Hacettepe Journal of
Mathematics & Statistics

Hacet. J. Math. Stat.
Volume XX (x) (2019), 1 – 15
DOI : 10.15672/HJMS.xx

Research Article

Fractional strong metric dimension of convex polytopes and its applications

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Abstract

The fractional versions of various metric related parameters have recently gained importance due to their applications in the fields of sensor networking, robot navigation and linear optimization problems. Convex polytopes are collection of those polytopes of Euclidean space which are their convex subsets. They have key importance in the field of network designing due to their stable and resilient structure which aids optimal data transfer. The identification and removal of components (nodes) of a communication network causing abruption in its flow is of key importance for optimal data transmission. These components are referred as strong resolving neighbourhood (SRNs) in graph theory and assigning least weight to these components aids the computation of fractional strong metric dimension (FSMD). In this paper, we compute FSMD for certain convex polytopes which include $\mathbb{P}_n$, $\mathbb{P}_n^1$ and $\mathbb{P}_n^2$. In this regard, it is shown that for $n \geq 3$, FSMD of $\mathbb{P}_n$ and $\mathbb{P}_n^2$ is n and $\frac{3n}{2}$, respectively. Also, FSMD of $\mathbb{P}_n^1$ is n when n is odd and $\frac{3n}{2}$ when n is even. Finally, an application of FSMD in the context of internet connection networks is furnished.

Mathematics Subject Classification (2020). 05C90, 05C35, 05C12

Keywords. convex polytope, metric dimension, fractional strong metric dimension

1. Introduction and preliminaries

Convex polytopes are fascinating mathematical models that form building blocks of many mathematical theories such as algebraic geometry, combinatorial optimization and linear programming. These structures have captured the attention of mathematicians and scientists due importance and application in many fields, including computer science, economics, physics and chemistry. In this era of automation and computerization, robotics and machineries substitute the human labour for cost efficient production of output required to accommodate the demand of markets with minimal use of these landmarks. In this regard, distance based parameters of graphs have utmost importance and application in several areas. One of the distance based parameters is metric dimension, having applications in real life including network discovery [3], pharmaceutical chemistry [4],

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Received: 30.11.2022; Accepted: 08.04.2024

optimizations [14] and robot navigation. Slater [21] introduced the concept of metric dimension under the name location number in 1975. Later, in 1976, Harary and Melter [8] independently gave the concept and named it metric dimension. In [4], metric dimension was formulated as integer programming problem. Some developments in metric dimension can be seen in [4, 6–8]. Different variants of metric dimension have been introduced by researchers, some of them are local metric dimension [17], strong metric dimension [20], fractional metric dimension [2] and etc. The main focus of researchers was to discuss these variants of metric dimension on different families of graphs. In [20], Sebö and Tannier introduced a stronger variant of metric dimension of a graph known as strong metric dimension. Further in [18], it was identified by Oellermann and Fransen that the calculation of strong metric dimension is NP hard problem. In 2001, Currie and Oellermann [6] used the linear relaxation of the integer programming problem for metric dimension to define the concept of fractional metric dimension. The fractional metric dimension was further established by Arumugam and Mathew [2] in 2012 by introducing the notion of resolving neighbourhoods. In [16], FMD of generalized Jahangir graphs was computed. In 2013, Kang and Yi [13] introduced the notion of FSMD by imposing a stronger condition on fractional metric dimension and also calculated it for some important families of graphs. It was further argued that using the same scheme described in [7], the problem of finding FSMD can be translated as linear programming problem. The localized version of fractional metric dimension have been described in [1] and [15]. A stronger variant of local fractional metric dimension is discussed in [10].

Let $A = (V(A), E(A))$ be a finite, undirected, connected and simple graph where $|E(A)|$ and $|V(A)|$ are known as the order and size of the graph. The open neighbourhood of any vertex $x \in V(A)$ is the collection of all the vertices of A adjacent to x, denoted by $N(x)$. For any two of vertices $x, y \in V(A)$, the distance $d(x, y)$, is the length of shortest $x - y$ path in A also known as geodesic. If for every vertex a in $N(x)$ where $x \in V(A)$, the condition $d(a, y) \leq d(x, y)$ holds, then x is maximally distant from $y \in V(A)$. For the pair of vertices $x, y \in V(A)$, x is said to be mutually maximally distant from y (x MMD y) only if x is maximally distant from y and y is maximally distant from x. For an ordered subset $S = \{x_i; 1 \leq i \leq k\}$ of $V(A)$, we refer to the ordered t-vector $r(x|S) = (d(v, x_i))_{i=1}^{t}$ as representation of x with respect to S. The set S is called a resolving set for the graph A if $r(x|S) = r(y|S)$ implies that $x = y$ for all $x, y \in V(A)$. The cardinality of resolving set of A with least number of elements is known as metric dimension of A, denoted by $\dim(A)$. A vertex z strongly resolves two vertices x_1 and x_2 if x_1 belongs to a shortest $x_2 - z$ path, or if x_2 belongs to a shortest $x_1 - z$ path. A vertex set S of A is a strong resolving set of A if every two distinct vertices of A are strongly resolved by some vertex of S. The cardinality of strong resolving set having minimum number elements is known as strong metric dimension of A, represented by $\mathrm{sdim}(A)$. The set $S\{x_1, x_2\}$ consisting of all vertices z from the vertex set of A such that x_1 lies on $x_2 - z$ geodesic or x_2 lies on $x_1 - z$ geodesic is known as the strong resolving neighbourhood (SRN) of x_1 and x_2. A real valued function $\alpha : V(A) \to [0, 1]$ is a strong resolving function of A if for any distinct pair of vertices $x_1, x_2 \in A$, the weight $\alpha(S\{x_1, x_2\})$ is greater than or equal to 1. The FSMD of A is denoted by $\mathrm{sdim}_{\mathrm{f}}(A)$ and defined as $\min\{\alpha(V(A)) : \alpha$ is a strong resolving function of $A\}$. The SRNs for a graph A with least cardinality and its compliment are given by $\mathcal{S}(A) = \{S^* | \ S^*$ is the SRN such that $|S^*| = \gamma(A)\}$, where $\gamma(A)$ is the cardinality of smallest SRN of A. Also, $\overline{\mathcal{S}}(A) = \{S' | \ S'$ is the SRN of A not in $\mathcal{S}(A)\}$. The technique used in this paper to compute FSMD of a graph is mentioned below:

Theorem 1.1. [11] *Let $\Omega(A) = \mathcal{S}(A) \cup \overline{\mathcal{S}}(A)$ be collection of all SRNs of graph A such that for every distinct pair of vertices $x, y \in V(A)$ $|S\{x, y\} \cap (\cup_{S^* \in \mathcal{S}(A)} S^*)| \geq \gamma(A)$ where $\gamma(A)$ is the cardinality of strong resolving neighbourhoods with minimum number of elements.*

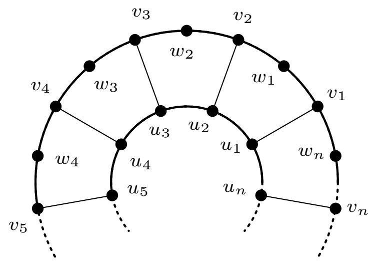

Figure 1. The Convex Polytope $\mathbb{P}_n$

Then

$$\mathrm{sdim}_f(A) = \sum_{t=1}^{\beta(A)} \frac{1}{\gamma(A)}, \ \textit{where } \beta(A) = |\cup_{S^* \in \mathcal{S}(A)} S^*|.$$

Recent developments regarding the computation of metric variants for convex polytopes can be seen in [5], [9], [10] and [19]. Some bounds and results introduced by Kang [12] are mentioned below:

Lemma 1.2. [12] *Let A be a connected graph with order n. Then*

(1) $\mathrm{dim_f}(A) \leq \mathrm{sdim_f}(A)$,
(2) $\mathrm{sdim_f}(A) \leq \mathrm{sdim}(A)$
(3) $S\{a, b\} = \{a, b\}$ *if and only if the vertices a and b are MMD and $\gamma(A) = 2$, where $a \neq b \in V(A)$ and $n \geq 2$.*

The main contributions of the paper are as follows:

1.1. Main results:

The research conducted in this article leads to the following novel results.

Theorem 1.3. *For $n \geq 3$,*

(1) $\mathrm{sdim_f}(\mathbb{P}_n) = n$
(2) $\mathrm{sdim_f}(\mathbb{P}^1_n) = \begin{cases} \frac{3n}{2} & \textit{if } n \textit{ is even} \\ n & \textit{if } n \textit{ is odd} \end{cases}$
(3) $\mathrm{sdim_f}(\mathbb{P}^2_n) = \frac{3n}{2}$

In this paper, we compute FSMD of certain convex polytopes. In section 2 and 3, SRNs of the convex polytopes $\mathbb{P}_n$ and $\mathbb{P}^1_n$ are computed. Section 4 is devoted to the discussion of SRNs of the convex polytope $\mathbb{P}^2_n$. In section 5, FSMD of all convex polytopes discussed above is computed. Section 6 contains application of FSMD in the context of internet connection networks and finally the article is concluded in section 7.

2. Strong resolving neighbourhoods of convex polytope $\mathbb{P}_n$

In this section, SRNs of the convex polytope $\mathbb{P}_n$ will be calculated. The convex polytope $\mathbb{P}_n$ is obtained by attaching adjacent pentagons with a cycle. The vertex and edge set of $\mathbb{P}_n$ are $V(\mathbb{P}_n) = \{u_i, v_i, w_i \mid 1 \leq i \leq n\}$ and $E(\mathbb{P}_n) = \{u_iu_{i+1}, u_iv_i, v_iw_i, w_iv_{i+1} \mid 1 \leq i \leq n\}$, respectively, with the indices taken $\bmod n$. The convex polytope $\mathbb{P}_n$ is shown in Figure 1. In order to compute SRNs of the graph $\mathbb{P}_n$, its vertices are to be pairwise classified depending upon either being on the same cycle or on different cycle. In the following lemmas SRNs of pair of MMD vertices in $\mathbb{P}_n$ are computed.

Lemma 2.1. *Let $\mathbb{P}_n$ be a convex polytope where n is even. Then, $|S\{v_p, v_{p+\frac{n}{2}}\}| = |S\{w_p, w_{p+k}\}| = 2$ where $1 \le p \le n$ and $4 \le p+k \le \frac{n}{2}$. Also, $|(\bigcup\limits_{p=1}^{n} S\{v_p, v_{p+\frac{n}{2}}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{w_p, w_{p+k}\})| = 2n$.*

Proof. It is clear from the construction of the graph $\mathbb{P}_n$ that v_p and $v_{p+\frac{n}{2}}$ are MMD. Also, w_p and w_{p+k} are MMD in $\mathbb{P}_n$ therefore, in view of Lemma 1.2, the SRNs of pair of vertices v_p and $v_{p+\frac{n}{2}}$ is $\{v_p, v_{p+\frac{n}{2}}\}$. Also, the SRN $S\{w_p, w_{p+k}\} = \{w_p, w_{p+k}\}$. Here, $|S\{v_p, v_{p+\frac{n}{2}}\}| = |S\{w_p, w_{p+k}\}| = 2$. From the above it implies that $(\bigcup\limits_{p=1}^{n} S\{v_p, v_{p+\frac{n}{2}}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{w_p, w_{p+k}\}) = \{v_b, w_b \mid 1 \le b \le n\}$. Hence, $|(\bigcup\limits_{p=1}^{n} S\{v_p, v_{p+\frac{n}{2}}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{w_p, w_{p+k}\})| = 2n$. □

Lemma 2.2. *Let $\mathbb{P}_n$ be a convex polytope where n is odd. Then, $|S\{v_p, w_{p+\lfloor\frac{n}{2}\rfloor}\}| = |S\{w_p, w_{p+k}\}| = 2$ where $1 \le p \le n$ and $4 \le p+k \le \lceil\frac{n}{2}\rceil$. Also, $|(\bigcup\limits_{p=1}^{n} S\{v_p, w_{p+\lfloor\frac{n}{2}\rfloor}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{w_p, w_{p+k}\})| = 2n$.*

Proof. The vertices v_p and $w_{p+\lfloor\frac{n}{2}\rfloor}$ of the graph $\mathbb{P}_n$ are MMD. Also, the pair of vertices w_p and w_{p+k} are MMD in $\mathbb{P}_n$ therefore, in view of Lemma 1.2, the SRNs of pair of vertices v_p and $w_{p+\lfloor\frac{n}{2}\rfloor}$ is $\{v_p, w_{p+\lfloor\frac{n}{2}\rfloor}\}$. Also, $S\{w_p, w_{p+k}\} = \{w_p, w_{p+k}\}$. It is clear that $|S\{v_p, w_{p+\lfloor\frac{n}{2}\rfloor}\}| = |S\{w_p, w_{p+k}\}| = 2$. From the above it implies that $(\bigcup\limits_{p=1}^{n} S\{v_p, w_{p+\lfloor\frac{n}{2}\rfloor}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{w_p, w_{p+k}\}) = \{v_b, w_b \mid 1 \le b \le n\}$. Hence, $|(\bigcup\limits_{p=1}^{n} S\{v_p, w_{p+\lfloor\frac{n}{2}\rfloor}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{w_p, w_{p+k}\})| = 2n$. □

Consider

$$\mathcal{H}(\mathbb{P}_n) = \begin{cases} A & \text{if n is even} \\ B & \text{if n is odd} \end{cases} \tag{2.1}$$

where $|A| = |\bigcup\limits_{p=1}^{n} S\{v_p, v_{p+\lfloor\frac{n}{2}\rfloor}\} \bigcup (\bigcup\limits_{p=1}^{n} S\{w_p, w_{p+k}\}| = 2n$ and $|B| = |\bigcup\limits_{p=1}^{n} S\{v_p, w_{p+\lfloor\frac{n}{2}\rfloor}\} \bigcup (\bigcup\limits_{p=1}^{n} S\{w_p, w_{p+k}\}| = 2n$ where $4 \le p+k \le \lfloor\frac{n}{2}\rfloor$. In the following lemmas, we will show that the collection $\mathcal{H}(\mathbb{P}_n)$ coincides with the collection $\mathcal{S}(\mathbb{P}_n)$. Also, it will be shown that the intersection of $\bigcup\limits_{H^* \in \mathcal{H}(\mathbb{P}_n)} H^*$ with the SRNs of any other pair of vertices of $\mathbb{P}_n$ contain at-least $|H^*|$ vertices of $\mathbb{P}_n$. The symmetry of the convex polytope $\mathbb{P}_n$ allows us to consider the SRNs of only the pair of vertices discussed in the following lemmas.

The SRNs of pair of vertices in $\mathbb{P}_n$ lying on the inner cycle are computed in the following lemma.

Lemma 2.3. *Let $\mathbb{P}_n$ be a convex polytope. Then, $|H^*| \le |S\{u_i, u_{i+m}\}|$ where $1 \le i \le n$. Also, $|S\{u_i, u_{i+m}\} \bigcap A| \ge |H^*|$ for n even and $|S\{u_i, u_{i+m}\} \bigcap B| \ge |H^*|$ for n odd.*

Proof. In the view of Table 1, it can be seen that $|S\{u_i, u_{i+m}\} \bigcap A| \ge |H^*|$ for n even and $|S\{u_i, u_{i+m}\} \bigcap B| \ge |H^*|$ for n odd. □

In the following lemma, SRNs of pair of vertices of degree three lying in different cycles are considered.

Lemma 2.4. *Let $\mathbb{P}_n$ be a convex polytope. Then, $|H^*| \le |S\{u_i, v_{i+m}\}|$ where $1 \le i \le n$. Also $|S\{u_i, v_{i+m}\} \bigcap A| \ge |H^*|$ for n even and $|S\{u_i, v_{i+m}\} \bigcap B| \ge |H^*|$ for n odd.*

Table 1. SRNs $S\{u_i, u_{i+m}\}$ for $\mathbb{P}_n$.

m	$S\{u_i, u_{i+m}\}$ **when** n **is even**	$S\{u_i, u_{i+m}\}$ **when** n **is odd**
1	$\{w_i, w_{i+\frac{n}{2}}\}^c$	$\{w_i, u_{i+\lceil\frac{n}{2}\rceil}, v_{i+\lceil\frac{n}{2}\rceil}\}^c$
2	$\{u_a, v_a, w_i, w_a, u_b, v_b, w_b, w_{b-1}\| a = i+1, b = i + \frac{n}{2} + 1\}^c$	$\{u_a, v_a, w_i, w_a, u_b, u_{b+1}, v_b, v_{b+1}, w_b\| a = i+1, b = i + \lceil\frac{n}{2}\rceil\}^c$
$3 \leq m \leq \lfloor\frac{n}{2}\rfloor$	$\{u_x, v_x, w_y, w_q, u_p, v_p\|i+1 \leq x \leq i+m-1, i \leq y \leq i+m-1, i+\frac{n}{2}+1 \leq p \leq i+\frac{n}{2}+m-1, i+\frac{n}{2} \leq q \leq i+\frac{n}{2}+m-1\}^c$	$\{u_x, v_x, w_y, w_q, u_p, v_p\|i+1 \leq x \leq i+m-1, i \leq y \leq i+m-1, i+\lfloor\frac{n}{2}\rfloor+1 \leq p \leq i+\lfloor\frac{n}{2}\rfloor+m-1, i+\lfloor\frac{n}{2}\rfloor \leq q \leq i+\lfloor\frac{n}{2}\rfloor+m-1\}^c$

Proof. It can be seen from Table 2 that $|S\{u_i, v_{i+m}\} \bigcap A| \geq |H^*| = 2$ and for n even and $|S\{u_i, v_{i+m}\} \bigcap B| \geq |H^*|$ for n odd. □

Table 2. SRNs $S\{u_i, v_{i+m}\}$ for $\mathbb{P}_n$.

m	$S\{u_i, v_{i+m}\}$ **when** n **is even**	$S\{u_i, v_{i+m}\}$ **when** n **is odd**
0	$\{v_{i-1}, v_{i+1}, w_{i-2}, w_{i+1}\}^c$	$\{v_{i-1}, v_{i+1}, w_{i-2}, w_{i+1}\}^c$
1	$\{u_x, v_i, w_i, w_{i-1}\|i+1 \leq x \leq i+\frac{n}{2}\}^c$	$\{u_x, v_i, w_i, w_{i-1}, v_y, w_{y'}\| i+1 \leq x \leq i+m+\lfloor\frac{n}{2}\rfloor, i+m+1 \leq y \leq i+m+\lfloor\frac{n}{2}\rfloor, i+m+1 \leq y' \leq i+m+\lfloor\frac{n}{2}\rfloor-1\}^c$
$3 \leq m < \frac{n}{2}$	$\{u_x, v_y, v_z, w_{y'}, w_z\|i+1 \leq x \leq i+\frac{n}{2}+k-1, i+1 \leq y \leq i+k-1, i+k+1 \leq z \leq i+\frac{n}{2}+k-1, i \leq y' \leq i+k-1\}^c$	$\{u_x, v_y, v_z, w_{y'}, w_{z'}\|i+1 \leq x \leq i+m+\lfloor\frac{n}{2}\rfloor, i+1 \leq y \leq i+m-1, i+m+1 \leq z \leq i+m+\lfloor\frac{n}{2}\rfloor, i \leq y' \leq i+m-1, i+m+1 \leq z \leq i+m+\lfloor\frac{n}{2}\rfloor-1\}^c$
$\frac{n}{2}$	$\{u_i, v_i, v_{i+\frac{n}{2}}\}$	

In the following lemma, SRNs of pair of vertices with degree three lying on the outer cycle are computed.

Lemma 2.5. *Let $\mathbb{P}_n$ be a convex polytope. Then, $|H^*| \leq |S\{v_i, v_{i+m}\}|$ where $1 \leq i \leq n$. Also, $|S\{v_i, v_{i+m}\} \bigcap A| \geq |H^*|$ for n even and $|S\{v_i, v_{i+m}\} \bigcap B| \geq |H^*|$ for n odd.*

Proof. Due to symmetry of the graph $\mathbb{P}_n$, the cases discussed in Table 3 show that $|S\{v_i, v_{i+m}\} \bigcap A| \geq |H^*|$ for n even and $|S\{v_i, v_{i+m}\} \bigcap B| \geq |H^*|$ for n odd. □

Table 3. SRNs $S\{v_i, v_{i+m}\}$ for $\mathbb{P}_n$.

m	$S\{v_i, v_{i+m}\}$ **when** n **is even**	$S\{v_i, v_{i+m}\}$ **when** n **is odd**
1	$\{v_{i-1}, v_i, v_{i+1}, v_{i+2}, w_{i-1}, w_{i-2}, w_{i+1}, w_{i+2}\}$	$\{v_{i-1}, v_i, v_{i+1}, v_{i+2}, w_i, w_{i-1}, w_{i+1}, w_{i+2}\}$
$2 \leq m < \frac{n}{2}$	$\{v_i, v_{i+m}, w_{i-1}, w_{i+m}\}$	$\{v_i, v_{i+m}, w_{i-1}, w_{i+m}\}$
$\frac{n}{2}$	$\{v_i, v_{i+\frac{n}{2}}\}$	

SRNs of vertices with degree two and three lying on different cycles will be computed in the following lemma.

Lemma 2.6. *Let $\mathbb{P}_n$ be a convex polytope. Then, $|H^*| \leq |S\{u_i, w_{i+m}\}|$ where $1 \leq i \leq n$. Also, $|S\{u_i, w_{i+m}\} \bigcap A| \geq |H^*|$ for n even and $|S\{u_i, w_{i+m}\} \bigcap B| \geq |H^*|$ for n odd.*

Proof. Taking table 4 into consideration we have, $|S\{u_i, w_{i+m}\} \bigcap A| \geq |H^*|$ for n even and $|S\{u_i, w_{i+m}\} \bigcap B| \geq |H^*|$ for n odd. □

Table 4. SRNs $S\{u_i, w_{i+m}\}$ for $\mathbb{P}_n$.

m	$S\{u_i, w_{i+m}\}$ **when** n **is even**	$S\{u_i, w_{i+m}\}$ **when** n **is odd**
$0 \leq m \leq 2$	$\{u_t, v_s, w_x, v_{i+m}, v_{i+m-1}, w_{i+m-1}, w_{i+m-2} \mid i+1 \leq t \leq \frac{n}{2}+i+m, i+m+1 \leq s \leq \frac{n}{2}+i+m, i+m+1 \leq x \leq \frac{n}{2}+i+m-1\}^c$	$\{u_t, v_s, w_s, v_a, v_{a-1}, w_{a-1}, w_{a-2} \mid a = i+m, i+1 \leq t \leq \lfloor\frac{n}{2}\rfloor+i+m, i+m+1 \leq s \leq \lfloor\frac{n}{2}\rfloor+i+m\}^c$
$3 \leq m < \lfloor\frac{n}{2}\rfloor$	$\{u_t, v_s, w_x, v_y, w_{y'} \mid i+1 \leq t \leq \frac{n}{2}+i+m, i+m+1 \leq s \leq \frac{n}{2}+i+m, i+m+1 \leq x \leq \frac{n}{2}+i+m-1, i+m-(m-1) \leq y \leq i+m, (i+m)-m \leq y' \leq i+m-1\}^c$	$\{u_t, v_s, w_s, v_y, w_{y'} \mid i+1 \leq t \leq \lfloor\frac{n}{2}\rfloor+i+m, i+m+1 \leq s \leq \lfloor\frac{n}{2}\rfloor+i+m, i+m-(m-1) \leq y \leq i+m, i+m-m \leq y' \leq i+m-1\}^c$
$\lfloor\frac{n}{2}\rfloor$	$\{u_t, v_s, w_x, v_y, w_{y'} \mid i+1 \leq t \leq \frac{n}{2}+i+m, i+m+1 \leq s \leq \frac{n}{2}+i+m, i+m+1 \leq x \leq \frac{n}{2}+i+m-1, i+m-(m-1) \leq y \leq i+m, (i+m)-m \leq y' \leq i+m-1\}^c$	$\{u_i, v_i, w_{i+m}\}$

SRNs of vertices with degree two and three lying on same outer cycle will be computed in the following lemma.

Lemma 2.7. *Let $\mathbb{P}_n$ be a convex polytope. Then, $|H^*| \leq |S\{v_i, w_{i+m}\}|$ where $1 \leq i \leq n$. Also, $|S\{v_i, w_{i+m}\} \bigcap A| \geq |H^*|$ for n even and $|S\{v_i, w_{i+m}\} \bigcap B| \geq |H^*|$ for n odd.*

Proof. In view of Table 5 we have, $|S\{v_i, w_{i+m}\} \bigcap A| \geq |H^*|$ for n even and $|S\{v_i, w_{i+m}\} \bigcap B| \geq |H^*|$ for n odd. □

Table 5. SRNs $S\{v_i, w_{i+m}\}$ for $\mathbb{P}_n$.

m	$S\{v_i, w_{i+m}\}$ when n is even	$S\{v_i, w_{i+m}\}$ when n is odd
0	$\{u_t, v_s, w_{s'} \mid i+1 \leq t \leq \frac{n}{2}+i, 4 \leq s \leq \frac{n}{2}+i, 4 \leq s' \leq \frac{n}{2}\}^c$	$\{u_t, v_s, w_s \mid i+1 \leq t \leq \lfloor\frac{n}{2}\rfloor+i, 4 \leq s \leq \lfloor\frac{n}{2}\rfloor+i\}^c$
1	$\{u_t, v_s, w_{s'}, v_{i+1}, w_i \mid 1 \leq t \leq n, 4 \leq s \leq n-1, 4 \leq s' \leq n-2\}^c$	$\{u_t, v_s, w_{s'}, v_{i+1}, w_i \mid 1 \leq t \leq n, 4 \leq s \leq n-1, 4 \leq s' \leq n-2\}^c$
$2 \leq m \leq \lfloor\frac{n}{2}\rfloor - i$	$\{v_i, w_{i+m}, w_{i-1}\}$	$\{v_i, w_{i+m}, w_{i-1}\}$
$\lceil\frac{n}{2}\rceil - i$	$\{v_i, w_{i+m}, w_{i-1}\}$	$\{v_i, w_{i+m}\}$

SRNs of vertices with degree two on outer cycle will be computed in the following lemma.

Lemma 2.8. *Let $\mathbb{P}_n$ be a convex polytope. Then, $|H^*| \leq |S\{w_i, w_{i+m}\}|$ where $1 \leq i \leq n$. Also, $|S\{w_i, w_{i+m}\} \bigcap A| \geq |H^*|$ for n even and $|S\{w_i, w_{i+m}\} \bigcap B \ \geq |H^*|$ for n odd.*

Proof. It can be seen from Table 6 that $|S\{w_i, w_{i+m}\} \bigcap A| \geq |H^*|$ for n even and $|S\{w_i, w_{i+m}\} \bigcap B| \geq |H^*|$ for n odd. □

Table 6. SRNs $S\{w_i, w_{i+m}\}$ for $\mathbb{P}_n$.

m	$S\{w_i, w_{i+m}\}$ **when** n **is even**	$S\{w_i, w_{i+m}\}$ **when** n **is odd**
$2 \leq i+m \leq 3$	$\{u_t, v_s, v_{s'}, w_x \mid 1 \leq t \leq n, i+1 \leq s \leq i+m, i+4 \leq s' \leq n+m-2, i+4 \leq x \leq n+m-3\}^c$	$\{u_t, v_s, v_{s'}, w_x, w_y \mid 1 \leq t \leq n, i+1 \leq s \leq i+m, i+4 \leq s' \leq n+m-2, i+4 \leq x \leq n-2, i+1 \leq y \leq i+m-1\}^c$
1	$\{u_t, v_s, w_{s'}, v_{i+1}, w_i \mid 1 \leq t \leq n, 4 \leq s \leq n-1, 4 \leq s' \leq n-2\}^c$	$\{u_t, v_s, w_{s'}, v_{i+1}, w_i \mid 1 \leq t \leq n, 4 \leq s \leq n-1, 4 \leq s' \leq n-2\}^c$
$2 \leq m \leq \lfloor\frac{n}{2}\rfloor - i$	$\{v_i, w_{i+m}, w_{i-1}\}$	$\{v_i, w_{i+m}, w_{i-1}\}$
$\lceil\frac{n}{2}\rceil - i$	$\{v_i, w_{i+m}, w_{i-1}\}$	$\{v_i, w_{i+m}\}$

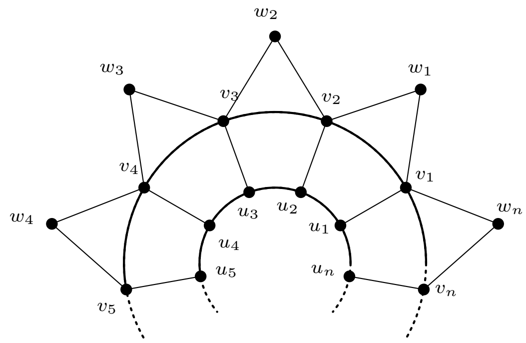

Figure 2. The convex polytope $\mathbb{P}_n^1$

3. Strong resolving neighbourhoods of convex polytope $\mathbb{P}_n^1$

In this section, SRNs of the convex polytope $\mathbb{P}_n^1$ will be calculated. The convex polytope $\mathbb{P}_n^1$ is formed by joining the degree three vertices that are adjacent to degree two vertices, v_t and v_{t+1} by an edge in $\mathbb{P}_n$. The order and size of $\mathbb{P}_n^1$ are given by $|V(\mathbb{P}_n^1)| = |\{u_i, v_i, w_i \mid 1 \le i \le n\}| = 3n$ and $|E(\mathbb{P}_n^1)| = |\{u_iu_{i+1}, u_iv_i, v_iv_{i+1}, v_iw_i, w_iv_{i+1} \mid 1 \le i \le n\}| = 5n$, respectively with the indices taken $\bmod n$. The convex polytope $\mathbb{P}_n^1$ is shown in Figure 2. In order to compute SRNs of the graph $\mathbb{P}_n^1$, its vertices are to be pairwise classified depending upon either being on the same cycle or on different cycle or on the outer triangles of $\mathbb{P}_n^1$.
In the following lemmas SRNs of pair of MMD vertices in $\mathbb{P}_n^1$ are computed.

Lemma 3.1. *Let $\mathbb{P}_n^1$ be a convex polytope, where n is even. Then,* $|S\{u_p, v_{p+\frac{n}{2}}\}| = |S\{u_p, w_{p+\frac{n}{2}-1}\}| = |S\{w_p, w_{p+k}\}| = 2$ *where* $1 \le p, k \le n$. *Also,* $|(\bigcup\limits_{p=1}^{n} S\{u_p, v_{p+\frac{n}{2}}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{w_p, w_{p+k}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{u_p, w_{p+\frac{n}{2}-1}\})| = 3n$.

Proof. From the construction of the graph $\mathbb{P}_n^1$ and in view of Lemma 1.2, the cardinality of SRNs of pair of vertices that are MMD is given by $|S\{u_p, v_{p+\frac{n}{2}}\}| = |S\{u_p, w_{p+\frac{n}{2}-1}\}|$ $=|S\{w_p, w_{p+k}\}| = 2$. From above it implies that $(\bigcup\limits_{p=1}^{n} S\{u_p, v_{p+\frac{n}{2}}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{w_p, w_{p+k}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{u_p, w_{p+\frac{n}{2}-1}\}) = \{u_b, v_b, w_b \mid 1 \le b \le n\}$. Hence, $|\bigcup\limits_{p=1}^{n} S\{u_p, v_{p+\frac{n}{2}}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{w_p, w_{p+k}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{u_p, w_{p+\frac{n}{2}-1}\})| = 3n$. □

Lemma 3.2. *Let $\mathbb{P}_n^1$ be a convex polytope, where n is odd. Then* $|S\{u_p, w_{p+\lfloor\frac{n}{2}\rfloor}\}| = |S\{w_p, w_{p+k}\}| = 2$ *where* $1 \le p, k \le n$. *Also,* $|(\bigcup\limits_{p=1}^{n} S\{u_p, w_{p+\lfloor\frac{n}{2}\rfloor}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{w_p, w_{p+k}\})| = 2n$.

Proof. The pair of vertices u_p, $w_{p+\lfloor\frac{n}{2}\rfloor}$ and w_p, w_{p+k} of the graph $\mathbb{P}_n^1$ are MMD. Therefore, in view of Lemma 1.2, the cardinality of SRNs of pair of vertices u_p, $w_{p+\lfloor\frac{n}{2}\rfloor}$ and w_p, w_{p+k} is given by $|S\{u_p, w_{p+\lfloor\frac{n}{2}\rfloor}\}| = |S\{w_p, w_{p+k}\}| = 2$. From the above it implies that $\bigcup\limits_{p=1}^{n} S\{u_p, w_{p+\lfloor\frac{n}{2}\rfloor}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{w_p, w_{p+k}\} = \{u_b, w_b \mid 1 \le b \le n\}$. Hence, $|\bigcup\limits_{p=1}^{n} S\{u_p, w_{p+\lfloor\frac{n}{2}\rfloor}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{w_p, w_{p+k}\}| = 2n$. □

Consider

$$\mathcal{H}(\mathbb{P}_n^1) = \begin{cases} C & \text{if n is even} \\ D & \text{if n is odd} \end{cases} \tag{3.1}$$

where $C = (\bigcup\limits_{p=1}^{n} S\{u_p, v_{p+\frac{n}{2}}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{w_p, w_{p+k}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{u_p, w_{p+\frac{n}{2}-1}\})$ and $D = (\bigcup\limits_{p=1}^{n} S\{u_p, w_{p+\lfloor\frac{n}{2}\rfloor}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{w_p, w_{p+k}\})$. In the following lemmas, we will show that the collection $\mathcal{H}(\mathbb{P}_n^1)$ coincides with the collection $\mathcal{S}(\mathbb{P}_n^1)$. Also, it will be shown that the intersection of $\bigcup\limits_{H^* \in \mathcal{H}(\mathbb{P}_n^1)} H^*$ with the SRNs of any other pair of vertices of $\mathbb{P}_n^1$ contain at-least $|H^*|$ vertices of $\mathbb{P}_n^1$. The symmetry of the convex polytope $\mathbb{P}_n^1$ allows us to consider the SRNs of only the pair of vertices discussed in the following lemmas.

The SRNs of pair of vertices in $\mathbb{P}_n^1$ lying on the same cycle (inner and outer) are computed in the following lemma.

Lemma 3.3. *Let $\mathbb{P}_n^1$ be a convex polytope. Then, $|H^*| \leq |S\{u_i, u_{i+m}\}| = |S\{v_i, v_{i+m}\}|$ where $1 \leq i \leq n$. Also, $|S\{u_i, u_{i+m}\} \bigcap C| \geq |H^*|$, $|S\{v_i, v_{i+m}\} \bigcap C| \geq |H^*|$ for n even and $|S\{u_i, u_{i+m}\} \bigcap D| \geq |H^*|$, $|S\{v_i, v_{i+m}\} \bigcap D| \geq |H^*|$ for n odd.*

Proof. It can be seen from Table 7 that $|S\{u_i, u_{i+m}\} \bigcap C| \geq |H^*|$, $|S\{v_i, v_{i+m}\} \bigcap C| \geq |H^*|$ for n even and $|S\{u_i, u_{i+m}\} \bigcap D| \geq |H^*|$, $|S\{v_i, v_{i+m}\} \bigcap D| \geq |H^*|$ for n odd. □

Table 7. SRNs $S\{u_i, u_{i+m}\}$ and $S\{v_i, v_{i+m}\}$ for $\mathbb{P}_n$.

m	$S\{u_i, u_{i+m}\}/S\{v_i, v_{i+m}\}$ **when** n **is even**	$S\{u_i, u_{i+m}\}/S\{v_i, v_{i+m}\}$ **when** n **is odd**
1	$\{w_i, w_{i+\frac{n}{2}}\}^c$	$\{w_i, u_{i+\lceil\frac{n}{2}\rceil}, v_{i+\lceil\frac{n}{2}\rceil}\}^c$
2	$\{u_a, v_a, w_{a-1}, w_a, u_{b+1}, v_{b+1}, w_b, w_{b+1} \mid a = i+1, b = i+\frac{n}{2}\}^c$	$\{u_a, v_a, w_{a-1}, w_a, u_c, u_{c+1}, v_c, v_{c+1}, w_c \mid c = i+\lceil\frac{n}{2}\rceil\}^c$
$3 \leq m \leq \lfloor\frac{n}{2}\rfloor$	$\{u_x, v_x, w_y, w_q, u_p, v_p \mid i+1 \leq x \leq i+m-1, i \leq y \leq i+m-1, i+\frac{n}{2}+1 \leq p \leq i+\frac{n}{2}+m-1, i+\frac{n}{2} \leq q \leq i+\frac{n}{2}+m-1\}^c$	$\{u_x, v_x, w_y, w_p, u_p, v_p \mid i+1 \leq x \leq i+m-1, i \leq y \leq i+m-1, c \leq p \leq c+m-2\}^c$

In the following lemma, SRNs of pair of vertices of degree three lying in different cycles are considered.

Lemma 3.4. *Let $\mathbb{P}_n^1$ be a convex polytope. Then, $|H^*| \leq |S\{u_i, v_{i+m}\}|$ where $1 \leq i \leq n$. Also, $|S\{u_i, v_{i+m}\} \bigcap C| \geq |H^*|$ for n even and $|S\{u_i, v_{i+m}\} \bigcap D| \geq |H^*|$ for n odd.*

Proof. It can be seen from Table 8 that $|S\{u_i, v_{i+m}\} \bigcap C| \geq |H^*|$ for n even and $|S\{u_i, v_{i+m}\} \bigcap D| \geq |H^*|$ for n odd. □

Table 8. SRNs $S\{u_i, v_{i+m}\}$ for $\mathbb{P}_n^1$.

m	$S\{u_i, v_{i+m}\}$ **when** n **is even**	$S\{u_i, v_{i+m}\}$ **when** n **is odd**
0	$V(\mathbb{P}_n^1)$	$V(\mathbb{P}_n^1)$
1	$\{u_x, v_y, w_z \mid i+1 \leq x \leq i+\frac{n}{2}, i+\frac{n}{2}+1 \leq y \leq i+n, i+\frac{n}{2} \leq z \leq i+n\}^c$	$\{u_x, v_y, v_z, w_y, w_z \mid i+1 \leq x \leq i+\lceil\frac{n}{2}\rceil+m-1, i \leq y \leq i+m-1, i+\lceil\frac{n}{2}\rceil+m-1 \leq z \leq n\}^c$
$2 \leq m < \lfloor\frac{n}{2}\rfloor$	$\{u_x, v_y, w_z \mid i+1 \leq x \leq i+\frac{n}{2}+m-1, i+\frac{n}{2}+1 \leq y \leq n+i+m-1, i+\frac{n}{2} \leq z \leq n+i+m-1\}^c$	$\{u_x, v_y, v_z, w_y, w_z \mid i+1 \leq x \leq i+\lceil\frac{n}{2}\rceil+m-1, i \leq y \leq i+m-1, i+\lceil\frac{n}{2}\rceil \leq z \leq n+i-1\}^c$
$\lfloor\frac{n}{2}\rfloor$	$\{u_i, v_{i+\frac{n}{2}}\}$	$\{u_i, v_{i+\lfloor\frac{n}{2}\rfloor}, w_{i+\lfloor\frac{n}{2}\rfloor}\}$

SRNs of vertices with degree two and three lying on different cycles will be computed in the following lemmas.

Lemma 3.5. *Let $\mathbb{P}_n^1$ be a convex polytope. Then, $|H^*| \leq |S\{u_i, w_{i+m}\}|$ where $1 \leq i \leq n$. Also, $|S\{u_i, w_{i+m}\} \bigcap C| \geq |H^*|$ for n even and $|S\{u_i, w_{i+m}\} \bigcap D| \geq |H^*|$ for n odd.*

Proof. In view of Table 9 we conclude, $|S\{u_i, w_{i+m}\} \bigcap C| \geq |H^*|$ for n even and $|S\{u_i, w_{i+m}\} \bigcap D| \geq |H^*|$ for n odd. □

Table 9. SRNs $S\{u_i, w_{i+m}\}$ for $\mathbb{P}_n^1$

m	$S\{u_i, w_{i+m}\}$ **when n is even**	$S\{u_i, w_{i+m}\}$ **when n is odd**
0	$\{u_t, w_i \| i + \frac{n}{2} + m + 1 \leq t \leq i\}$	$\{u_t, w_{i+m} \| i + \lceil \frac{n}{2} \rceil \leq t \leq n + i\}$
$1 \leq m < 3$	$\{u_t, w_i \| i + \frac{n}{2} + m + 1 \leq t \leq i\}$	$\{u_t, w_{i+m}; i + \lceil \frac{n}{2} \rceil + m \leq t \leq n + i\}$
$3 \leq m < \lfloor \frac{n}{2} \rfloor$	$\{u_i, w_{i+m}\}$	$\{u_t, w_{i+m}; i + \lceil \frac{n}{2} \rceil + m \leq t \leq n + i\}$
$\lfloor \frac{n}{2} \rfloor$	$\{u_i, w_{i+m}\}$	$\{u_i, w_{i+m}\}$

Lemma 3.6. *Let $\mathbb{P}_n^1$ be a convex polytope. Then, $|H^*| \leq |S\{v_i, w_{i+m}\}|$ where $1 \leq i \leq n$. Also, $|S\{v_i, w_{i+m}\} \bigcap C| \geq |H^*|$ for n even and $|S\{v_i, w_{i+m}\} \bigcap D| \geq |H^*|$ for n odd.*

Proof. It can be seen from Table 10 that $|S\{v_i, w_{i+m}\} \bigcap C| \geq |H^*|$ for n even and $|S\{v_i, w_{i+m}\} \bigcap D| \geq |H^*|$ for n odd. □

Table 10. SRNs $S\{v_i, w_{i+m}\}$ for $\mathbb{P}_n^1$.

m	$S\{v_i, w_{i+m}\}$ **when n is even**	$S\{v_i, w_{i+m}\}$ **when n is odd**
0	$\{u_t, v_t, w_s, w_{s'} \| i + 1 \leq t \leq i + \frac{n}{2} + m,$ $i \leq s \leq i + m - 1, i + m + 1 \leq s \leq i + \frac{n}{2} + m - 1\}^c$	$\{u_t, v_t, w_t \| i + 1 \leq t \leq \lfloor \frac{n}{2} \rfloor + i\}^c$
$1 \leq m \leq \lfloor \frac{n}{2} \rfloor$	$\{u_t, v_t, w_s, w_{s'} \| i + 1 \leq t \leq i + \frac{n}{2} + m,$ $i \leq s \leq i + m - 1, i + m + 1 \leq s \leq i + \frac{n}{2} + m - 1\}^c$	$\{u_t, v_t, w_x, w_s \| i + 1 \leq t \leq \lceil \frac{n}{2} \rceil + m,$ $i + 2 \leq s \leq \lceil \frac{n}{2} \rceil + m, i \leq s \leq i + m - 1\}^c$

Lemma 3.7. *Let $\mathbb{P}_n^1$ be a convex polytope. Then, $|H^*| \leq |S\{w_i, w_{i+m}\}|$ where $1 \leq i \leq n$. Also, $|S\{w_i, w_{i+m}\} \bigcap C| \geq |H^*|$ when n is even and $|S\{w_i, w_{i+m}\} \bigcap D| \geq |H^*|$ when n is odd.*

Proof. The SRNs of the vertices w_i and w_{i+m} are $S\{w_i, w_{i+m}\} = \{u_i, w_{i+m}\}$. It will follows from the above that $|S\{w_i, w_{i+m}\} \bigcap C| \geq |H^*|$ for $\mathbb{P}_n^1$ when n is even and $|S\{w_i, w_{i+m}\} \bigcap D| \geq |H^*|$ when n is odd. □

4. Strong resolving neighbourhoods of convex polytope $\mathbb{P}_n^2$

In this section, SRNs of the convex polytope $\mathbb{P}_n^2$ will be calculated. $\mathbb{P}_n^2$ is formed by adding the edges $v_t v_{t+1}$ and $v_t u_{t+1}$ in $\mathbb{P}_n$. The order and size of $\mathbb{P}_n^2$ are represented as $|V(\mathbb{P}_n^2)| = |\{u_i, v_i, w_i \mid 1 \leq i \leq n\}| = 3n$ and $|E(\mathbb{P}_n^2)| = |\{u_i u_{i+1}, u_i v_i, v_i v_{i+1}, v_i w_i, w_i v_{i+1}, v_i u_{i+1} \mid 1 \leq i \leq n\}| = 6n$ respectively, with the indices taken mod n. The convex polytope $\mathbb{P}_n^2$ is shown in Figure 3. In order to compute SRNs of the graph $\mathbb{P}_n^2$, its vertices are to be pairwise classified depending upon either being on the same cycle or on different cycle or on the outer triangles of $\mathbb{P}_n^2$.
In the following lemmas SRNs of pair of MMD vertices in $\mathbb{P}_n^2$ are computed.

Lemma 4.1. *Let $\mathbb{P}_n^2$ be a convex polytope, where n is even. Then, $|S\{u_p, u_{p+\frac{n}{2}}\}| = |S\{v_p, v_{p+\frac{n}{2}}\}| = |S\{u_p, w_{p+\frac{n}{2}}\}| = |S\{w_p, w_{p+k}\}| = 2$ where $1 \leq p, k \leq n$. Also, $|(\bigcup_{p=1}^{n} S\{u_p, u_{p+\frac{n}{2}}\}) \bigcup (\bigcup_{p=1}^{n} S\{v_p, v_{p+\frac{n}{2}}\}) \bigcup (\bigcup_{p=1}^{n} S\{u_p, w_{p+\frac{n}{2}}\}) \bigcup (\bigcup_{p=1}^{n} S\{w_p, w_{p+k}\})| = 3n$.*

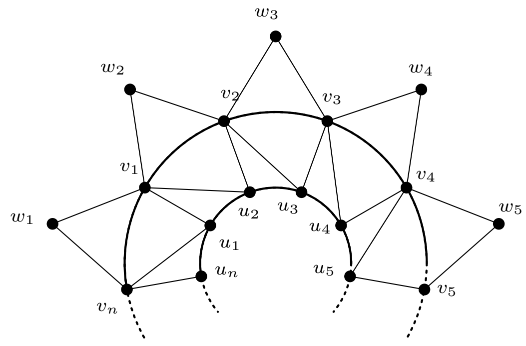

Figure 3. The Convex Polytope $\mathbb{P}_n^2$

Proof. In view of Lemma 1.2, the SRNs of pair of vertices that are mutually maximally distant in $\mathbb{P}_n^2$ are given by $|S\{u_p, u_{p+\frac{n}{2}}\}| = |S\{v_p, v_{p+\frac{n}{2}}\}| = |S\{u_p, w_{p+\frac{n}{2}}\}| = |S\{w_p, w_{p+k}\}| = 2$. From above it implies that $(\bigcup\limits_{p=1}^{n} S\{u_p, u_{p+\frac{n}{2}}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{v_p, v_{p+\frac{n}{2}}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{u_p, w_{p+\frac{n}{2}}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{w_p, w_{p+k}\}) = \{u_b, v_b, w_b \mid 1 \le b \le n\}$. Hence, $|(\bigcup\limits_{p=1}^{n} S\{u_p, u_{p+\frac{n}{2}}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{v_p, v_{p+\frac{n}{2}}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{u_p, w_{p+\frac{n}{2}}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{w_p, w_{p+k}\})| = 3n$. □

Lemma 4.2. *Let $\mathbb{P}_n^2$ be a convex polytope, where n is odd. Then $|S\{u_p, v_{p+\lfloor\frac{n}{2}\rfloor}\}| = |S\{v_p, w_{p+\lfloor\frac{n}{2}\rfloor}\}| = |S\{u_p, w_{p+t}\}| = |S\{w_p, w_{p+k}\}| = 2$ where $1 \le p, k \le n$ and $\lfloor\frac{n}{2}\rfloor - 1 \le t \le \lfloor\frac{n}{2}\rfloor$. Also, $|(\bigcup\limits_{p=1}^{n} S\{u_p, v_{p+\lfloor\frac{n}{2}\rfloor}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{v_p, w_{p+\lfloor\frac{n}{2}\rfloor}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{u_p, w_{p+t}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{w_p, w_{p+k}\})| = 3n$.*

Proof. When n is odd, then for $\mathbb{P}_n^2$ in view of Lemma 1.2, the cardinality of SRNs of pair of MMD vertices is given by $|S\{u_p, v_{p+\lfloor\frac{n}{2}\rfloor}\}| = |S\{v_p, w_{p+\lfloor\frac{n}{2}\rfloor}\}| = |S\{u_p, w_{p+t}\}| = |S\{w_p, w_{p+k}\}| = 2$. From the above it implies that $|(\bigcup\limits_{p=1}^{n} S\{u_p, v_{p+\lfloor\frac{n}{2}\rfloor}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{v_p, w_{p+\lfloor\frac{n}{2}\rfloor}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{u_p, w_{p+t}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{w_p, w_{p+k}\})| = |\{u_b, w_b \mid 1 \le b \le n\}| = 3n$. □

Consider

$$\mathcal{H}(\mathbb{P}_n^2) = \begin{cases} C' & \text{if n is even} \\ D' & \text{if n is odd} \end{cases} \tag{4.1}$$

where $C' = (\bigcup\limits_{p=1}^{n} S\{u_p, u_{p+\frac{n}{2}}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{v_p, v_{p+\frac{n}{2}}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{u_p, w_{p+\frac{n}{2}}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{w_p, w_{p+k}\})$ and $D' = (\bigcup\limits_{p=1}^{n} S\{u_p, v_{p+\lfloor\frac{n}{2}\rfloor}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{v_p, w_{p+\lfloor\frac{n}{2}\rfloor}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{u_p, w_{p+t}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{w_p, w_{p+k}\})$. In the following lemmas, we will show that the collection $\mathcal{H}(\mathbb{P}_n^2)$ coincides with the collection $\mathcal{S}(\mathbb{P}_n^2)$. Also, it will be shown that the intersection of $\bigcup\limits_{H^* \in \mathcal{H}(\mathbb{P}_n^2)} H^*$ with the SRNs of any other pair of vertices of $\mathbb{P}_n^2$ contain at-least $|H^*|$ vertices of $\mathbb{P}_n^2$. The symmetry of $\mathbb{P}_n^2$ allows us to consider the SRNs of only the pair of vertices discussed in the following lemmas. The SRNs of pair of vertices in $\mathbb{P}_n^2$ lying on the same cycle are computed in the following lemma.

Lemma 4.3. *Let $\mathbb{P}_n^2$ be a convex polytope. Then, $|H^*| \le |S\{u_i, u_{i+m}\}|$ where $1 \le i \le n$. Also, $|S\{u_i, u_{i+m}\} \bigcap C'| \ge |H^*|$ for n even and $|S\{u_i, u_{i+m}\} \bigcap D'| \ge |H^*|$ for n odd.*

Proof. In view of the Table 11, it can be seen that $|S\{u_i, u_{i+m}\} \bigcap C'| \geq |H^*|$ for n even and $|S\{u_i, u_{i+m}\} \bigcap D'| \geq |H^*|$ for n odd. □

Table 11. SRNs $S\{u_i, u_{i+m}\}$ for $\mathbb{P}_n^2$.

m	$S\{u_i, u_{i+m}\}$ **when** n **is even**	$S\{u_i, u_{i+m}\}$ **when** n **is odd**
1	$\{v_i, w_i, w_{i-1}, v_{i+\frac{n}{2}}\}^c$	$\{v_i, w_i, w_{i-1}, w_{i+\lfloor\frac{n}{2}\rfloor}\}^c$
2	$\{u_{i+1}, v_i, v_{i+1}, w_{i-1}, w_i, w_{i+1}, u_{i+\frac{n}{2}+1}, v_{i+\frac{n}{2}}, v_{i+\frac{n}{2}+1}, w_{i+\frac{n}{2}}\}^c$	$\{u_{i+1}, u_{i+\lceil\frac{n}{2}\rceil}, u_{i+\lceil\frac{n}{2}\rceil+1}, v_i, v_{i+1}, v_{i+\lceil\frac{n}{2}\rceil}, w_{i-1}, w_i, w_{i+1}, w_{i+\lceil\frac{n}{2}\rceil-1}, w_{i+\lceil\frac{n}{2}\rceil}\}^c$
$3 \leq m \leq \lfloor\frac{n}{2}\rfloor$	$\{u_a, v_b, w_b, w_q, u_p, v_{p'} \| i+1 \leq a \leq i+m-1, i \leq b \leq i+m-1, i-1 \leq b' \leq i+m-1, i+\frac{n}{2}+1 \leq p \leq i+\frac{n}{2}+m-1, i+\frac{n}{2} \leq p' \leq i+\frac{n}{2}+m-1, i+\frac{n}{2} \leq q \leq i+\frac{n}{2}+m-2\}^c$	$\{u_x, v_y, w_{y'}, w_q, u_p, v_{p'} \| i+1 \leq x \leq i+m-1, i \leq y \leq i+m-1, i-1 \leq y' \leq i+m-1, i+\lceil\frac{n}{2}\rceil \leq p \leq i+\lceil\frac{n}{2}\rceil+m-1, i+\lceil\frac{n}{2}\rceil \leq p' \leq i+\lceil\frac{n}{2}\rceil+m-2, i+\lceil\frac{n}{2}\rceil-1 \leq q \leq i+\lceil\frac{n}{2}\rceil+m-2\}^c$

The SRNs of vertices with degree four and degree six in $\mathbb{P}_n^2$ lying on different cycles are computed in the following lemma.

Lemma 4.4. *Let $\mathbb{P}_n^2$ be a convex polytope. Then, $|H^*| \leq |S\{u_i, v_{i+m}\}|$ where $1 \leq i \leq n$. Also, $|S\{u_i, v_{i+m}\} \bigcap C'| \geq |H^*|$ for n even and $|S\{u_i, v_{i+m}\} \bigcap D'| \geq |H^*|$ for n odd.*

Proof. Taking Table 12 into consideration we imply that, $|S\{u_i, v_{i+m}\} \bigcap C'| \geq |H^*|$ for n even and $|S\{u_i, v_{i+m}\} \bigcap D'| \geq |H^*|$ for n odd. □

Table 12. SRNs $S\{u_i, v_{i+m}\}$ for $\mathbb{P}_n^2$.

m	$S\{u_i, v_{i+m}\}$ **when** n **is even**	$S\{u_i, v_{i+m}\}$ **when** n **is odd**
0	$\{u_a, v_b, w_{b'} \| i+1 \leq a \leq i+\frac{n}{2}, i+\frac{n}{2} \leq b \leq i+n-1, i+\frac{n}{2} \leq b' \leq i+n-2\}^c$	$\{u_a, v_b, w_{b'} \| i+1 \leq a \leq i+\lfloor\frac{n}{2}\rfloor, i+\lceil\frac{n}{2}\rceil \leq b \leq i+n-1, i+\lfloor\frac{n}{2}\rfloor \leq b' \leq i+n-2\}^c$
$1 \leq m < \frac{n}{2}$	$\{u_a, v_b, w_b \| i+1 \leq a \leq i+m+\frac{n}{2}, i+\frac{n}{2} \leq b \leq i+m-1\}^c$	$\{u_a, v_b, w_b \| i+1 \leq a \leq i+m+\lfloor\frac{n}{2}\rfloor, i+\lceil\frac{n}{2}\rceil \leq b \leq i+m-1, i+\lfloor\frac{n}{2}\rfloor \leq b \leq i+m-1\}^c$
$\lfloor\frac{n}{2}\rfloor$	$\{u_a, v_b, w_b \| i+1 \leq a \leq i+m+\frac{n}{2}, i+\frac{n}{2} \leq b \leq i+m-1\}^c$	$\{u_i, v_{i+\lfloor\frac{n}{2}\rfloor}\}$

Lemma 4.5. *Let $\mathbb{P}_n^2$ be a convex polytope. Then, $|H^*| \leq |S\{u_i, w_{i+m}\}|$ where $1 \leq i \leq n$. Also, $|S\{u_i, w_{i+m}\} \bigcap C'| \geq |H^*|$ for n even and $|S\{u_i, w_{i+m}\} \bigcap D'| \geq |H^*|$ for n odd.*

Proof. The SRNs of u_i and w_{i+m} are $S\{u_i, w_{i+m}\} = \{w_{i+m}, u_a | i+m+\lceil\frac{n}{2}\rceil+1 \leq a \leq i+n\}$ where $0 \leq m \leq \lfloor\frac{n}{2}\rfloor - 1$. It implies from the above that, $|S\{u_i, w_{i+m}\} \bigcap C'| \geq |H^*|$ for n even and $|S\{u_i, w_{i+m}\} \bigcap D'| \geq |H^*|$ for n odd. □

Lemma 4.6. *Let $\mathbb{P}_n^2$ be a convex polytope. Then, $|H^*| \leq |S\{v_i, v_{i+m}\}|$ where $1 \leq i \leq n$. Also, $|S\{v_i, v_{i+m}\} \bigcap C'| \geq |H^*|$ for n even and $|S\{v_i, v_{i+m}\} \bigcap D'| \geq |H^*|$ for n odd.*

Proof. By considering Table 13 we have, $|S\{v_i, v_{i+m}\} \bigcap C'| \geq |H^*|$ for n even and $|S\{v_i, v_{i+m}\} \bigcap D'| \geq |H^*|$ for n odd. □

Table 13. SRNs $S\{v_i, v_{i+m}\}$ for $\mathbb{P}_n^2$.

m	$S\{v_i, v_{i+m}\}$ **when** n **is even**	$S\{v_i, v_{i+m}\}$ **when** n **is odd**
1	$\{u_{i+1}, u_{i+\frac{n}{2}+1}, w_i, w_{i+\frac{n}{2}}\}^c$	$\{u_{i+1}, v_{i+\lfloor\frac{n}{2}\rfloor+1}, w_i\}^c$
$2 \leq m < \lfloor\frac{n}{2}\rfloor$	$\{u_a, u_b, v_{a'}, v_{b'}, w_c, w_d \| i+1 \leq a \leq i+m, i+1 \leq a' \leq i+m-1, i \leq c \leq i+m-1, i+\frac{n}{2}+1 \leq b \leq i+m+\frac{n}{2}, i+\frac{n}{2}+1 \leq b' \leq i+m+\lceil\frac{n}{2}\rceil-1, i+\lceil\frac{n}{2}\rceil \leq d \leq i+m+\lceil\frac{n}{2}\rceil-1\}^c$	$\{u_a, u_b, v_{a'}, v_{b'}, w_c, w_d \| i+1 \leq a \leq i+m, i+1 \leq a' \leq i+m-1, i \leq c \leq i+m-1, i+\lceil\frac{n}{2}\rceil+1 \leq b \leq i+m+\lceil\frac{n}{2}\rceil-1, i+\lceil\frac{n}{2}\rceil \leq b' \leq i+m+\lceil\frac{n}{2}\rceil-1, i+\lceil\frac{n}{2}\rceil \leq d \leq i+m+\lceil\frac{n}{2}\rceil-2\}^c$
$\lfloor\frac{n}{2}\rfloor$	$\{v_i, v_{i+m}\}$	$\{u_a, u_b, v_{a'}, v_{b'}, w_c, w_d \| i+1 \leq a \leq i+m, i+1 \leq a' \leq i+m-1, i \leq c \leq i+m-1, i+\lceil\frac{n}{2}\rceil+1 \leq b \leq i+m+\lceil\frac{n}{2}\rceil-1, i+\lceil\frac{n}{2}\rceil \leq b' \leq i+m+\lceil\frac{n}{2}\rceil-1, i+\lceil\frac{n}{2}\rceil \leq d \leq i+m+\lceil\frac{n}{2}\rceil-2\}^c$

Let A be a finite, undirected, connected and simple graph. Then, FSMD of A will be computed with the following steps of the algorithm:

Step I: Adjacency Matrix

The input adjacency matrix of the graph A, denoted by $B = [b_{ij}]$ is formed in such a way that $b_{ij} = 1$, if x_i and x_j are adjacent nodes in A, otherwise and $b_{ij} = 0$;

Step II: Matrix of Distances

Compute the matrix of distances given by $D(A)$;

Step III: SRNs

Compute the collection $S\{x_1, x_2\}$ of nodes $y \in V(A)$, for every pair of adjacent nodes x_1, x_2 such that x_1 lies on $x_2 - y$ geodesic or x_2 lies on $x_1 - y$ geodesic;

Step IV: Least cardinality of SRN

Calculate $\gamma(A) = \min\{|S\{x_1, x_2\}|\}$;

Step V: Collection of all SRNs with least cardinality

Compute $\mathcal{S}(A) = \{S\{x_1, x_2\}|\ |S\{x_1, x_2\}| = \gamma(A)\}$ and $\beta(A) = |\bigcup \mathcal{S}(A)|$;

Step VI: Computation of FSMD

If for all $x_1, x_2 \in E(A)$, $|S\{x_1, x_2\} \bigcap \left(\bigcup \mathcal{S}(A)\right)| \geq \gamma(A)$, then compute

$$\mathrm{sdim_f}(A) = \sum_{t=1}^{\beta(A)} \frac{1}{\gamma(A)}$$

Lemma 4.7. *Let $\mathbb{P}_n^2$ be a convex polytope. Then, $|H^*| \leq |S\{v_i, w_{i+m}\}|$ where $1 \leq i \leq n$. Also, $|S\{v_i, w_{i+m}\} \bigcap C'| \geq |H^*|$ for n even and $|S\{v_i, w_{i+m}\} \bigcap D'| \geq |H^*|$ for n odd.*

Proof. In view of Table 14 we have, $|S\{v_i, w_{i+m}\} \bigcap C'| \geq |H^*|$ for n even and $|S\{v_i, w_{i+m}\} \bigcap D'| \geq |H^*|$ for n odd. $\square$

Table 14. SRNs $S\{v_i, w_{i+m}\}$ for $\mathbb{P}_n^2$.

m	$S\{v_i, w_{i+m}\}$ **when** n **is even**	$S\{v_i, w_{i+m}\}$ **when** n **is odd**
0	$\{u_a, v_b, w_c \mid i+2 \leq a \leq i+\frac{n}{2}, i+1 \leq b \leq i+\frac{n}{2}, i+1 \leq c \leq i+\frac{n}{2}-1\}^c$	$\{u_a, v_b, w_b \mid i+2 \leq a \leq i+\lceil\frac{n}{2}\rceil, i+1 \leq b \leq i+\lceil\frac{n}{2}\rceil-1\}^c$
$1 \leq m < \lfloor\frac{n}{2}\rfloor$	$\{u_a, v_a, w_b, w_c \mid i+1 \leq a \leq i+m+\frac{n}{2}, i \leq b \leq i+m-1, i+m+1 \leq c \leq i+m+\frac{n}{2}-1\}^c$	$\{u_a, v_b, w_c, w_d \mid i+1 \leq a \leq i+m+\lceil\frac{n}{2}\rceil, i+1 \leq b \leq i+m+\lceil\frac{n}{2}\rceil-1, i \leq c \leq i+m-1, i+m+1 \leq d \leq i+m+\lceil\frac{n}{2}\rceil-1\}^c$
$\lfloor\frac{n}{2}\rfloor$	$\{u_{i+1}, v_i, w_i, w_{i+m}\}$	$\{v_i, w_{i+m}\}$

For the computation of FSMD of any graph, we have the following algorithm in view of Theorem1.1 which could be used with the help of Matlab or other simulation tools:

5. Fractional strong metric dimension of graphs

In this section, FSMD of certain convex polytopes is computed.

Theorem 5.1. *For $n \geq 3$, then* $\mathrm{sdim_f}(\mathbb{P}_n) = n$.

Proof. In view of Lemma 2.1 and 2.2, for n even $|S\{v_p, v_{p+\frac{n}{2}}\}| = |S\{w_p, w_{p+k}\}| = 2$ where $1 \leq p \leq n$ and $4 \leq p+k \leq \frac{n}{2}$ and for n odd $|S\{v_p, w_{p+\lfloor\frac{n}{2}\rfloor}\}| = |S\{w_p, w_{p+k}\}| = 2$ where $1 \leq p \leq n$ and $4 \leq p+k \leq \lceil\frac{n}{2}\rceil$. Also, from Lemma 2.3 to Lemma 2.8, $|H^*| \leq |S\{x, y\}|\ \forall\, x, y \in V(\mathbb{P}_n)$. Therefore,

$$\mathcal{S}(\mathbb{P}_n) = \begin{cases} A & \text{if n is even} \\ B & \text{if n is odd} \end{cases} \text{ and } \beta(\mathbb{P}_n) = \begin{cases} |A| = 2n & \text{if n is even} \\ |B| = 2n & \text{if n is odd} \end{cases}$$

where $A = (\bigcup_{p=1}^{n} S\{v_p, v_{p+\frac{n}{2}}\}) \bigcup (\bigcup_{p=1}^{n} S\{w_p, w_{p+k}\})$ and $B = (\bigcup_{p=1}^{n} S\{v_p, w_{p+\lfloor\frac{n}{2}\rfloor}\}) \bigcup (\bigcup_{p=1}^{n}$

$S\{w_p, w_{p+k}\})$. Hence, from Theorem 1.1,

$$\mathrm{sdim_f}(\mathbb{P}_n) = \sum_{z=1}^{\beta(\mathbb{P}_n)} \frac{1}{\gamma(\mathbb{P}_n)} = n.$$

□

Theorem 5.2. *For $n \geq 3$, $\mathrm{sdim_f}(\mathbb{P}_n^1) = \begin{cases} \frac{3n}{2} & \text{if } n \text{ is even} \\ n & \text{if } n \text{ is odd} \end{cases}$*

Proof. In view of Lemma 3.1 and 3.2, for n even $|S\{u_p, v_{p+\frac{n}{2}}\}| = |S\{u_p, w_{p+\frac{n}{2}-1}\}| = |S\{w_p, w_{p+k}\}| = 2$ and for n odd $|S\{u_p, w_{p+\lfloor\frac{n}{2}\rfloor}\}| = |S\{w_p, w_{p+k}\}| = 2$ where $1 \leq p, k \leq n$. Also, from Lemma 3.3 to Lemma 3.7, $|H^*| \leq |S\{x, y\}|\ \forall\ x, y \in V(\mathbb{P}_n^1)$. Therefore, $\mathcal{S}(\mathbb{P}_n^1) = \begin{cases} C & \text{if } n \text{ is even} \\ D & \text{if } n \text{ is odd} \end{cases}$ and $\beta(\mathbb{P}_n^1) = \begin{cases} |C| = 3n & \text{if n is even} \\ |D| = 2n & \text{if n is odd} \end{cases}$ where $C = (\bigcup\limits_{p=1}^{n} S\{u_p, v_{p+\frac{n}{2}}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{w_p, w_{p+k}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{u_p, w_{p+\frac{n}{2}-1}\})$ and $D = (\bigcup\limits_{p=1}^{n} S\{u_p, w_{p+\lfloor\frac{n}{2}\rfloor}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{w_p, w_{p+k}\})$. Hence, from Theorem 1.1,

$$\mathrm{sdim_f}(\mathbb{P}_n^1) = \sum_{z=1}^{\beta(\mathbb{P}_n^1)} \frac{1}{\gamma(\mathbb{P}_n^1)} = \begin{cases} \frac{3n}{2} & \text{if n is even} \\ n & \text{if n is odd} \end{cases}.$$

□

Theorem 5.3. *For $n \geq 3$, $\mathrm{sdim_f}(\mathbb{P}_n^2) = \frac{3n}{2}$.*

Proof. In view of Lemma 4.1 and 4.2, for n even $|S\{u_p, u_{p+\frac{n}{2}}\}| = |S\{v_p, v_{p+\frac{n}{2}}\}| = |S\{u_p, w_{p+\frac{n}{2}}\}| = |S\{w_p, w_{p+k}\}| = 2$ and for n odd $|S\{u_p, v_{p+\lfloor\frac{n}{2}\rfloor}\}| = |S\{v_p, w_{p+\lfloor\frac{n}{2}\rfloor}\}| = |S\{u_p, w_{p+k}\}| = |S\{w_p, w_{p+k}\}| = 2$ where $1 \leq p, k \leq n$. Also, from Lemma 4.3 to Lemma 4.7, $|H^*| \leq |S\{x, y\}|\ \forall\ x, y \in V(\mathbb{P}_n^2)$. Therefore, $\mathcal{S}(\mathbb{P}_n^2) = \begin{cases} C' & \text{if n is even} \\ D' & \text{if n is odd} \end{cases}$ and $\beta(\mathbb{P}_n^2) = \begin{cases} |C'| = 3n & \text{if n is even} \\ |D'| = 3n & \text{if n is odd} \end{cases}$ where $C' = (\bigcup\limits_{p=1}^{n} S\{u_p, u_{p+\frac{n}{2}}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{v_p, v_{p+\frac{n}{2}}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{u_p, w_{p+\frac{n}{2}}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{w_p, w_{p+k}\})$ and $D' = (\bigcup\limits_{p=1}^{n} S\{u_p, v_{p+\lfloor\frac{n}{2}\rfloor}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{v_p, w_{p+\lfloor\frac{n}{2}\rfloor}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{u_p, w_{p+t}\}) \bigcup (\bigcup\limits_{p=1}^{n} S\{w_p, w_{p+k}\})$. Hence, from Theorem 1.1,

$$\mathrm{sdim_f}(\mathbb{P}_n^2) = \sum_{z=1}^{\beta(\mathbb{P}_n^2)} \frac{1}{\gamma(\mathbb{P}_n^2)} = \frac{3n}{2}.$$

□

6. Applications

Consider an example of data flow in a region through internet connection network. The objective of the study is that the flow of data should be optimal which is possible only if it is at a uniform rate. The internet connection is potentially affected by the following factors:

- If data transmission is required by transmission devices that is far from the node that has router installed on it, then the data transmission will be delayed.
- If the data transmission devices increase then more routers will be required to ensure data flow at a uniform rate.

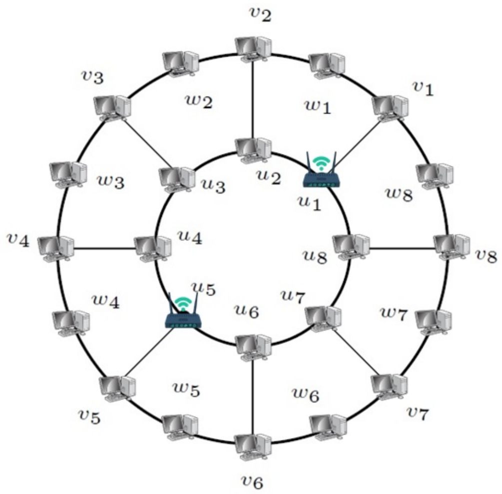

Figure 4. FSMD in Internet connection Network

This network can be expressed in terms of FSMD as follows. In an internet connection network, data transmission devices are considered as nodes and interconnections between these devices are represented by edges. In order to achieve an optimal data flow, the nodes which are not suitable for the installation of routers for data transmission are to be identified and excluded. The collection of such nodes are referred as SRNs of a graph and allocating least dependency on these nodes accounts for FSMD. As an illustrative case, we have an internet connection network consisting of different data transmission devices and routers at nodes of $\mathbb{P}_8$ as shown in the Figure 4. In view of Algorithm **??**, the routers are not to be placed at the vertices v_i and w_i where $1 \leq i \leq n$ on the outer cycle of $\mathbb{P}_8$. Each router is placed such that data transmission takes place smoothly. This results in a cost efficient Wifi hotspot (internet connection network) which allows electronic devices to connect to the internet and exchange data wirelessly through radio waves.

7. Conclusion

Convex polytopes are an interesting mathematical structures having importance and application in the fields chemistry, physics, economics and computer science. These mathematical objects have captured the attention of mathematicians and researchers for centuries. The fractional versions of various metric related parameters have been extensively studied by reseachers due to their applications in the fields of robot navigation, sensor networking and chemistry. In this paper, FSMD of certain convex polytopes is computed. A combinatorial technique is used to determine exact values of FSMD for $\mathbb{P}_n$, $\mathbb{P}_n^1$ and $\mathbb{P}_n^2$. The obtained results lead us to the conclusion that the structures of these convex polytopes have FSMD that depends on the value of n. An application of FSMD is discussed for the optimal data flow in a region through internet connection network. The computation of FSMD is an NP hard problem due to which in many cases authors are only able to compute it for different classes or bounds for it rather than exact values for general graphs. This underscore the significance of the computed results. In future, we are interested to compute FMD and FSMD for some other convex polytopes and wheel related graphs.

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