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RANDOM FIXED POINT RESULTS FOR GENERALIZED ASYMPTOTICALLY NONEXPANSIVE RANDOM OPERATORS

Isa YILDIRIM[1] and Muhammed Emin BATUHAN[2]

[1]Department of Mathematics, Faculty of Science, Atatürk University, 25240 Erzurum, TÜRKİYE

[2]Graduate School of Natural and Applied Sciences, Atatürk University, 25240 Erzurum, TÜRKİYE

Abstract. In this paper, we define an implicit random iterative process with errors for three finite families of generalized asymptotically nonexpansive random operators. We also prove some convergence theorems using this iteration method in separable Banach spaces.

1. Introduction and Preliminaries

Random analysis is one of the most important areas of mathematics. Particularly, random techniques have a very common use in pure and applied mathematics. Hans [9] and Spacek [17] proved random fixed point results for random contraction mappings on separable metric spaces. Later, many authors have worked on it using different operator classes and different spaces. Some of them are given in these references [1], [3], [4], [10], [11], [12] and [13].

Let $(\mho, \Sigma)$ be a measurable space and $\mathfrak{X}$ be a real Banach space. Assume that E is an operator from $\mho \times \mathfrak{X}$ to $\mathfrak{X}$. Here, the m-th iterate $E(\ell, E(\ell, \ldots, E(\ell, u_0)))$ of E is denoted by as $E^m(\ell, u_0)$.

Definition 1. *Let f be a mapping on $\mho$. If for any Borel subset $\mathfrak{X} \subset \mathbb{R}$ the set $f^{-1}(\mathfrak{X})$ is measurable, the mapping f is called measurable.*

Definition 2. *Let E be an operator from $\mho \times \mathfrak{X}$ to $\mathfrak{X}$. If $E(\cdot, u_0) : \mho \to \mathfrak{X}$ is measurable for every $u_0 \in \mathfrak{X}$, then it is called a random operator.*

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[1] isayıldırım@atauni.edu.tr-Corresponding author; 0000-0001-6165-716X;

[2] emin.batuhan@gmail.com; 0000-0001-9318-4627.

Definition 3. *Let E be an operator from $\mho \times \mathfrak{X}$ to $\mathfrak{X}$. If $E(\ell, \cdot) : \mathfrak{X} \to \mathfrak{X}$ is continuous for each $\ell \in \mho$, it is continuous.*

Definition 4. *Let E be an operator from $\mho \times \mathfrak{X}$ to $\mathfrak{X}$. If $E(\ell, p(\ell)) = p(\ell), \forall \ell \in \mho$, p is called a random fixed point of the random operator E. Here, $p : \mho \to \mathfrak{X}$ is a measurable function. We denote by $RF(E)$ the set of random fixed points of E.*

Definition 5. ([2]) Let $\mathfrak{X}$ be a separable Banach space and Θ be a nonempty subset of $\mathfrak{X}$. Assume that $E : \mho \times \Theta \to \Theta$ is a random operator. Then E is called

(i) nonexpansive random operator if

$$\|E(\ell, u_0) - E(\ell, \upsilon_0)\| \leq \|u_0 - \upsilon_0\| \text{ for all } u_0, \upsilon_0 \in \Theta \text{ and for each } \ell \in \mho,$$

(ii) asymptotically nonexpansive random operator if there exists a sequence of measurable functions $r_m : \mho \to [1, \infty)$ with $\lim_{m\to\infty} r_m(\ell) = 1$ for each $\ell \in \mho$ such that

$$\|E^m(\ell, u_0) - E^m(\ell, \upsilon_0)\| \leq r_m(\ell)\|u_0 - \upsilon_0\|, \forall u_0, \upsilon_0 \in \Theta, m \in \mathbb{N},$$

(iii) asymptotically quasi-nonexpansive random operator if there exists a sequence of measurable functions $r_m : \mho \to [0, \infty)$ with $\lim_{m\to\infty} r_m(\ell) = 0, \forall \ell \in \mho$ such that

$$\|E^m(\ell, \eta(\ell)) - p(\ell)\| \leq (1 + r_m(\ell)) \|\eta(\ell) - p(\ell)\|,$$

where $p : \mho \to \Theta$ is a random fixed point of E and $\eta : \mho \to \Theta$ is a measurable mapping.

(iv) uniformly L-Lipschitzian random operator if for all $u_0, \upsilon_0 \in \Theta$ and for all $\ell \in \mho$

$$\|E^m(\ell, u_0) - E^m(\ell, \upsilon_0)\| \leq L\|u_0 - \upsilon_0\|,$$

where, $m \geq 1$ and $L > 0$.

(v) semi-compact random operator if for a sequence of measurable mappings $\{\varrho_m\}$ from $\mho$ to Θ, with $\lim_{m\to\infty} \|\varrho_m(\ell) - E(\ell, \varrho_m(\ell))\| = 0$ for all $\ell \in \mho$, we have a subsequence $\{\varrho_{m_k}\}$ of $\{\varrho_m\}$ such that $\varrho_{m_k}(\ell) \to \varrho(\ell)$ for each $\ell \in \mho$, where ϱ is a measurable mapping from $\mho$ to Θ.

In 1995, Choudhury gave the random Ishikawa iteration method and he proved some random fixed point theorems using this method in Hilbert spaces. Thus he contributed to the development of random iteration schemes. Later, some authors introduced different iteration methods for random fixed points of different operator classes ([2], [5], [6], [7], [8], [14], [15]). In 2005, Beg and Abbas [2] introduced the following implicit iteration process for weakly contractive and asymptotically nonexpansive random operators in Banach spaces. They also showed that this iteration method converges to the common random fixed point of a finite family of asymptotically quasi-nonexpansive random operators in Banach spaces.

Throughout the rest of this paper, I denote the set of the first $\aleph$ natural numbers, that is, $I = \{1, 2, \ldots, \aleph\}$. Also, $F = \bigcap_{i=1}^{\aleph} [RF(\mathcal{S}_i) \cap RF(E_i) \cap RF(\mathcal{K}_i)]$ shows the set of common fixed points of three finite families of mappings $\{\mathcal{S}_i : i \in I\}$, $\{E_i : i \in I\}$ and $\{\mathcal{K}_i : i \in I\}$.

Let $E_i : \mho \times \Theta \to \Theta$ be a finite family of random operators and $\varrho_0 : \mho \to \Theta$ be any measurable function. Let us define the sequence of functions $\{\varrho_m\}$ as follows:

$$\varrho_m(\ell) = \alpha_m \varrho_{m-1}(\ell) + (1 - \alpha_m) E_{i(m)}^{k(m)}(\ell, \varrho_m(\ell)) \tag{1}$$

where $m = (k-1)\aleph + i$.

In 2007, Plubtieng et al. [14] introduced the following implicit iteration method and they obtained some convergence results for a common random fixed point of a finite family of asymptotically quasi-nonexpansive random operators under some conditions in uniformly convex separable Banach spaces.

Let $E_i : \mho \times \Theta \to \Theta$ be a finite family of random operators and $\varrho_0 : \mho \to \Theta$ be any measurable function. Also, let's assume that the sequence $\{f_m\}$ consists of measurable mappings from $\mho$ to Θ. For all $m \geq 1$ and $\forall \ell \in \mho$,

$$\varrho_m(\ell) = \alpha_m \varrho_{m-1}(\ell) + (1 - \alpha_m) E_{i(m)}^{k(m)}(\ell, \varrho_m(\ell)) + f_m(\ell)$$

where $m = (k-1)\aleph + i$ and each $\{f_m(\ell)\}$ is summable sequence in Θ, that is, the series $\sum_{m=1}^{\infty} \|f_m(\ell)\|$ is convergent.

Afterwards, Banerjee and Choudhury [1] constructed an implicit random iterative process with errors for a finite family of asymptotically nonexpansive random operators in real Banach space. They also proved that this process converges to the common random fixed point of such operators in the setting of uniformly convex Banach spaces. Their iteration process is as follows:

Let $E_i : \mho \times \Theta \to \Theta$ be a finite family of random operators and $\varrho_0 : \mho \to \Theta$ be a measurable function. For all $m \geq 1$ and $\forall \ell \in \mho$,

$$\begin{aligned} \varrho_m(\ell) &= \alpha_m \varrho_{m-1}(\ell) + \beta_m E_{i(m)}^{k(m)}(\ell, \eta_m(\ell)) + \gamma_m f_m(\ell) \\ \eta_m(\ell) &= a_m \varrho_m(\ell) + b_m E_{i(m)}^{k(m)}(\ell, \varrho_m(\ell)) + c_m g_m(\ell), \end{aligned} \tag{2}$$

where $\{\alpha_m\}, \{\beta_m\}, \{\gamma_m\}, \{a_m\}, \{b_m\}, \{c_m\}$ are sequences in $[0,1]$ with $\alpha_m + \beta_m + \gamma_m = a_m + b_m + c_m = 1$ and $\{f_m\}, \{g_m\}$ are bounded sequences of measurable functions from $\mho$ to Θ.

Based on the above studies, we first present the idea of the generalized asymptotically nonexpansive random operators. We also give an implicit iteration method using three finite families of these operator classes. Then, we obtain some convergence results using this iteration process.

Definition 6. *Let $\mathfrak{X}$ be a separable Banach space and Θ be a nonempty subset of $\mathfrak{X}$. Also, let's assume that $E : \mho \times \Theta \to \Theta$ is a random operator. Then E is said to be a generalized asymptotically nonexpansive random operator if there exist two sequences of measurable functions $\mu_m(\ell) : \mho \to [0,\infty)$, $\nu_m(\ell) : \mho \to [0,\infty)$ with $\lim_{m\to\infty} \mu_m(\ell) = 0$ and $\lim_{m\to\infty} \nu_m(\ell) = 0$ for each $\ell \in \mho$ such that*

$$\|E^m(\ell, u_0) - E^m(\ell, \upsilon_0)\| \leq \|u_0 - \upsilon_0\| + \mu_m(\ell)\|u_0 - \upsilon_0\| + \nu_m(\ell)$$

for all $u_0, \upsilon_0 \in \Theta$ and for each $\ell \in \mho$.

Remark 1. *From above definitions, we can see that every asymptotically nonexpansive random operator is generalized asymptotically nonexpansive random operator. But, its converse is not true in general. We know also that every asymptotically nonexpansive random operator with $RF(T) \neq \emptyset$ is asymptotically quasi-nonexpansive random operator and every asymptotically nonexpansive and asymptotically quasi-nonexpansive random operator is uniformly L-Lipschitzian random operator.*

Definition 7. *Let $\mathfrak{X}$ be a separable Banach space and Θ be a nonempty subset of $\mathfrak{X}$ and $\mathcal{S}_i, E_i, \mathcal{K}_i : \mho \times \Theta \to \Theta$ be three finite families of random operators. Also, suppose that $\varrho_0 : \mho \to \Theta$ is a measurable function.*

Then our iteration method with errors is defined as follows:

$$\begin{cases} \varrho_1(\ell) = \alpha_1 \mathcal{S}_1(\ell, \varrho_0(\ell)) + \beta_1 E_1\left(\ell, a_1\varrho_1(\ell) + b_1\mathcal{K}_1\left(\ell, \varrho_1(\ell)\right) + c_1 g_1(\ell)\right) \\ \qquad +\gamma_1 f_1(\ell) \\ \varrho_2(\ell) = \alpha_2 \mathcal{S}_2(\ell, \varrho_1(\ell)) + \beta_2 E_2\left(\ell, a_2\varrho_2(\ell) + b_2\mathcal{K}_2\left(\ell, \varrho_2(\ell)\right) + c_2 g_2(\ell)\right) \\ \qquad +\gamma_2 f_2(\ell) \\ \ldots \\ \varrho_{\aleph}(\ell) = \alpha_{\aleph} \mathcal{S}_{\aleph}(\ell, \varrho_{\aleph-1}(\ell)) + \beta_{\aleph} E_{\aleph}(\ell, a_{\aleph}\varrho_{\aleph}(\ell) + b_{\aleph}\mathcal{K}_{\aleph}\left(\ell, \varrho_{\aleph}(\ell)\right) \\ \qquad +c_{\aleph} g_{\aleph}(\ell)) + \gamma_{\aleph} f_{\aleph}(\ell) \\ \varrho_{\aleph+1}(\ell) = \alpha_{\aleph+1} \mathcal{S}_{\aleph+1}(\ell, \varrho_{\aleph}(\ell)) + \beta_{\aleph+1} E_1^2(\ell, a_{\aleph+1}\varrho_{\aleph+1}(\ell) \\ \qquad\quad +b_{\aleph+1}\mathcal{K}_1^2\left(\ell, \varrho_{\aleph+1}(\ell)\right) + c_{\aleph+1} g_{\aleph+1}(\ell)) + \gamma_{\aleph+1} f_{\aleph+1}(\ell) \\ \ldots \\ \varrho_{\aleph}(\ell) = \alpha_{2\aleph} \mathcal{S}_{2\aleph}(\ell, \varrho_{2\aleph-1}(\ell)) + \beta_{2\aleph} E_{\aleph}^2(\ell, a_{2\aleph}\varrho_{2\aleph}(\ell) \\ \qquad +b_{2\aleph}\mathcal{K}_{\aleph}^2\left(\ell, \varrho_{\aleph}(\ell)\right) + c_{2\aleph} g_{2\aleph}(\ell)) + \gamma_{2\aleph} f_{\aleph}(\ell) \\ \varrho_{2\aleph+1}(\ell) = \alpha_{2\aleph+1} \mathcal{S}_{2\aleph+1}(\ell, \varrho_{2\aleph}(\ell)) + \beta_{2\aleph+1} E_1^3(\ell, a_{2\aleph+1}\varrho_{2\aleph+1}(\ell) \\ \qquad\quad +b_{2\aleph+1}\mathcal{K}_1^3\left(\ell, \varrho_{2\aleph+1}(\ell)\right) + c_{2\aleph+1} g_{2\aleph+1}(\ell)) + \gamma_{2\aleph+1} f_{2\aleph+1}(\ell) \\ \ldots \end{cases}$$

We can write compact form of above iteration as follows:

$$\begin{cases} \varrho_m(\ell) = \alpha_m \mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{m-1}(\ell)) + \beta_m E_{i(m)}^{k(m)}\left(\ell, \eta_m(\ell)\right) + \gamma_m f_m(\ell) \\ \eta_m(\ell) = a_m \varrho_m(\ell) + b_m \mathcal{K}_{i(m)}^{k(m)}\left(\ell, \varrho_m(\ell)\right) + c_m g_m(\ell), \quad m \geq 1, \quad \forall \ell \in \mho \end{cases} \tag{3}$$

where $\{\alpha_m\}, \{\beta_m\}, \{\gamma_m\}, \{a_m\}, \{b_m\}, \{c_m\}$ are sequences in $[0,1]$ with $\alpha_m + \beta_m + \gamma_m = a_m + b_m + c_m = 1$ and $\{f_m\}, \{g_m\}$ are bounded sequences of measurable functions from $\mho$ to Θ.

Lemma 1. ([18]) Let $\{\mu_m\}, \{v_m\}$ and $\{\delta_m\}$ be sequences of nonnegative real numbers such that

$$\mu_{m+1} \leq (1 + \delta_m)\, \mu_m + v_m.$$

If $\sum \delta_m < \infty$ and $\sum v_m < \infty$, then

(*i*) $\lim_{m\to\infty} \mu_m$ exists,
(*ii*) $\lim_{m\to\infty} \mu_m = 0$ whenever $\liminf_{m\to\infty} \mu_m = 0$.

Lemma 2. ([16]) Let $\mathfrak{X}$ be a uniformly convex Banach space and $\{u_{0m}\}$ and $\{\upsilon_{0m}\}$ be two sequences in $\mathfrak{X}$ such that $\limsup_{m\to\infty}\|u_{0m}\|\leq r$, $\limsup_{m\to\infty}\|\upsilon_{0m}\|\leq r$ and $\lim_{m\to\infty}\|\ell_m u_{0m}+(1-\ell_m)\upsilon_{0m}\|=r$ satisfying for any $r\geq 0$. Also, let's assume that $0<p\leq\ell_m\leq q<1$. Then $\lim_{m\to\infty}\|u_{0m}-\upsilon_{0m}\|=0$.

2. Main results

Now, we will give some convergence theorems for generalized asymptotically nonexpansive random operators using our implicit random iteration scheme with errors.

Theorem 1. *Let $\mathfrak{X}$ be a separable Banach space and Θ be a nonempty closed convex subset of $\mathfrak{X}$. Let $\mathcal{S}_i, E_i, \mathcal{K}_i : \mho\times\Theta\to\Theta$ be generalized asymptotically nonexpansive random operators with the sequence of measurable mappings $\{r_{i_m}\}$: $\mho\to[1,\infty)$ satisfying $\sum_{m=1}^{\infty}(r_{i_m}(\ell)-1)<\infty$ for all $\ell\in\mho$ and for all $i\in I$. Suppose that $F\neq\emptyset$. Let the iteration $\{\varrho_m\}$ be defined by (3) with the additional assumption $0<\alpha\leq\alpha_m,\beta_m\leq\beta<1$ and $\sum_{m=1}^{\infty}\gamma_m<\infty,\sum_{m=1}^{\infty}c_m<\infty$. Then $\{\varrho_m\}$ converges strongly to a common random fixed point of the random operators $\mathcal{S}_i$,E_i and $\mathcal{K}_i$ if and only if for all $\ell\in\mho,\liminf_{m\to\infty}d(\varrho_m(\ell),F)=0$, where $d(\varrho_m(\ell),F)=\inf\{\|\varrho_m(\ell)-\varrho(\ell)\|:\varrho\in F\}$.*

Proof. Let ϱ be a fixed point, that is $\varrho\in F$. Since $\{f_m\}$ and $\{g_m\}$ are bounded sequences, we can write for each $\ell\in\mho$,

$$M(\ell)=\sup_{m\geq1}\|f_m(\ell)-\varrho(\ell)\|\vee\sup_{m\geq1}\|g_m(\ell)-\varrho(\ell)\|.$$

It is clear that $M(\ell)<\infty$ for each $\ell\in\mho$. Also assume that $r_m(\ell)=\{\max r_{i_m}(\ell): i=1,2,\ldots,\aleph\}$ for each $m\geq 1$. From the condition $\sum_{m=1}^{\infty}(r_{i_m}(\ell)-1)<\infty$ for each $\ell\in\mho$, we have that $\sum_{m=1}^{\infty}(r_m(\ell)-1)<\infty$. Using (3), we obtain that

$$\begin{aligned}
\|\eta_m(\ell)-\varrho(\ell)\| &= \left\|a_m\varrho_m(\ell)+b_m\mathcal{K}_{i(m)}^{k(m)}(\ell,\varrho_m(\ell))+c_m g_m(\ell)-\varrho(\ell)\right\| \qquad (4)\\
&\leq a_m\|\varrho_m(\ell)-\varrho(\ell)\|+b_m\left\|\mathcal{K}_{i(m)}^{k(m)}(\ell,\varrho_m(\ell))-\varrho(\ell)\right\|\\
&\quad +c_m\|g_m(\ell)-\varrho(\ell)\|\\
&\leq a_m\|\varrho_m(\ell)-\varrho(\ell)\|+b_m r_{k(m)}(\ell)\|\varrho_m(\ell)-\varrho(\ell)\|\\
&\quad +b_m v_m(\ell)+c_m M(\ell)\\
&= a_m\|\varrho_m(\ell)-\varrho(\ell)\|+b_m(1+\mu_m(\ell))\|\varrho_m(\ell)-\varrho(\ell)\|\\
&\quad +b_m v_m(\ell)+c_m M(\ell),\\
&\leq (1+\mu_m(\ell))\|\varrho_m(\ell)-\varrho(\ell)\|+b_m v_m(\ell)+c_m M(\ell).
\end{aligned}$$

where $\mu_m(\ell)=r_{k(m)}(\ell)-1$.Also,

$$\begin{aligned}
\|\varrho_m(\ell)-\varrho(\ell)\| &= \left\|\alpha_m\mathcal{S}_{i(m)}^{k(m)}(\ell,\varrho_{m-1}(\ell)+\beta_m E_{i(m)}^{k(m)}(\ell,\eta_m(\ell))+\gamma_m f_m(\ell)-\varrho(\ell)\right\|\\
&\leq \alpha_m\left\|\mathcal{S}_{i(m)}^{k(m)}(\ell,\varrho_{m-1}(\ell)-\varrho(\ell)\right\|+\beta_m\left\|E_{i(m)}^{k(m)}(\ell,\eta_m(\ell))-\varrho(\ell)\right\|
\end{aligned}$$

$$
\begin{aligned}
& +\gamma_m \|f_m(\ell)-\varrho(\ell)\| \\
\leq\ & \alpha_m r_{k(m)}(\ell)\left\|\varrho_{m-1}(\ell)-\varrho(\ell)\right\|+\alpha_m v_m(\ell) \\
& +\beta_m r_{k(m)}(\ell)\|\eta_m(\ell)-\varrho(\ell)\|+\gamma_m M(\ell)+\beta_m v_m(\ell) \\
\leq\ & \alpha_m(1+\mu_m(\ell))\left\|\varrho_{m-1}(\ell)-\varrho(\ell)\right\| \\
& +\beta_m(1+\mu_m(\ell))\left[(1+\mu_m(\ell))\|\varrho_m(\ell)-\varrho(\ell)\|+c_m M(\ell)\right] \\
& +\alpha_m v_m(\ell)+\beta_m v_m(\ell)+\gamma_m M(\ell) \\
=\ & \alpha_m(1+\mu_m(\ell))\left\|\varrho_{m-1}(\ell)-\varrho(\ell)\right\| \\
& +\beta_m(1+\mu_m(\ell))^2\|\varrho_m(\ell)-\varrho(\ell)\| \\
& +\beta_m c_m(1+\mu_m(\ell))M(\ell)+\beta_m v_m(\ell)+\beta_m(1+\mu_m(\ell))b_m v_m(\ell) \\
& +\alpha_m v_m(\ell)+\gamma_m M(\ell) \\
\leq\ & \alpha_m(1+\mu_m(\ell))\left\|\varrho_{m-1}(\ell)-\varrho(\ell)\right\| \\
& +(1-\alpha_m)(1+p_m(\ell))\|\varrho_m(\ell)-\varrho(\ell)\| \\
& +\left[\beta_m c_m(1+\mu_m(\ell))+\gamma_m\right]M(\ell)+\beta_m v_m(\ell) \\
& +\beta_m(1+\mu_m(\ell))b_m v_m(\ell)+\alpha_m v_m(\ell), \\
\leq\ & \alpha_m\left\|\varrho_{m-1}(\ell)-\varrho(\ell)\right\|+(1-\alpha_m+p_m(\ell))\|\varrho_m(\ell)-\varrho(\ell)\| \\
& +\left[\beta_m c_m(1+\mu_m(\ell))+\gamma_m\right]M(\ell)+\beta_m v_m(\ell) \\
& +\beta_m(1+\mu_m(\ell))b_m v_m(\ell)+\alpha_m v_m(\ell).
\end{aligned}
$$

where $p_m(\ell)=2\mu_m(\ell)+\mu_m(\ell)^2$. This implies that

$$
\begin{aligned}
\|\varrho_m(\ell)-\varrho(\ell)\| \leq\ & \left\|\varrho_{m-1}(\ell)-\varrho(\ell)\right\|+\frac{p_m(\ell)}{\alpha_m}\|\varrho_m(\ell)-\varrho(\ell)\| \\
& +\frac{\beta_m c_m(1+\mu_m(\ell))+\gamma_m}{\alpha_m}M(\ell)+v_m(\ell) \\
& +\frac{\beta_m v_m(\ell)+\beta_m(1+\mu_m(\ell))b_m v_m(\ell)}{\alpha_m} \\
\leq\ & \left\|\varrho_{m-1}(\ell)-\varrho(\ell)\right\|+\frac{p_m(\ell)}{\alpha}\|\varrho_m(\ell)-\varrho(\ell)\| \\
& +\frac{\beta_m c_m(1+\mu_m(\ell))+\gamma_m}{\alpha}M(\ell)+v_m(\ell) \\
& +\frac{\beta_m v_m(\ell)+\beta_m(1+\mu_m(\ell))b_m v_m(\ell)}{\alpha}
\end{aligned}
$$

and

$$
\|\varrho_m(\ell)-\varrho(\ell)\| \leq \frac{\alpha}{\alpha-p_m(\ell)}v_m(\ell)\left\|\varrho_{m-1}(\ell)-\varrho(\ell)\right\| \tag{5}
$$

$$
\begin{aligned}
&+\left(\frac{\begin{array}{c}\beta_m c_m\left(1+\mu_m(\ell)\right)+\gamma_m+\beta_m v_m(\ell)\\ +\beta_m\left(1+\mu_m(\ell)\right)b_m v_m(\ell)+v_m(\ell)\end{array}}{\alpha-p_m(\ell)}\right)M(\ell)\\
=\;&\left(1+\frac{p_m(\ell)}{(\alpha-p_m(\ell))v_m(\ell)}\right)\left\|\varrho_{m-1}(\ell)-\varrho(\ell)\right\|\\
&+\left(\frac{\begin{array}{c}\beta_m c_m\left(1+\mu_m(\ell)\right)+\gamma_m+\beta_m v_m(\ell)\\ +\beta_m\left(1+\mu_m(\ell)\right)b_m v_m(\ell)+v_m(\ell)\end{array}}{\alpha-p_m(\ell)}\right)M(\ell).
\end{aligned}
$$

From the condition $\sum_{m=1}^{\infty}\left(r_{k(m)}(\ell)-1\right)<\infty$ for each $\ell\in\mho$, we know that $\sum_{m=1}^{\infty}\mu_m(\ell)<\infty$ and hence $\sum_{m=1}^{\infty}p_m(\ell)<\infty$. So $\lim_{m\to\infty}p_m(\ell)=0$ for each $\ell\in\mho$. From the definition of generalized asymptotically nonexpansive random operator, we also have $\lim_{m\to\infty}v_m(\ell)=0$ for each $\ell\in\mho$. Then for $\ell\in\mho$, there exists $m_1\in\aleph$ such that $p_m(\ell)<\frac{\alpha}{2}$ for all $m\geq m_1$. Thus from (5) we have that, for all $m\geq m_1$

$$
\begin{aligned}
\left\|\varrho_m(\ell)-\varrho(\ell)\right\| \;\leq\;&\left(1+2\frac{p_m(\ell)}{\alpha v_m(\ell)}\right)\left(1+\mu_m(\ell)\right)\left\|\varrho_{m-1}(\ell)-\varrho(\ell)\right\| \qquad (6)\\
&+\left(\frac{\begin{array}{c}\beta_m c_m\left(1+\mu_m(\ell)\right)+\gamma_m+\beta_m v_m(\ell)\\ +\beta_m\left(1+\mu_m(\ell)\right)b_m v_m(\ell)+v_m(\ell)\end{array}}{\alpha}\right)2M(\ell)\\
=\;&\left(1+\lambda_m(\ell)\right)\left\|\varrho_{m-1}(\ell)-\varrho(\ell)\right\|+\sigma_m(\ell),
\end{aligned}
$$

where

$$\lambda_m(\ell)=2\frac{p_m(\ell)}{\alpha v_m(\ell)}\left(1+\mu_m(\ell)\right)+\mu_m(\ell)$$

and

$$\sigma_m(\ell)=\frac{\beta_m c_m\left(1+\mu_m(\ell)\right)+\gamma_m+\beta_m v_m(\ell)+\beta_m\left(1+\mu_m(\ell)\right)b_m v_m(\ell)+v_m(\ell)}{\alpha}2M(\ell).$$

Therefore $\sum_{m=1}^{\infty}\lambda_m(\ell)<\infty$ and $\sum_{m=1}^{\infty}\sigma_m(\ell)<\infty$. This implies that

$$d\left(\varrho_m(\ell),F\right)\leq 1+\lambda_m(\ell)d\left(\varrho_{m-1}(\ell),F\right)+\sigma_m(\ell).$$

Using Lemma 2, we obtain that $\lim_{m\to\infty}d\left(\varrho_m(\ell),F\right)$ exists for each $\ell\in\mho$. Moreover, from the condition of the theorem we have for all $\ell\in\mho$,

$$\lim_{m\to\infty}d\left(\varrho_m(\ell),F\right)=0.$$

We can see that the sequence $\{\varrho_m(\ell)\}$ is a Cauchy sequence for each $\ell\in\mho$ using a similar method as in [2]. Therefore $\varrho_m(\ell)\to p(\ell)$ as $m\to\infty$ for each $\ell\in\mho$, where $p:\mho\to F$. Next, we will prove that $p\in F$. Since for each $\ell\in\mho$, $\varrho_m(\ell)\to p(\ell)$ as $m\to\infty$ there exists $m_3\in\aleph$ such that $\left\|\varrho_m(\ell)-p(\ell)\right\|<\frac{\epsilon}{3(1+r_1(\ell))}$ for all

$m \geq m_3$. Since $\lim_{m\to\infty} d(\varrho_m(\ell), F) = 0$ for each $\ell \in \mho$, there exists $m_4 \in \aleph$ such that $d(\varrho_m(\ell), F) < \frac{\epsilon}{3(1+r_1(\ell))}$ for all $m \geq m_4$. So there exists $\varrho^* \in F$ such that $\|\varrho_m(\ell) - \varrho^*(\ell)\| \leq \frac{\epsilon(\ell))}{3(1+r_1(\ell))}$ for all $m \geq m_4$. Since $\lim_{m\to\infty} \nu_m(\ell) = 0$ for each $\ell \in \mho$, there exists $m_5 \in \aleph$ such that $v_m(\ell) < \frac{\epsilon}{3(1+r_1(\ell))}$ for all $m \geq m_5$. Let $m_6 = \max\{m_3, m_4, m_5\}$. Now for all $l \in I$ and for all $m \geq m_6$

$$\begin{aligned}
\|\mathcal{S}_l(\ell, p(\ell)) - p(\ell)\| &\leq \|\mathcal{S}_l(\ell, p(\ell)) - \varrho^*(\ell)\| + \|\varrho^*(\ell) - p(\ell)\| \\
&\leq \|\mathcal{S}_l(\ell, p(\ell)) - \mathcal{S}_l(\ell, \varrho^*(\ell))\| + \|\varrho^*(\ell) - p(\ell)\| \\
&\leq r_1(\ell)\|\varrho^*(\ell) - p(\ell)\| + v_1(\ell) + \|\varrho^*(\ell) - p(\ell)\| \\
&= (1 + r_1(\ell))\|\varrho^*(\ell) - p(\ell)\| + v_1(\ell) \\
&\leq (1 + r_1(\ell))\|\varrho^*(\ell) - \varrho_m(\ell)\| + (1 + r_1(\ell))\|\varrho_m(\ell) - p(\ell)\| \\
&\quad + (1 + r_1(\ell))\, v_1(\ell) \\
&< (1 + r_1(\ell))\frac{\epsilon}{3(1 + r_1(\ell))} + (1 + r_1(\ell))\frac{\epsilon}{3(1 + r_1(\ell))} \\
&\quad + (1 + r_1(\ell))\frac{\epsilon}{3(1 + r_1(\ell))} \\
&= \epsilon
\end{aligned}$$

which implies that $\mathcal{S}_l(\ell, p(\ell)) = p(\ell)$ for all $l \in I$ and for each $\ell \in \mho$. Similarly, we can show that $E_l(\ell, p(\ell)) = p(\ell)$ and $\mathcal{K}_l(\ell, p(\ell)) = p(\ell)$ for all $l \in I$ and for each $\ell \in \mho$. Therefore, we can say that $p \in F$. That is, $\{\varrho_m\}$ converges strongly to a common random fixed point of $\mathcal{S}_i$, E_i and $\mathcal{K}_i$. □

Lemma 3. *Let $\mathfrak{X}$ be a uniformly convex separable Banach space and Θ be a nonempty closed convex subset of $\mathfrak{X}$. Let $\mathcal{S}_i, E_i, \mathcal{K}_i : \mho \times \Theta \to \Theta$ be uniformly L-Lipschitzian generalized asymptotically nonexpansive random operators with the sequence of measurable mappings $\{r_{i_m}\} : \mho \to [1, \infty)$ satisfying $\sum_{m=1}^{\infty}(r_{i_m}(\ell) - 1) < \infty$ for each $\ell \in \mho$ and for all $i \in I$. Suppose that $F \neq \emptyset$. Let the iteration $\{\varrho_m\}$ be defined by (3) with the additional assumption $0 < \alpha \leq \alpha_m, \beta_m \leq \beta < 1$ and $\sum_{m=1}^{\infty}\gamma_m < \infty, \sum_{m=1}^{\infty} c_m < \infty$. Then*

$$\lim_{m\to\infty}\|\varrho_m(\ell) - \mathcal{S}_l(\ell, \varrho_m(\ell))\| = 0\ ,\ \lim_{m\to\infty}\|\varrho_m(\ell) - E_l(\ell, \varrho_m(\ell))\| = 0$$

and

$$\lim_{m\to\infty}\|\varrho_m(\ell) - \mathcal{K}_l(\ell, \varrho_m(\ell))\| = 0$$

for each $\ell \in \mho$ and for all $l = 1, 2, \ldots, \aleph$.

Proof. Let $\varrho \in F$ be arbitrary. Since $\{f_m\}, \{g_m\}$ are bounded sequences of measurable functions from $\mho$ to Θ, so we can write as follows,

$$M(\ell) = \sup_{m\geq 1}\|f_m(\ell) - \varrho(\ell)\| \vee \sup_{m\geq 1}\|g_m(\ell) - \varrho(\ell)\|.$$

From above the equality, we have $M(\ell) < \infty$ for each $\ell \in \mho$. Assume that $r_m(\ell) = \{\max r_{i_m}(\ell) : i = 1, 2, \ldots, \aleph\}$ for each $m \geq 1$. This implies that $\sum_{m=1}^{\infty}(r_m(\ell) - 1) <$

∞ for each $\ell \in \mho$. Using (6) we know that

$$\|\varrho_m(\ell) - \varrho(\ell)\| \leq (1 + \lambda_m(\ell)) \left\|\varrho_{m-1}(\ell) - \varrho(\ell)\right\| + \sigma_m(\ell),$$

where $\sum_{m=1}^{\infty} \lambda_m(\ell) < \infty$ and $\sum_{m=1}^{\infty} \sigma_m(\ell) < \infty$. From Lemma 1, we obtain that $\lim_{m\to\infty} \|\varrho_m(\ell) - \varrho(\ell)\|$ exists for all $\varrho \in F$ and for each $\ell \in \mho$. We suppose that $\lim_{m\to\infty} \|\varrho_m(\ell) - \varrho(\ell)\| = a_\ell$. From (4), we have that

$$\|\eta_m(\ell) - \varrho(\ell)\| \leq (1 + \mu_m(\ell)) \|\varrho_m(\ell) - \varrho(\ell)\| + b_m v_m(\ell) + c_m M(\ell).$$

From the above inequality, we obtain that

$$\limsup_{m\to\infty} \|\eta_m(\ell) - \varrho(\ell)\| \leq a_\ell \text{ for each } \ell \in \mho. \tag{7}$$

Also

$$\begin{aligned} a_\ell &= \lim_{m\to\infty} \|\varrho_m(\ell) - \varrho(\ell)\| \qquad (8)\\ &= \lim_{m\to\infty} \left\|\alpha_m \mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{m-1}(\ell) + \beta_m E_{i(m)}^{k(m)}(\ell, \eta_m(\ell)) + \gamma_m f_m(\ell) - \varrho(\ell)\right\| \\ &= \lim_{m\to\infty} \left\| \begin{array}{c} (1-\beta_m)\left(\mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{m-1}(\ell) - \varrho(\ell) + \gamma_m(f_m(\ell) - \mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{m-1}(\ell)))\right) \\ +\beta_m\left(E_{i(m)}^{k(m)}(\ell, \eta_m(\ell)) - \varrho(\ell) + \gamma_m\left(f_m(\ell) - \mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{m-1}(\ell))\right)\right) \end{array} \right\|. \end{aligned}$$

For all $\ell \in \mho$, we have

$$\begin{aligned} &\left\|\mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{m-1}(\ell) - \varrho(\ell) + \gamma_m\left(f_m(\ell) - \mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{m-1}(\ell))\right)\right\| \\ \leq\ &\left\|\mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{m-1}(\ell)) - \varrho(\ell)\right\| + \gamma_m \left\|f_m(\ell) - \mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{m-1}(\ell))\right\|. \end{aligned}$$

Taking limsup on the both sides of above inequality, we obtain that

$$\begin{aligned} &\limsup_{m\to\infty} \left\| \begin{array}{c} \mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{m-1}(\ell)) - \varrho(\ell) \\ +\gamma_m\left(f_m(\ell) - \mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{m-1}(\ell))\right) \end{array} \right\| \qquad (9) \\ \leq\ &\limsup_{m\to\infty} \left(\left\|\mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{m-1}(\ell)) - \varrho(\ell)\right\| + \gamma_m \left\|f_m(\ell) - \mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{m-1}(\ell))\right\|\right) \\ \leq\ &\limsup_{m\to\infty} \left(\begin{array}{c} (1 + \mu_m(\ell)) \left\|\varrho_{m-1}(\ell) - \varrho(\ell)\right\| + v_m(\ell) \\ +\gamma_m \left\|f_m(\ell) - \mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{m-1}(\ell))\right\| \end{array} \right) = a_\ell. \end{aligned}$$

Also, we can write the following inequality

$$\begin{aligned} &\left\|E_{i(m)}^{k(m)}(\ell, \eta_m(\ell)) - \varrho(\ell) + \gamma_m\left(f_m(\ell) - \mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{m-1}(\ell))\right)\right\| \\ \leq\ &\left\|E_{i(m)}^{k(m)}(\ell, \eta_m(\ell)) - \varrho(\ell)\right\| + \gamma_m \left\|f_m(\ell) - \mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{m-1}(\ell))\right\| \\ \leq\ &r_{k(m)}(\ell) \|\eta_m(\ell) - \varrho(\ell)\| + v_m(\ell) + \gamma_m r_{k(m)}(\ell) \left\|f_m(\ell) - \varrho_{m-1}(\ell)\right\| + v_m(\ell). \end{aligned}$$

Taking again limsup at the above inequality, we get

$$\begin{aligned}
&\limsup_{m\to\infty}\left\|\begin{array}{c} E_{i(m)}^{k(m)}(\ell,\eta_m(\ell))-\varrho(\ell) \\ +\gamma_m\left(f_m(\ell)-\mathcal{S}_{i(m)}^{k(m)}(\ell,\varrho_{m-1}(\ell))\right)\end{array}\right\| \qquad (10)\\
\leq\ &\limsup_{m\to\infty}\left(\begin{array}{c}(1+\mu_m(\ell))\left\|\eta_m(\ell)-\varrho(\ell)\right\| \\ +v_m(\ell)+\gamma_m\left\|\left(f_m(\ell)-\mathcal{S}_{i(m)}^{k(m)}(\ell,\varrho_{m-1}(\ell))\right)\right\|\end{array}\right)\leq a_\ell.
\end{aligned}$$

From (8),(9),(10) and Lemma 2, we obtain that

$$\lim_{m\to\infty}\left\|E_{i(m)}^{k(m)}(\ell,\eta_m(\ell))-\mathcal{S}_{i(m)}^{k(m)}(\ell,\varrho_{m-1}(\ell))\right\|=0 \qquad (11)$$

for each $\ell\in\mho$. For each $\ell\in\mho$, we have

$$\begin{aligned}
&\left\|\varrho_m(\ell)-\mathcal{S}_{i(m)}^{k(m)}(\ell,\varrho_{m-1}(\ell))\right\| \qquad (12)\\
=\ &\left\|\begin{array}{c}\alpha_m\mathcal{S}_{i(m)}^{k(m)}(\ell,\varrho_{m-1}(\ell))+\beta_m E_{i(m)}^{k(m)}(\ell,\eta_m(\ell)) \\ +\gamma_m f_m(\ell)-\mathcal{S}_{i(m)}^{k(m)}(\ell,\varrho_{m-1}(\ell))\end{array}\right\| \\
\leq\ &\beta_m\left\|E_{i(m)}^{k(m)}(\ell,\eta_m(\ell))-\mathcal{S}_{i(m)}^{k(m)}(\ell,\varrho_{m-1}(\ell))\right\| \\
&+\gamma_m\left\|f_m(\ell)-\mathcal{S}_{i(m)}^{k(m)}(\ell,\varrho_{m-1}(\ell))\right\| \\
\to\ &0 \text{ as } m\to\infty.
\end{aligned}$$

Hence for each $\ell\in\mho$ and for all $l\in I$,

$$\lim_{m\to\infty}\left\|\varrho_m(\ell)-\mathcal{S}_{i(m)}^{k(m)}(\ell,\varrho_{m+l}(\ell))\right\|=0.$$

Since

$$\begin{aligned}
\left\|\varrho_m(\ell)-E_{i(m)}^{k(m)}(\ell,\eta_m(\ell))\right\| \leq\ &\left\|\varrho_m(\ell)-\mathcal{S}_{i(m)}^{k(m)}(\ell,\varrho_{m-1}(\ell))\right\| \\
&+\left\|\mathcal{S}_{i(m)}^{k(m)}(\ell,\varrho_{m-1}(\ell))-E_{i(m)}^{k(m)}(\ell,\eta_m(\ell))\right\|,
\end{aligned}$$

by using (11),(12), we obtain that

$$\lim_{m\to\infty}\left\|\varrho_m(\ell)-E_{i(m)}^{k(m)}(\ell,\eta_m(\ell))\right\|=0 \qquad (13)$$

for each $\ell\in\mho$. We also have

$$\begin{aligned}
&\left\|\eta_m(\ell)-\varrho(\ell)\right\| \\
=\ &\left\|a_m\varrho_m(\ell)+b_m\mathcal{K}_{i(m)}^{k(m)}(\ell,\varrho_m(\ell))+c_m g_m(\ell)-\varrho_m(\ell)\right\| \\
\leq\ &b_m\left\|\mathcal{K}_{i(m)}^{k(m)}(\ell,\varrho_m(\ell))-\varrho_m(\ell)\right\|+c_m\left\|g_m(\ell)-\varrho_m(\ell)\right\|.
\end{aligned}$$

Using (7), we have that $\limsup_{m\to\infty}\left\|\eta_m(\ell)-\varrho(\ell)\right\|\leq a_\ell$ for each $\ell\in\mho$. Also, we have

$$\liminf_{m\to\infty}\left\|\eta_m(\ell)-\varrho(\ell)\right\| \leq \liminf_{m\to\infty}\alpha_m r_m(\ell)\left\|\varrho_{m-1}(\ell)-\varrho(\ell)\right\|$$

$$+\beta_m r_m(\ell)\|\eta_m(\ell)-\varrho(\ell)\|+\gamma_m\|f_m(\ell)-\varrho(\ell)\|$$

which implies that

$$a_\ell \le \alpha_m a_\ell + \beta_m \liminf_{m\to\infty}\|\eta_m(\ell)-\varrho_m(\ell)\|.$$

From above inequality,

$$\begin{aligned}\frac{(1-\alpha_m)a_\ell}{\beta_m} &\le \liminf_{m\to\infty}\|\eta_m(\ell)-\varrho_m(\ell)\| \\ a_\ell &\le \liminf_{m\to\infty}\|\eta_m(\ell)-\varrho_m(\ell)\|.\end{aligned}$$

Now

$$\begin{aligned}a_\ell &= \lim_{m\to\infty}\|\eta_m(\ell)-\varrho(\ell)\| \\ &= \lim_{m\to\infty}\left\|a_m\varrho_m(\ell)+b_m\mathcal{K}_{i(m)}^{k(m)}(\ell,\varrho_m(\ell))+c_m g_m(\ell)-\varrho(\ell)\right\| \\ &= \lim_{m\to\infty}\left\|\begin{array}{c}(1-b_m)\left[\varrho_m(\ell)-\varrho(\ell)+c_m g_m(\ell)-\varrho_m(\ell)\right] \\ +b_m\left[\mathcal{K}_{i(m)}^{k(m)}(\ell,\varrho_m(\ell))-\varrho(\ell)+c_m g_m(\ell)-\varrho_m(\ell)\right]\end{array}\right\|.\end{aligned}$$

So

$$\begin{aligned}&\limsup_{m\to\infty}\|\varrho_m(\ell)+\varrho(\ell)+c_m(g_m(\ell)-\varrho_m(\ell))\| \\ \le\ &\limsup_{m\to\infty}\|\varrho_m(\ell)+\varrho(\ell)\|+c_m\|(g_m(\ell)-\varrho_m(\ell))\| \\ \le\ &a_\ell\end{aligned}$$

and

$$\begin{aligned}&\limsup_{m\to\infty}\left\|\mathcal{K}_{i(m)}^{k(m)}(\ell,\varrho_m(\ell))-\varrho(\ell)+c_m g_m(\ell)-\varrho_m(\ell)\right\| \\ \le\ &\limsup_{m\to\infty}\left\|\mathcal{K}_{i(m)}^{k(m)}(\ell,\varrho_m(\ell))-\varrho(\ell)\right\|+c_m\|g_m(\ell)-\varrho_m(\ell)\| \\ \le\ &\limsup_{m\to\infty} r_m(\ell)\|\varrho_m(\ell)-\varrho(\ell)\|+\upsilon_m(\ell)+c_m\|g_m(\ell)-\varrho_m(\ell)\| \\ \le\ &a_\ell.\end{aligned}$$

Taking Lemma 2

$$\left\|\mathcal{K}_{i(m)}^{k(m)}(\ell,\varrho_m(\ell))-\varrho_m(\ell)\right\|\to 0 \text{ as } m\to\infty$$

Using (13), we obtain that

$$\|\varrho_m(\ell)-\eta_m(\ell)\|\to 0 \text{ as } m\to\infty.$$

We have

$$\begin{aligned}&\left\|\mathcal{S}_{i(m)}^{k(m)}(\ell,\varrho_{m-1}(\ell))-E_m(\ell,\varrho_m(\ell))\right\| \qquad (14) \\ \le\ &\left\|\mathcal{S}_{i(m)}^{k(m)}(\ell,\varrho_{m-1}(\ell))-E_{i(m)}^{k(m)}(\ell,\eta_m(\ell))\right\|+\left\|E_{i(m)}^{k(m)}(\ell,\eta_m(\ell))-E_m(\ell,\varrho_m(\ell))\right\|\end{aligned}$$

$$
\begin{aligned}
&\leq \left\|\mathcal{S}_{i(m)}^{k(m)}(\ell,\varrho_{m-1}(\ell)) - E_{i(m)}^{k(m)}(\ell,\eta_m(\ell))\right\| + L\left\|E_{i(m)}^{k(m)-1}(\ell,\eta_m(\ell)) - \varrho_m(\ell)\right\| \\
&= \sigma_m(\ell) + L\left\|E_{i(m)}^{k(m)-1}(\ell,\eta_m(\ell)) - \varrho_m(\ell)\right\|
\end{aligned}
$$

where $\sigma_m(\ell) = \left\|\mathcal{S}_{i(m)}^{k(m)}(\ell,\varrho_{m-1}(\ell)) - E_{i(m)}^{k(m)}(\ell,\eta_m(\ell))\right\|$ for each $\ell \in \mho$. From (11), we get that $\sigma_m(\ell) \to 0$ for each $\ell \in \mho$ as $m \to \infty$. We also have

$$
\begin{aligned}
&\left\|E_{i(m)}^{k(m)-1}(\ell,\eta_m(\ell)) - \varrho_m(\ell)\right\| \qquad (15)\\
\leq\ &\left\|E_{i(m)}^{k(m)-1}(\ell,\eta_m(\ell)) - E_{i(m-\aleph)}^{k(m)-1}\left(\ell,\varrho_{m-\aleph}(\ell)\right)\right\| \\
&+\left\|E_{i(m-\aleph)}^{k(m)-1}\left(\ell,\varrho_{m-\aleph}(\ell)\right) - E_{i(m-\aleph)}^{k(m)-1}\left(\ell,\eta_{m-\aleph}(\ell)\right)\right\| \\
&+\left\|E_{i(m-\aleph)}^{k(m)-1}\left(\ell,\eta_{m-\aleph}(\ell)\right) - \mathcal{S}_{i(m-\aleph)}^{k(m-\aleph)}(\ell,\varrho_{(m-\aleph)-1}(\ell))\right\| \\
&+\left\|\mathcal{S}_{i(m-\aleph)}^{k(m-\aleph)}(\ell,\varrho_{(m-\aleph)-1}(\ell)) - \varrho_m(\ell)\right\|.
\end{aligned}
$$

for each $m > \aleph, m = (m-\aleph)(modN)$. Again since $m = (k(m)-1)\aleph + i(m)$, we have $k(m-\aleph) = k(m) - 1$ and $i(m-\aleph) = i(m)$. Using (15), we can write

$$
\begin{aligned}
&\left\|E_{i(m)}^{k(m)-1}(\ell,\eta_m(\ell)) - \varrho_m(\ell)\right\| \qquad (16)\\
\leq\ &\left\|E_{i(m-\aleph)}^{k(m-\aleph)}(\ell,\eta_m(\ell)) - E_{i(m-\aleph)}^{k(m-\aleph)}\left(\ell,\varrho_{m-\aleph}(\ell)\right)\right\| \\
&+\left\|E_{i(m-\aleph)}^{k(m-\aleph)}\left(\ell,\varrho_{m-\aleph}(\ell)\right) - E_{i(m-\aleph)}^{k(m-\aleph)}\left(\ell,\eta_{m-\aleph}(\ell)\right)\right\| \\
&+\left\|E_{i(m-\aleph)}^{k(m-\aleph)}\left(\ell,\eta_{m-\aleph}(\ell)\right) - \mathcal{S}_{i(m-\aleph)}^{k(m-\aleph)}(\ell,\varrho_{(m-\aleph)-1}(\ell))\right\| \\
&+\left\|\mathcal{S}_{i(m-\aleph)}^{k(m-\aleph)}(\ell,\varrho_{(m-\aleph)-1}(\ell)) - \varrho_m(\ell)\right\| \\
\leq\ &L\left\|\eta_m(\ell) - \varrho_{m-\aleph}(\ell)\right\| + L\left\|\varrho_{m-\aleph}(\ell) - \eta_{m-\aleph}(\ell)\right\| + \sigma_{m-\aleph}(\ell) \\
&+\left\|\mathcal{S}_{i(m-\aleph)}^{k(m-\aleph)}(\ell,\varrho_{(m-\aleph)-1}(\ell)) - \varrho_m(\ell)\right\|.
\end{aligned}
$$

Also, we have

$$
\begin{aligned}
\left\|\varrho_m(\ell) - \varrho_{m-1}(\ell)\right\| &= \left\|\begin{array}{c}\alpha_m\mathcal{S}_{i(m)}^{k(m)}(\ell,\varrho_{m-1}(\ell)) + \beta_m E_{i(m)}^{k(m)}(\ell,\eta_m(\ell)) \\ +\gamma_m f_m(\ell) - \varrho_{m-1}(\ell)\end{array}\right\| \\
&\leq \alpha_m\left\|\mathcal{S}_{i(m)}^{k(m)}(\ell,\varrho_{m-1}(\ell)) - \varrho_{m-1}(\ell)\right\| \\
&\quad +\beta_m\left\|E_{i(m)}^{k(m)}(\ell,\eta_m(\ell)) - \varrho_{m-1}(\ell)\right\| \\
&\leq \alpha_m(\left\|\mathcal{S}_{i(m)}^{k(m)}(\ell,\varrho_{m-1}(\ell)) - \varrho_m(\ell)\right\| + \left\|\varrho_m(\ell) - \varrho_{m-1}(\ell)\right\|) \\
&\quad +\beta_m(\left\|E_{i(m)}^{k(m)}\ell,(\eta_m(\ell)) - \varrho_m(\ell)\right\| + \left\|\varrho_m(\ell) - \varrho_{m-1}(\ell)\right\|)
\end{aligned}
$$

and

$$\begin{aligned}
&= \alpha_m \left\|\mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{m-1}(\ell)) - \varrho_m(\ell)\right\| + \beta_m \left\|E_{i(m)}^{k(m)}(\ell, \eta_m(\ell)) - \varrho_m(\ell)\right\| \\
&\quad +(\alpha_m + \beta_m) \left\|\varrho_m(\ell) - \varrho_{m-1}(\ell)\right\| \\
&\Longrightarrow (1 - \alpha_m - \beta_m) \left\|\varrho_m(\ell) - \varrho_{m-1}(\ell)\right\| \leq \alpha_m \left\|\mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{m-1}(\ell)) - \varrho_m(\ell)\right\| \\
&\quad +\beta_m \left\|E_{i(m)}^{k(m)}(\ell, \eta_m(\ell)) - \varrho_m(\ell)\right\| \\
&\Longrightarrow \left\|\varrho_m(\ell) - \varrho_{m-1}(\ell)\right\| \leq \frac{\begin{array}{c}\alpha_m \left\|\mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{m-1}(\ell)) - \varrho_m(\ell)\right\| \\ +\beta_m \left\|E_{i(m)}^{k(m)}(\ell, \eta_m(\ell)) - \varrho_m(\ell)\right\|\end{array}}{(1 - \alpha_m - \beta_m)} \\
&\leq \frac{\begin{array}{c}\alpha_m \left\|\mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{m-1}(\ell)) - \varrho_m(\ell)\right\| \\ +\beta_m \left\|E_{i(m)}^{k(m)}(\ell, \eta_m(\ell)) - \varrho_m(\ell)\right\|\end{array}}{1 - 2\beta_m} \\
&\Longrightarrow \left\|\varrho_m(\ell) - \varrho_{m-1}(\ell)\right\| \to 0 \text{ as } m \to \infty \text{ for each } \ell \in \mho.
\end{aligned}$$

So from (14) and (16) we have for each $\ell \in \mho$,

$$\begin{aligned}
&\left\|\mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{m-1}(\ell)) - E_m\left(\ell, \varrho_m(\ell)\right)\right\| \\
\leq\ & \sigma_m(\ell) + L^2 \left\|\eta_m(\ell) - \varrho_{m-\aleph}(\ell)\right\| + L^2 \left\|\varrho_{m-\aleph}(\ell) - \eta_{m-\aleph}(\ell)\right\| + L\sigma_{m-\aleph}(\ell) \\
& +L \left\|\mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{(m-\aleph)-1}(\ell)) - \varrho_m(\ell)\right\| \\
\leq\ & \sigma_m(\ell) + L^2 \left(\left\|\eta_m(\ell) - \varrho_m(\ell)\right\| + \left\|\varrho_m(\ell) - \varrho_{m-\aleph}(\ell)\right\|\right) + L^2 \left\|\varrho_{m-\aleph}(\ell) - \eta_{m-\aleph}(\ell)\right\| \\
& +L\sigma_{m-\aleph}(\ell) + L \left\|\mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{(m-\aleph)-1}(\ell)) - \varrho_m(\ell)\right\|.
\end{aligned}$$

It follows that

$$\left\|\mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{m-1}(\ell)) - E_m\left(\ell, \varrho_m(\ell)\right)\right\| \to 0 \text{ as } m \to \infty \tag{17}$$

By (17) and (12) we obtain that

$$\begin{aligned}
& \left\|\varrho_m(\ell) - E_m\left(\ell, \varrho_m(\ell)\right)\right\| \qquad (18) \\
\leq\ & \left\|\varrho_m(\ell) - \mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{m-1}(\ell))\right\| + \left\|\mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{m-1}(\ell)) - E_m\left(\ell, \varrho_m(\ell)\right)\right\| \\
\to\ & 0 \text{ as } m \to \infty
\end{aligned}$$

Now for each $l \in \{1, 2, \ldots, \aleph\}$, by using (18) we get that

$$\begin{aligned}
\left\|\varrho_m(\ell) - \mathcal{S}_{m+l}\left(\ell, \varrho_m(\ell)\right)\right\| &\leq \left\|\varrho_m(\ell) - \varrho_{m+l}(\ell)\right\| + \left\|\varrho_{m+l}(\ell) - \mathcal{S}_{m+l}\left(\ell, \varrho_{m+l}(\ell)\right)\right\| \\
&\quad + \left\|\mathcal{S}_{m+l}\left(\ell, \varrho_{m+l}(\ell)\right) - \mathcal{S}_{m+l}\left(\ell, \varrho_m(\ell)\right)\right\| \\
&\leq \left\|\varrho_m(\ell) - \varrho_{m+l}(\ell)\right\| + \left\|\varrho_{m+l}(\ell) - \mathcal{S}_{m+l}\left(\ell, \varrho_{m+l}(\ell)\right)\right\| \\
&\quad +L \left\|\varrho_{m+l}(\ell) - \varrho_m(\ell)\right\|
\end{aligned}$$

$$
\begin{aligned}
&\leq \left\|\varrho_m(\ell)-\varrho_{m+l}(\ell)\right\|+\left\|\varrho_{m+l}(\ell)-\varrho_{m+l-1}(\ell)\right\| \\
&\quad +\left\|\varrho_{m+l-1}(\ell)-\mathcal{S}_{m+l}\left(\ell, \varrho_{m+l}(\ell)\right)\right\| \\
&\quad +L\left\|\varrho_{m+l}(\ell)-\varrho_m(\ell)\right\| \\
&\leq \left\|\varrho_m(\ell)-\varrho_{m+l}(\ell)\right\|+\left\|\varrho_{m+l}(\ell)-\varrho_{m+l-1}(\ell)\right\| \\
&\quad +\left\|\varrho_{m+l-1}(\ell)-\varrho_{m+l}(\ell)\right\| \\
&\quad +\left\|\varrho_{m+l-1}(\ell)-\mathcal{S}_{m+l}\left(\ell, \varrho_{m+l}(\ell)\right)\right\| \\
&\quad +L\left\|\varrho_{m+l}(\ell)-\varrho_m(\ell)\right\| \\
&\rightarrow 0 \text{ as } m \rightarrow \infty \text{ for each } \ell \in \mho.
\end{aligned}
$$

Therefore we have

$$\lim_{m\to\infty}\left\|\varrho_m(\ell)-\mathcal{S}_l\left(\ell, \varrho_m(\ell)\right)\right\|=0$$

for each $\ell \in \mho$ and for each $l \in I$. Similarly we have

$$\lim_{m\to\infty}\left\|\varrho_m(\ell)-E_l\left(\ell, \varrho_m(\ell)\right)\right\|=0 \text{ and } \lim_{m\to\infty}\left\|\varrho_m(\ell)-\mathcal{K}_l\left(\ell, \varrho_m(\ell)\right)\right\|=0$$

for each $\ell \in \mho$ and for each $l \in I$. □

Definition 8. *Let $\mathcal{S}_i, E_i, \mathcal{K}_i : \mho \times \Theta \to \Theta$ be continuous random operators with $F \neq \emptyset$. They is said to satisfy Condition (B^*) if there is a nondecreasing function f on $[0,\infty)$ with $f(0)=0$ and $f(t)>0$ for each $t \in (0,\infty)$ such that for each $\ell \in \mho$*

$$
\begin{aligned}
f(d(u_0(\ell),F)) &\leq \max_{1\leq i\leq \aleph}\{\|u_0(\ell)-\mathcal{S}_i(\ell,u_0(\ell))\|\} \\
\text{or } f(d(u_0(\ell),F)) &\leq \max_{1\leq i\leq \aleph}\{\|u_0(\ell)-E_i(\ell,u_0(\ell))\|\} \\
\text{or } f(d(u_0(\ell),F)) &\leq \max_{1\leq i\leq \aleph}\{\|u_0(\ell)-\mathcal{K}_i(\ell,u_0(\ell))\|\}
\end{aligned}
$$

where $u_0 : \mho \to \Theta$ is a measurable function.

Theorem 2. *Let $\mathfrak{X}$ be a uniformly convex separable Banach space and Θ be a nonempty closed convex subset of $\mathfrak{X}$. Let $\mathcal{S}_i, E_i, \mathcal{K}_i : \mho \times \Theta \to \Theta$ be uniformly L-Lipschitzian generalized asymptotically nonexpansive random operators with the sequence of measurable mappings $\{r_{i_m}\} : \mho \to [1,\infty)$ satisfying $\sum_{m=1}^{\infty}(r_{i_m}(\ell)-1) < \infty$ for each $\ell \in \mho$ and for all $i \in I$. Suppose that $F \neq \emptyset$. Let the iteration $\{\varrho_m\}$ be defined by (3) with the additional assumption $0 < \alpha \leq \alpha_m, \beta_m \leq \beta < 1$ and $\sum_{m=1}^{\infty}\gamma_m < \infty, \sum_{m=1}^{\infty} c_m < \infty$. If the families $\mathcal{S}_i, E_i$ and $\mathcal{K}_i$ satisfies Condition (B^*) for each $\ell \in \mho$, then $\{\varrho_m\}$ converges strongly to a common random fixed point of $\mathcal{S}_i, E_i$ and $\mathcal{K}_i$.*

Proof. From Theorem 1, we know that $\lim_{m\to\infty} d\left(\varrho_m(\ell), F\right)$ exists for each $\ell \in \mho$. Using Lemma 3 and Condition (B^*), we have that

$$\lim_{m\to\infty} f\left(d\left(\varrho_m(\ell), F\right)\right)=0.$$

From definition of f, we have $\lim_{m\to\infty} d\left(\varrho_m(\ell), F\right)=0$. Hence the result follows by Theorem 1. □

Theorem 3. *Let $\mathfrak{X}$ be a uniformly convex separable Banach space and Θ be a nonempty closed convex subset of $\mathfrak{X}$. Let $\mathcal{S}_i, E_i, \mathcal{K}_i : \mho \times \Theta \to \Theta$ be uniformly L-Lipschitzian generalized asymptotically nonexpansive random operators with the sequence of measurable mappings $\{r_{i_m}\} : \mho \to [1,\infty)$ satisfying $\sum_{m=1}^{\infty} (r_{i_m}(\ell) - 1) < \infty$ for each $\ell \in \mho$ and for all $i \in I$. Suppose that $F \neq \emptyset$ and at least one of member of the families $\mathcal{S}_i, E_i$ and $\mathcal{K}_i$ is semi-compact random operator. Let the iteration $\{\varrho_m\}$ be defined by (3) with the additional assumption $0 < \alpha \leq \alpha_m, \beta_m \leq \beta < 1$ and $\sum_{m=1}^{\infty} \gamma_m < \infty, \sum_{m=1}^{\infty} c_m < \infty$, then $\{\varrho_m\}$ converges strongly to a common random fixed point of $\mathcal{S}_i, E_i$ and $\mathcal{K}_i$.*

Proof. From Lemma 3, we know that $\lim_{m\to\infty} \|\varrho_m(\ell) - \mathcal{S}_l(\ell, \varrho_m(\ell))\| = 0$ for each $\ell \in \mho$ and for each $l \in I$. Assume that $\mathcal{S}_1$ is semi-compact random operator. Then there exists a subsequence $\{\varrho_{m_k}(\ell)\}$ of $\{\varrho_m(\ell)\}$ such that $\varrho_{m_k}(\ell) \to \varrho(\ell)$ for each $\ell \in \mho$, where ϱ is a measurable mapping from $\mho$ to Θ. Thus

$$\begin{aligned} \|\varrho(\ell) - \mathcal{S}_l(\ell, \varrho(\ell))\| &= \lim_{k\to\infty} \left\|\varrho_{m_k}(\ell) - \mathcal{S}_l\left(\ell, \varrho_{m_k}(\ell)\right)\right\| \\ &= 0 \quad \text{for each } \ell \in \mho \text{ and for each } l \in I. \end{aligned}$$

It follows that $\varrho \in F$. Since $\{\varrho_m(\ell)\}$ has a subsequence $\{\varrho_{m_k}(\ell)\}$ such that $\varrho_{m_k}(\ell) \to \varrho(\ell)$ for each $\ell \in \mho$, we have that $\liminf_{m\to\infty} d(\varrho_m(\ell), F) = 0$. Hence the result follows by Theorem 1 . □

Remark 2. *i) Theorem 1, Lemma 3 and Theorems 2-3 are also valid for asymptotically nonexpansive random operators and uniformly L-Lipschitzian asymptotically nonexpansive random operators. If we take $\nu_m(\ell) = 0$ for each $\ell \in \mho$ and for all $m \geq 1$, the conclusions of our theorems are immediate.*
ii) Taking $\mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{m-1}(\ell)) = \varrho_{m-1}(\ell)$ for each $\ell \in \mho$ and $\mathcal{K} = E$ at the implicit iteration process (3), this reduces to the iteration process (2). So, Theorem 1, Lemma 3 and Theorems 2-3 extend and improve Theorem 3.1*, Lemma* 3.1 *and Theorems* 3.2-3.3 *of [1] for three finite families of generalized asymptotically nonexpansive random operators.*
iii) Taking $\mathcal{S}_{i(m)}^{k(m)}(\ell, \varrho_{m-1}(\ell)) = \varrho_{m-1}(\ell)$, $f_m(\ell) = 0$ for each $\ell \in \mho$, $a_m = b_m = c_m = 0$ for all $m \in \mathbb{N}$ at the implicit iteration process (3), we get that the iteration process (1). Thus, our results extend Theorem 4.1 *and Theorem* 4.2 *of [2] respectively. Moreover, our results extend and improve the corresponding results of [14].*

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