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Tensor Product of Phase Retrievable Frames

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Research Article

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Abstract
Frame vectors in the tensor product of Hilbert spaces that accomplish phase retrieval can be characterized. In this article, we determine the conditions under which the tensor product of vectors may do phase retrieval. Given that tensor product of two frames always implies a frame in the tensor product of Hilbert spaces, we particularly concentrate on finding conditions for phase retrieval in the tensor product of Hilbert spaces.

Keywords: Frame vectors, Tensor product, Phase retrieval

Fazları Geri Alınabilen Frame Vektörlerinin Tensör Çarpımı

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Öz
Hilbert uzaylarının tensör çarpımındaki faz geri dönüşünü gerçekleştiren frame vektörleri karakterize edilmektedir. Bu makalede, vektörlerin tensör çarpımının hangi koşullar altında faz geri getirme yapabileceği belirlenmektedir. İki frame setinin tensör çarpımının her zaman Hilbert uzaylarının tensör çarpımında bir frame ima ettiği göz önüne alındığında, bu çalışmada özellikle Hilbert uzaylarının tensör çarpımında frame vektörlerinin faz geri dönüşü için gerekli koşullar belirlenmiştir.

Anahtar Kelimeler: Frame vektörleri, Tensör çarpımı, Faz geri alma

Introduction

Given a signal x and an orthonormal basis $\{u_i\}_{i\in I}$ in a Hilbert space H, we know that the sequence of measurements $\{\langle x, u_i\rangle\}_{i\in I}$ allows us to reconstruct the signal x by using the orthonormal basis $\{u_i\}_{i\in I}$. The reconstruction of the signal x is not possible if we lose some of these measurements. Because of the redundancy, frame vectors can be possible used as a solution to the reconstruction of a signal if some of the coefficients are lost or cannot be measured. Frame vectors can be thought of as a generalization of orthonormal vectors with many other advantages. Duffin and Schaeffer [1] introduced frame theory for

separable Hilbert spaces in the context of nonharmonic Fourier series in 1952. Duffin and Schaeffer's definition was an abstraction of Gabor's concept for signal processing in [2].
Signal reconstruction is very important notion in engineering but it is a significant challenge if there is a partial loss of information. Given a signal x and a frame $\{x_i\}_{i\in I}$ in a Hilbert space H, we are not able to construct the exact signal x if we only have the phaseless measurements $\{|\langle x, x_i\rangle|\}_{i\in I}$.
Casazza, Balan and Edidin [3] introduced the concept of phase retrieval for Hilbert spaces in 2006. Phase retrieval is the idea of reconstruction of a signal without using phase.
They showed that given the intensity measurements $\{|\langle x, x_i\rangle|\}_{i\in I}$ from a redundant linear system, x and αx with $|\alpha| = 1$ cannot be distinguish from phaseless measurements. Phase retrieval has applications in areas such that X-ray crystallography [4, 5], speech recognition [6], electron microscopy [7, 8] and many of other areas. Tensor product is so often used in approximation theory. Folland in [9], Kadison and Ringrose in [10] gave the representation of the tensor product $H_1 \otimes H_2$ of Hilbert spaces H_1 and H_2 as the space of bounded antilinear maps from H_2 into H_1. G. Upender Reddy, N. Gopal Reddy and B. Krishna Reddy in [11], Amir Khosravi and M.S. Asgari in [12] introduced the definition of the frames vectors in the tensor products of Hilbert spaces. Both of the papers showed that given the two sequences $\{x_i\}_{i\in I}$ in H_1 and $\{y_j\}_{j\in J}$ in H_2, the tensor product $\{x_i \otimes y_j\}_{i\in I, j\in J}$ is a frame for the tensor product of Hilbert spaces $H_1 \otimes H_2$ if and only if the set of vectors $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$ is a frame in H_1 and H_2, respectively. Tensor product of dual frames is studied in [13] by Ya-Hui Wang and Yun-Zhang Li. Later, Samineh Zakeri and Ahmad Ahmedi in [14] investigated the conditions under which the tensor product of two frames is a scalable frame. A. Razghandi and R. Raisi Tousi gave an explicit expression to the reconstruction of a signal from magnitudes of frame coefficients of the tensor product dual frames.
In this paper, we study the property of phase retrievability of frame vectors in tensor product of Hilbert spaces. We examine the conditions under which a frame is phase retrievable in the tensor product of Hilbert spaces. The main results of this paper provide that given the two sequences $\{x_i\}_{i\in I}$ in H_1 and $\{y_j\}_{j\in J}$ in H_2, the tensor product of $\{x_i \otimes y_j\}_{i\in I, j\in J}$ is a phase retrievable frame for the tensor product of Hilbert spaces $H_1 \otimes H_2$ if and only if the set of vectors $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$ is a phase retrievable frame in H_1 and H_2, respectively. We also give an explicit expression to show that when a set of vectors in tensor product of Hilbert spaces does not do phase retrieval in $H_1 \otimes H_2$. The organization of the paper is as follows. In section 2, we give some basic definitions and information about frame theory, tensor product of Hilbert spaces and phase retrieval which are necessary to understand context of the paper. In section 3, we define phase retrieval in tensor product of Hilbert spaces. We give the conditions under which tensor product of frames vectors does phase retrieval in $H_1 \otimes H_2$. We prove that most of the results in [12] also holds for the tensor product of phase retrievable frames.

Preliminaries

In this section, we give basic definitions and some important properties of the frame theory, phase retrievable frames and the tensor product of Hilbert spaces in the literature. We refer the reader [3, 9, 10, 15–19] for more details. In the rest of this paper H will denote a separable Hilbert space, I and J are a countable index set.

Definition 1. [1] A collection of vectors $\{x_i\}_{i\in I}$ in a Hilbert space H is called a frame if there exists

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two constants $0 < A \leq B < \infty$ such that

$$A||x||^2 \leq \sum_i |\langle x, x_i\rangle|^2 \leq B||x||^2 \quad \text{for all } x \in H.$$

A and B are called the lower and the upper frame bounds. A frame is called a tight frame if $A = B$ and a Parseval frame if $A = B = 1$.

Definition 2. [3] A set of vectors $\{x_i\}_{i\in I}$ in a Hilbert space H does phase retrieval if for all $x, y \in H$ satisfying $|\langle x, x_i\rangle| = |\langle y, x_i\rangle|$ for all i, then $x = \alpha y$ for some scalar α with $|\alpha| = 1$.

There are different ways to define the tensor product of Hilbert spaces, but we will use the following. Folland in [9], Kadison and Ringrose in [10] defined the tensor product of Hilbert spaces H_1 and H_2 as a space of all antilinear operators. An operator $T : H_2 \longrightarrow H_1$ is said to be **antilinear** if

$$T(\alpha x + \beta y) = \bar{\alpha}T(x) + \bar{\beta}T(y)$$

for any $x, y \in H_2$ and $\alpha, \beta \in \mathbb{C}$. The adjoint T^* of a bounded antilinear map T is defined by

$$\langle T^*x, y\rangle = \langle Ty, x\rangle.$$

The tensor product $H_1 \otimes H_2$ of the Hilbert spaces H_1 and H_2 is the set of all antilinear maps $T : H_2 \longrightarrow H_1$ such that $\sum_j ||Tv_j||^2 < \infty$ for any orthonormal basis $\{v_j\}$ for H_2. By Theorem (7.12) in [9], $H_1 \otimes H_2$ is a Hilbert space with the norm

$$|||T|||^2 = \sum_j ||Tv_j||^2$$

and the associated inner product

$$\langle T_1, T_2\rangle = \sum_j \langle T_1 v_j, T_2 v_j\rangle$$

where $\{v_j\}$ is any orthonormal basis for H_2 and T_1, T_2 be the antilinear maps from H_2 onto H_1. For any $x \in H_1$ and $y \in H_2$, the map defined in [9] by

$$(x \otimes y)(t) = \langle y, t\rangle x, \quad t \in H_2$$

belongs to $H_1 \otimes H_2$ and

$$|||x \otimes y||| = ||x||\,||y||,$$

$$\langle x \otimes y, x^* \otimes y^*\rangle = \langle x, x^*\rangle\langle y, y^*\rangle \quad \text{for all } x, x^* \in H_1, y, y^* \in H_2.$$

Tensor product of operators have the following results in [9, Theorem 7.18].
Let H_1 and H_2 be Hilbert spaces. For all $C, C^{'} \in B(H_1)$ and $D, D^{'} \in B(H_2)$, we have

1. $||C \otimes D|| = ||C||||D||$

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2. $(C \otimes D)(C' \otimes D') = (CC') \otimes (DD')$

3. $(C \otimes D)^* = D^* \otimes C^*$

4. $(C \otimes D)(x \otimes y) = Cx \otimes Dy \quad$ for all $\quad x \in H_1,\, y \in H_2$

Now, we want to state the definition of a frame in the tensor product of Hilbert spaces.

Definition 3. [11] Let $\{x_i\}_{i\in I}$ and $\{y_j\}_{i\in J}$ be the set of vectors in the Hilbert spaces H_1 and H_2, respectively. The tensor product $\{x_i \otimes y_j\}_{i\in I, j\in J}$ is called a frame for the tensor product of Hilbert spaces $H_1 \otimes H_2$ if there exists two constants $0 < A \le B < \infty$ such that

$$A|||x \otimes y|||^2 \le \sum_{i,j} |\langle x \otimes y, x_i \otimes y_j\rangle|^2 \le B|||x \otimes y|||^2 \quad \text{for all } x \otimes y \in H_1 \otimes H_2.$$

A and B are called the lower and the upper frame bounds for $\{x_i \otimes y_j\}_{i\in I, j\in J}$.

Tensor Product of Phase Retrievable Frames

In this section, we extend the definition of phase retrieval in [3] to the tensor product of Hilbert spaces. We give the main results of phase retrievable frames in the tensor product of Hilbert spaces and show that we can carry out the frame vector results in [12] to the phase retrievable frames in the tensor product of Hilbert spaces.

Definition 4. Let $\{T_i\}_{i\in I}$ be a collection of operators in $H_1 \otimes H_2$. We say that the collection of $\{T_i\}_{i\in I}$ does phase retrieval if for all $x_1 \otimes y_1, x_2 \otimes y_2 \in H_1 \otimes H_2$ satisfying

$$|\langle x_1 \otimes y_1, T_i\rangle| = |\langle x_2 \otimes y_2, T_i\rangle| \quad \text{for all } i$$

then $(x_1 \otimes y_1) = \alpha(x_2 \otimes y_2)$ for some scalar α with $|\alpha| = 1$.

For any vector $x_i \in H_1$ and $y_j \in H_2$, since the map defined in [9] by

$$(x_i \otimes y_j)(t) = \langle y_j, t\rangle x_i, \quad t \in H_2$$

belongs to $H_1 \otimes H_2$, we can state the following theorem.

Theorem 1. Tensor product $\{x_i \otimes y_j\}_{\{i,j\}}$ of $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$ is a phase retrievable frame for $H_1 \otimes H_2$ if and only if the set of vectors $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$ does phase retrieval in H_1 and H_2, respectively.

Proof. Let us assume that $\{x_i \otimes y_j\}_{\{i,j\}}$ does phase retrievable for $H_1 \otimes H_2$. To prove that $\{x_i\}_{i\in I}$ does phase retrieval in H_1, for any $z_1, z_2 \in H_1$ that satisfies $|\langle z_1, x_i\rangle| = |\langle z_2, x_i\rangle|$ for all i, we want to show that there exists a scalar α such that $z_1 = \alpha z_2$ with $|\alpha| = 1$.
For each i and j, we have

$$\langle z_1 \otimes y_j, x_i \otimes y_j\rangle = \langle z_1, x_i\rangle\langle y_j, y_j\rangle \quad \text{and} \quad \langle z_2 \otimes y_j, x_i \otimes y_j\rangle = \langle z_2, x_i\rangle\langle y_j, y_j\rangle.$$

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Since we have $|\langle z_1, x_i\rangle| = |\langle z_2, x_i\rangle|$ for all i by the assumption, this allows us to write

$$|\langle z_1 \otimes y_j, x_i \otimes y_j\rangle| = |\langle z_z \otimes y_j, x_i \otimes y_j\rangle|$$

for all i and j. We assume that $\{x_i \otimes y_j\}_{\{i,j\}}$ does phase retrievable for $H_1 \otimes H_2$. This implies that there exists a scalar α such that $(z_1 \otimes y_j) = \alpha(z_2 \otimes y_j)$ with $|\alpha| = 1$. For $y_j \in H_2$, we have

$$\begin{aligned}(z_1 \otimes y_j)(y_j) = \alpha(z_2 \otimes y_j)(y_j) &\Longrightarrow \langle y_j, y_j\rangle z_1 = \alpha\langle y_j, y_j\rangle z_2 \\ &\Longrightarrow z_1 = \alpha z_2.\end{aligned}$$

This says that the set of vectors $\{x_i\}_{i\in I}$ does phase retrieval in H_1. A similar argument works for the set of vectors $\{y_j\}_{j\in J}$ to do phase retrieval in H_2.
To prove other direction, suppose the set of vectors $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$ does phase retrieval in H_1 and H_2, respectively. For any $z_1 \otimes t_1, z_2 \otimes t_2 \in H_1 \otimes H_2$ that satisfies

$$|\langle z_1 \otimes t_1, x_i \otimes y_j\rangle| = |\langle z_2 \otimes t_2, x_i \otimes y_j\rangle| \tag{1}$$

for all i and j, we can write

$$|\langle z_1 \otimes t_1, x_i \otimes y_j\rangle| = |\langle z_1, x_i\rangle\langle t_1, y_j\rangle| = |\langle z_1, x_i\rangle||\langle t_1, y_j\rangle|$$

$$|\langle z_2 \otimes t_2, x_i \otimes y_j\rangle| = |\langle z_2, x_i\rangle\langle t_2, y_j\rangle| = |\langle z_2, x_i\rangle||\langle t_2, y_j\rangle|.$$

By the equality in (1), we have

$$|\langle z_1, x_i\rangle||\langle t_1, y_j\rangle| = |\langle z_2, x_i\rangle||\langle t_2, y_j\rangle| \Longrightarrow \frac{|\langle z_1, x_i\rangle|}{|\langle z_2, x_i\rangle|} = \frac{|\langle t_2, y_j\rangle|}{|\langle t_1, y_j\rangle|} \tag{2}$$

for all i and j. Assume that $\frac{|\langle z_1, x_i\rangle|}{|\langle z_2, x_i\rangle|} = \beta \neq 0$ for some fixed i. Since the equality in (2) holds for all i and j, this implies that

$$\frac{|\langle z_1, x_i\rangle|}{|\langle z_2, x_i\rangle|} = \frac{|\langle t_2, y_j\rangle|}{|\langle t_1, y_j\rangle|} = \beta \neq 0$$

for all i and j. Therefore, we have $|\langle z_1, x_i\rangle| = \beta|\langle z_2, x_i\rangle|$ for all i and $|\langle t_2, y_j\rangle| = \beta|\langle t_1, y_j\rangle|$ for all j. Since we assume that the set of vectors $\{x_i\}_{i\in I}$ and $\{y_j\}_{i\in J}$ does phase retrieval in H_1 and H_2, respectively, we can say that there exists α_1, α_2 such that $z_1 = \alpha_1\beta z_2$ and $t_2 = \alpha_2\beta t_1$ with $|\alpha_1| = |\alpha_2| = 1$. For any $t \in H_2$, we have

$$\begin{aligned}(z_1 \otimes t_1)(t) = \langle t_1, t\rangle z_1 &= \langle \frac{t_2}{\alpha_2\beta}, t\rangle \alpha_1\beta z_2 \\ &= \frac{\alpha_1\beta}{\alpha_2\beta}\langle t_2, t\rangle z_2 \\ &= \frac{\alpha_1}{\alpha_2}(z_2 \otimes t_2)(t).\end{aligned}$$

Since there exists a scalar $\alpha = \frac{\alpha_1}{\alpha_2}$ such that $(z_1 \otimes t_1)(t) = \alpha(z_2 \otimes t_2)(t)$ together with $|\alpha| = |\frac{\alpha_1}{\alpha_2}| =$

$\frac{|\alpha_1|}{|\alpha_2|} = 1$, this shows that the tensor product $\{x_i \otimes y_j\}_{\{i,j\}}$ of $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$ is a phase retrievable frame for $H_1 \otimes H_2$.

Theorem 2. If tensor product $\{x_i \otimes y_j\}_{\{i,j\}}$ of $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$ is a phase retrievable frame for $H_1 \otimes H_2$, then $\overline{\text{span}}\{(x_i \otimes y_j)t\}_{\{i,j\}} = H_1$ for every nonzero vector $t \in H_2$.

Proof. We show the proof by contrapositive. Suppose there exists $t \in H_2$ such that

$\overline{\text{span}}\{(x_i \otimes y_j)t\}_{\{i,j\}} \neq H_1$.

This says that there exists a nonzero vector $z \in H_1$ such that $\langle z, (x_i \otimes y_j)t\rangle = 0 \quad \forall i, j$.

$$\langle z, (x_i \otimes y_j)t\rangle = \langle z, \langle y_j, t\rangle x_i\rangle = \langle z, x_i\rangle\langle t, y_j\rangle = 0 \tag{3}$$

For any nonzero scalar c with $|c| \neq 1$, let $z_c = cz$.
By the equality in (3),

$$\begin{aligned}
|\langle x_i \otimes y_j, z \otimes t\rangle| &= |\langle x_i, z\rangle\langle y_j, t\rangle| \\
&= |\langle x_i, z\rangle|.|\langle y_j, t\rangle| \\
&= |\langle x_i, \frac{1}{c} z_c\rangle|.|\langle y_j, t\rangle| \\
&= \frac{1}{|\bar{c}|}|\langle x_i, z_c\rangle|.|\langle y_j, t\rangle| \\
&= \frac{1}{|\bar{c}|}|\langle x_i \otimes y_j, z_c \otimes t\rangle| = 0.
\end{aligned}$$

That is $|\langle x_i \otimes y_j, z \otimes t\rangle| = |\langle x_i \otimes y_j, z_c \otimes t\rangle|$ for all i, j. On the other hand, we know that $(z_c \otimes t) = c(z \otimes t)$ for nonzero scalar c with $|c| \neq 1$. This implies that $(z_c \otimes t) \neq \alpha(z \otimes t)$ for any α with $|\alpha| = 1$. Therefore, tensor product $\{x_i \otimes y_j\}_{\{i,j\}}$ of $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$ is not a phase retrievable frame for $H_1 \otimes H_2$.

Theorem 3. Let $\{T_i\}_{i\in I}$ be a phase retrievable frame for $H_1 \otimes H_2$. Then for any $x \in H_1$ and $y \in H_2$, the set of vectors $\{T_i y\}_{i\in I}$ and $\{T_i^* x\}_{i\in I}$ does phase retrieval for H_1 and H_2, respectively.

Proof. Suppose that given $x_1, x_2 \in H_1$, we have $|\langle x_1, T_i y\rangle| = |\langle x_2, T_i y\rangle|$ for all i. For any orthonormal basis $\{u_\ell\}_{\ell\in L}$ in H_2,

$$\begin{aligned}
\langle x_1, T_i y\rangle &= \langle x_1, T_i(\sum_\ell \langle y, u_\ell\rangle u_\ell)\rangle = \langle x_1, \sum_\ell \overline{\langle y, u_\ell\rangle} T_i u_\ell\rangle \qquad (T_i \text{ is an antilinear map}) \\
&= \sum_\ell \langle y, u_\ell\rangle\langle x_1, T_i u_\ell\rangle = \sum_\ell \langle\langle y, u_\ell\rangle x_1, T_i u_\ell\rangle \\
&= \sum_\ell \langle (x_1 \otimes y)(u_\ell), T_i u_\ell\rangle \\
&= \langle x_1 \otimes y, T_i\rangle.
\end{aligned}$$

Similarly, we have $\langle x_2, T_i y\rangle = \langle x_2 \otimes y, T_i\rangle$ for all i.
Hence, $|\langle x_1 \otimes y, T_i\rangle| = |\langle x_2 \otimes y, T_i\rangle|$ for all i and $\{T_i\}_{i\in I}$ is a phase retrievable frame for $H_1 \otimes H_2$.

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This says that there exists α such that $(x_1 \otimes y) = \alpha(x_2 \otimes y)$ with $|\alpha| = 1$. For the given $y \in H_2$, we can write

$$\begin{aligned}(x_1 \otimes y)(y) = \alpha(x_2 \otimes y)(y) &\Longrightarrow \langle y, y\rangle x_1 = \alpha\langle y, y\rangle x_2 \\ &\Longrightarrow x_1 = \alpha x_2.\end{aligned}$$

This concludes that the set vectors $\{T_i y\}_{i\in I}$ does phase retrieval for H_1. A similar argument works for $\{T_i^* x\}_{i\in I}$ to do phase retrieval in H_2.

Corollary 1. Let T be an operator in $H_1 \otimes H_2$ such that $range(T) = H_1$ and $\{Q_j\}_{j\in J}$ be a phase retrievable frame for $H_2 \otimes H_1$. Then for any $x \in H_1$, the set of vectors $\{TQ_j(x)\}_j$ does phase retrieval in H_1.

Proof. For any $x \in H_1$ and any given $x_1, x_2 \in H_1$ which satisfies the condition $|\langle x_1, TQ_j(x)\rangle| = |\langle x_2, TQ_j(x)\rangle|$ for all j, we have

$$|\langle x_1, TQ_j(x)\rangle| = |\langle x_2, TQ_j(x)\rangle| \Rightarrow |\langle T^* x_1, Q_j(x)\rangle| = |\langle T^* x_2, Q_j(x)\rangle| \text{ for all } j.$$

Since we assume that $\{Q_j\}_{j\in J}$ is a phase retrievable frame for $H_2 \otimes H_1$, by Theorem (3), the set of vectors $\{Q_j(x)\}_{j\in J}$ is also phase retrievable frame for H_1. This says that $T^* x_1 = \alpha T^* x_2$ for some α with $|\alpha| = 1$. For any $y \in H_2$, we can write

$$\begin{aligned}\langle T^* x_1, y\rangle = \langle \alpha T^* x_2, y\rangle &\Rightarrow \langle x_1, Ty\rangle = \langle \alpha x_2, Ty\rangle \\ &\Rightarrow \langle x_1 - \alpha x_2, Ty\rangle = 0\end{aligned}$$

When $range(T) = H_1$, we can choose the $y \in H_2$ such that $Ty = x_1 - \alpha x_2$. Hence, $\langle x_1 - \alpha x_2, x_1 - \alpha x_2\rangle = ||x_1 - \alpha x_2||^2 = 0 \Rightarrow x_1 = \alpha x_2$ and this says that the set of vectors $\{TQ_j(x)\}_j$ does phase retrieval in H_1.

Corollary 2. If $\{T_i\}_{i\in I}$ is a phase retrievable frame for $H_1 \otimes H_2$, then for any $x \in H_1$ and $y \in H_2$, the set of tensor products $\{T_i(y \otimes x)T_i\}$ does phase retrieval in $H_1 \otimes H_2$.

Proof. For any $x \in H_1$ and $y \in H_2$, we showed in Theorem (3) that $\{T_i y\}_{i\in I}$ and $\{T_i^* x\}_{i\in I}$ does phase retrieval for H_1 and H_2, respectively. By Theorem (1), tensor product of two phase retrievable frames are also phase retrievable. Hence, $\{(T_i y) \otimes (T_i^* x)\}$ does phase retrieval in $H_1 \otimes H_2$. For any $t \in H_2$, we have

$$\begin{aligned}((T_i y) \otimes (T_i^* x))(t) &= \langle T_i^* x, t\rangle T_i y \\ &= \langle T_i t, x\rangle T_i y \\ &= T_i(\overline{\langle T_i t, x\rangle} y) \\ &= T_i(\langle x, T_i t\rangle y) \\ &= T_i(y \otimes x)T_i(t).\end{aligned}$$

This proves that the set $\{T_i(y \otimes x)T_i\}$ does phase retrieval in $H_1 \otimes H_2$.

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Corollary 3. Let $T_1 \in B(H_1), T_2 \in B(H_2)$ be invertible operators. If $\{x_i\}_{i\in I}$ and $\{y_j\}_{j\in J}$ are phase retrievable frames for H_1 and H_2, respectively, then $\{T_1x_i \otimes T_2y_j\}_{i,j}$ does phase retrieval for $H_1 \otimes H_2$.

Proof. Let T_1 and T_2 be invertible operators on H_1 and H_2, respectively.
Since we have $(T_1 \otimes T_2)^{-1} = T_1^{-1} \otimes T_2^{-1}$ and

$$(T_1 \otimes T_2)(T_1 \otimes T_2)^{-1} = T_1T_1^{-1} \otimes T_2T_2^{-1} = I_{H_1} \otimes I_{H_2},$$

$T_1 \otimes T_2$ is an invertible operator on $H_1 \otimes H_2$. Theorem (4.8) in [15] says that phase retrievable frames are preserved under invertible operators. By Theorem 1, we show that tensor product of two phase retrievable frames is also a phase retrievable frame for $H_1 \otimes H_2$. Hence, $(T_1 \otimes T_2)(x_i \otimes y_j) = \{T_1x_i \otimes T_2y_j\}_{i,j}$ does phase retrieval for $H_1 \otimes H_2$.

Corollary 4. The set of $\{T_i\}_{i\in I}$ does not do phase retrieval for $H_1 \otimes H_2$ if there exists linearly independent vectors $u_1, u_2 \in H_1$ and $v_1, v_2 \in H_2$ such that $\langle u_1, T_iv_2\rangle + \langle u_2, T_iv_1\rangle = 0$ for all i .

Proof. Suppose $u_1, u_2 \in H_1$ and $v_1, v_2 \in H_2$ be linearly independent vectors such that $\langle u_1, T_iv_2\rangle + \langle u_2, T_iv_1\rangle = 0$. For any scalar $\alpha \in \mathbb{C}$ with $|\alpha| = 1$, we can define

$$x_1 = u_1 + u_2 \qquad y_1 = v_1 + v_2$$

$$x_2 = \frac{u_1 - u_2}{\sqrt{\alpha}} \qquad y_2 = \frac{v_1 - v_2}{\sqrt{\alpha}}$$

This implies that

$$x_1 \oplus y_1 = u_1 \oplus v_1 + u_1 \oplus v_2 + u_2 \oplus v_1 + u_2 \oplus v_2$$

$$x_2 \oplus y_2 = \frac{1}{\alpha}(u_1 \oplus v_1 - u_1 \oplus v_2 - u_2 \oplus v_1 + u_2 \oplus v_2).$$

By assumption, we have $\langle u_1, T_iv_2\rangle + \langle u_2, T_iv_1\rangle = 0$.
This says that $|\langle x_1 \oplus y_1, T_i\rangle| = |\frac{1}{\alpha}||\langle x_2 \oplus y_2, T_i\rangle| = |\langle x_2 \oplus y_2, T_i\rangle|$ for all $i \in I$. To show that the set of $\{T_i\}_{i\in I}$ does not do phase retrievable for $H_1 \otimes H_2$, we need to show that $x_1 \oplus y_1 \neq \alpha x_2 \oplus y_2$. To have a contradiction, suppose we have $x_1 \oplus y_1 = \alpha x_2 \oplus y_2$.
For any $t \in H_2$, we have

$$(x_1 \oplus y_1 - \alpha x_2 \oplus y_2)(t) = \frac{1}{\alpha + 1}(u_1 \oplus v_2 + u_2 \oplus v_1)(t) = 0.$$

$(u_1 \oplus v_2 + u_2 \oplus v_1)(t) = \langle v_2, t\rangle u_1 + \langle v_1, t\rangle u_2 = 0$ and $u_1, u_2 \in H_1$ are linearly independent vectors, this says that $\langle v_2, t\rangle = \langle v_1, t\rangle = 0$. Since we have $\langle v_2 - v_1, t\rangle = 0$ for any $t \in H_2$, this contradicts with the assumption that $v_1, v_2 \in H_2$ are linearly independent vectors. Hence, we have $x_1 \oplus y_1 \neq \alpha x_2 \oplus y_2$ and the set of $\{T_i\}_{i\in I}$ does not do phase retrievable for $H_1 \otimes H_2$.

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