \sin u \frac{\partial}{\partial x_2} - \frac{\partial}{\partial x_3} + \frac{\partial}{\partial y_3}, \\ v_3 &= \frac{\partial}{\partial x_3} + \cos u \frac{\partial}{\partial y_1} + \sin u \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3}. \end{aligned}$$

Then, we have

$$\begin{aligned} Fv_1 &= v \sin u \frac{\partial}{\partial x_1} - v \cos u \frac{\partial}{\partial x_2} - w \sin u \frac{\partial}{\partial y_1} + w \cos u \frac{\partial}{\partial y_2}, \\ Fv_2 &= -\cos u \frac{\partial}{\partial x_1} - \sin u \frac{\partial}{\partial x_2} + \frac{\partial}{\partial x_3} + \frac{\partial}{\partial y_3}, \\ Fv_3 &= -\frac{\partial}{\partial x_3} + \cos u \frac{\partial}{\partial y_1} + \sin u \frac{\partial}{\partial y_2} + \frac{\partial}{\partial y_3}. \end{aligned}$$

Let us put $D_1 = \text{span}\{v_1\}$ is a proper slant distribution with slant angle $\theta_1 = \cos^{-1} \left(\frac{w^2 - v^2}{w^2 + v^2} \right)$ and $D_2 = \text{span}\{v_2, v_3\}$ is again a proper slant distribution with slant angle $\theta_2 = \cos^{-1} \left(\frac{2}{3} \right)$. Hence the submanifold M defined by χ is a bi-slant submanifold.

It is easy to verify that both the distributions D_1 and D_2 are integrable. If we denote the integrable manifolds of D_1 and D_2 by M_T and M_θ , respectively. Then the metric tensor g of product manifold M is given by

$$g = g_{M_\theta} + (v^2 + w^2)g_{M_T}$$

where

$$g_{M_\theta} = 3(dv^2 + dw^2) \quad \text{and} \quad g_{M_T} = du^2.$$

Hence, M is a proper non-trivial warped product bi-slant submanifold of \mathbb{R}^6 with warping function $f = \sqrt{v^2 + w^2}$ and whose bi-slant angles $\theta_1, \theta_2 \neq 0, \frac{\pi}{2}$.

Conflicts of Interest: The authors declare no conflict of interest.

Acknowledgment. The authors would like to thank the anonymous reviewers for useful comments and suggestions.

References

- [1] F. R. Al-Solamy, V. A. Khan and S. Uddin, *Geometry of warped product semi-slant submanifolds of nearly Kaehler manifolds*, Results in Mathematics. **71**(3), 783-799, 2017.
- [2] M. Atceken, *Warped product semi-slant submanifolds in locally Riemannian product manifolds*, Bull. Aust. Math. Soc. **77**(2), 177-186, 2008.
- [3] M. Atceken, *Slant submanifolds of a Riemannian product manifold*, Acta Math. Sci. Ser B Engl. Ed. **30**(1), 215-224, 2010.
- [4] A. Bejancu, *Semi-invariant submanifolds of locally product Riemannian manifolds*, An. Univ. Timisoara. Ser Stiint. Math. **22**, 3-11, 1984.
- [5] R. L. Bishop and B. O'Neill, *Manifolds of negative curvature*, Trans. Am. Math. Soc. **145**, 1-49, 1969.
- [6] J. L. Cabrerizo, A. Carriazo, L.M. Fernandez and M. Fernandez, *Semi-Slant submanifolds of a Sasakian manifold*, Geom. Dedicata. **78**(2), 183-199, 1999.
- [7] B. Y. Chen, *Slant immersions*, Bull. Aust. Math. Soc. **41**(1), 135-147, 1990.

- [8] B. Y. Chen, *Geometry of warped product CR-submanifolds in Kaehler manifolds*, Monatsh. Math. **133**, 177-195, 2001.
- [9] B. Y. Chen, *Geometry of warped product CR-submanifolds in Kaehler manifolds II*, Monatsh. Math. **134**, 103-119, 2001.
- [10] B. Y. Chen, *Another general inequality for CR-warped products in complex space forms*, Hokkaido Math. J. **32**(2), 415-444, 2003.
- [11] B. Y. Chen, *CR-warped products in complex projective spaces with compact holomorphic factor*, Monatshefte für Mathematik. **141**(3), 177-186, 2004.
- [12] B. Y. Chen, *Geometry of warped product submanifolds: A survey*, J. Adv. Math. Stud. **6**(2), 1-8, 2013.
- [13] B. Y. Chen, *Differential geometry of warped product manifolds and submanifolds*, World Scientific. 2017.
- [14] B. Y. Chen and O. J. Garay, *Pointwise slant submanifolds in almost Hermitian manifolds*, Turk. J. Math. **36**, 630-640, 2012.
- [15] B. Y. Chen, M. H. Shahid and F. R. Al-Solamy, *Complex Geometry of Slant Submanifolds*, Springer. Berlin 2021.
- [16] B. Y. Chen and S. Uddin, *Warped product pointwise bi-slant submanifolds of Kaehler manifolds*, Publicationes Mathematicae. **92**(1-2), 183-199, 2018.
- [17] F. Etayo, *On quasi-slant submanifolds of an almost Hermitian manifold*, Publ. Math. Debrecen. **53**(1-2), 217-223, 1998.
- [18] M. Gülbahar, E. Kilic and S. S. ÇELİK *Special proper pointwise slant surfaces of a locally product Riemannian manifold*, Turk. J. Math. **39**(6), 884-899, 2015.
- [19] S. Hiepko, *Eine inner kennzeichnung der verzerrten produkte*, Math. Ann. **241**, 209-215, 1979.
- [20] Y. Kentaro and M. Kon, *Generic submanifolds*, Annali di Matematica Pura ed Applicata. **123**(1), 59-92, 1980.
- [21] V. A. Khan and M. A. Khan, *Semi-slant submanifolds of a nearly Kaehler manifold*, Turk. J. Math. **31**(4), 341-353, 2007.
- [22] H. Li and X. Liu, *Semi-slant submanifolds of a locally product manifold*, Georgian Math. J. **12**, 273-282, 2005.
- [23] M. A. Lone, M. H. Shahid and S. D. Sharma, *Hemi-slant submanifolds of nearly Kaehler manifolds*, Gulf Journal of Mathematics, **4**(3), 2016.
- [24] M. F. Naghi, M. S. Stankovic and F. Alghamdi, *Chens improved inequality for pointwise hemi-slant warped products in Kaehler manifolds*, Filomat. **34**(3), 807-814, 2020.
- [25] N. Papaghuic, *Semi-slant submanifolds of a Kahlerian manifold*, An. Stiint. Univ. Al. I. Cuza Iasi Mat. **40**(1), 55-61, 1994.
- [26] B. Sahin, *Slant submanifolds of an almost product Riemannian manifold*, J. Korean Math. Soc. **43**(4), 717-732, 2006.
- [27] B. Sahin, *Non-existence of warped product semi-slant submanifolds of Kaehler manifolds*, Geometriae Dedicata. **117**(1), 195-202, 2006.
- [28] B. Sahin, *Warped product semi-slant submanifolds of a locally product Riemannian manifold*, Studia. Sci. Math. Hungar. **46**(2), 169-184, 2009.
- [29] B. Sahin, *Warped product pointwise semi-slant submanifolds of Kaehler manifolds*, Port. Math. **70**, 252-268, 2013.
- [30] B. Sahin and M. Atceken, *Semi-invariant submanifolds of Riemannian product manifold*, Balkan J. Geom. Appl. **8**(1), 91-100, 2003.
- [31] S. Uddin, F. Alghamdi and F. R. Al-Solamy, *Geometry of warped product pointwise semi-slant submanifolds of locally product Riemannian manifolds*, Journal of Geometry and Physics. **152**, 103658, 2020.
- [32] S. Uddin, B. Y. Chen and F. R. Al-Solamy, *Warped product bi-slant immersions in Kaehler manifolds*, Mediterr. J. Math. **14**(2), 14-95, 2017.

- [33] K. Yano and M. Kon, *Structures on manifolds*, Series in Pure mathematics, World scientific Publishing Co. Singapore, 1984.