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Stability of Uncertain Equations of Volterra-Levin Type and an Uncertain Delay Differential Equation via Fixed Point Method

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Abstract

In this work four uncertain delay differential equations of Volterra-Levin type will be considered. Applying suitable contraction mapping and fixed point method, the stability of the equations will be studied. It will be shown that the solutions are bounded and, with additional condition, the solutions tend to zero. Also, a necessary and sufficient condition for the asymptotic stability of the solutions of an uncertain differential equation will be presented.

Keywords: Stability Lyapunov Fixed point techniques Uncertain delay differential equations.
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1. Introduction

Introducing uncertain measure, uncertain theory was founded by Liu in 2007 ([8]). Since then, the theory has been spread in various fields such as optimal control ([16]), game theory ([13]), finance ([5]), heat conduction ([14]), uncertain risk analysis ([7]), uncertain logic ([6]) and uncertain programming ([4]). Uncertain differential equations were introduced by Liu in [3]. In this work we are interested to study the stability of uncertain (delay) differential equations (UDDEs). First, uncertain differential equations (UDEs) and some of the previous results in this area will be presented.

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An uncertain process C_t is said to be a Liu process if
(i) $C_0 = 0$ and almost all sample paths are Lipschitz continuous,
(ii) C_t has stationary and independent increments,
(iii) every increment $C_{s+t} - C_s$ is a normal uncertain variable with expected value 0 and variance t^2 ([3]).

An uncertain differential equation (UDE) is a type of differential equation which is driven by Liu process as

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t, \tag{1.1}$$

where C_t is a Liu process. The existence and uniqueness of solutions of (1.1) is proved by Chen and Liu ([1]). Recently, the existence and uniqueness of solutions of uncertain linear system has been presented by the authors in [10]. Also, the Liouville formula and explicit solutions of uncertain homogeneous linear systems is proved by the authors in [11]. Moreover, the continuity and differentiability of solutions with respect to initial conditions and Peano theorem for uncertain differential equations have recently been presented in [9].

Definition 1.1. An uncertain delay differential equation (UDDE) is an uncertain differential equation in which the increment of the uncertain process depends on the values of the process in the past.

Stability is one of the most important problems in the study of UDEs. First, the concept of the stability will be reviewed. Let $X_t = (x_{t1}, x_{t2}, \ldots, x_{tn}) \in \mathbb{R}^n$, $P = (p_1, p_2, \ldots, p_n) \in \mathbb{R}^n$, $f = (f_1, f_2, \ldots, f_n)$ and $g = (g_1, g_2, \ldots, g_n)$ in which f_i and g_i are continuous and partially differentiable functions on $\{t \in [t_0, \infty), x_{ti} \in \mathbb{R}\}$. Suppose $X_t(P)$ be the solution of system

$$\begin{cases} \mathrm{d}X_t &= f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t, \\ X_{t_0} &= X_0 = P, \end{cases}$$

for all $t > 0$ and $P \in \mathbb{R}^n$. Now, this is a question that, how small changes in the initial conditions affect the long-term behavior of the solutions? If the system varies little under small perturbations of the initial position, then we say that motion of the system is stable. In this manuscript, applying the fixed point method, the concept of stability for UDDEs will be investigated.

The stability in measure, which is founded by Liu ([3]) in 2009, is defined as follows.

Definition 1.2. An UDE is said to be stable if for any $\epsilon > 0$ and $\varepsilon > 0$, there exists $\delta > 0$ such that for any solutions X_t and Y_t we have

$$M\{|X_t - Y_t| > \epsilon\} < \varepsilon, \quad \forall t > 0,$$

where $|X_0 - Y_0| < \delta$ and M is an uncertain measure ([3]).

Some stability results for Volterra-Hadamard random partial fractional integral equations have been given by the authors in [12]. Also, a sufficient condition about the stability of solutions of an UDE is given by Yao et al. in 2013 as follows ([15]).

Theorem 1.3. *([15]) The uncertain differential equation*

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t,$$

is stable if $f(t, X_t)$ and $g(t, X_t)$ satisfy the linear growth condition

$$|f(t, X_t)| + |g(t, X_t)| \leqslant K(1 + |X_t|), \quad \forall X_t \in \mathbb{R}, t \geqslant 0,$$

for some constant K and strong Lipschitz condition

$$|f(t, X_t) - f(t, Y_t)| + |g(t, X_t) - g(t, Y_t)| \leqslant L(t)|X_t - Y_t|, \quad \forall X_t, Y_t \in \mathbb{R}, t \geqslant 0,$$

for some bounded and integrable function $L(t)$ on $[0, \infty)$.

2. Preliminaries

In this section, to prove our main results, some needed preliminaries will be presented. First, consider the following theorem which is proven by Ji and Zhau in [2].

Theorem 2.1. *([2]) Let C_t be an n-dimensional Liu process and Y_t be an $m \times n$ Liu integrable uncertain matrix process. Then,*

$$\left| \int_a^b Y_t(\gamma)\mathrm{d}C_t(\gamma) \right| \leqslant K_\gamma \int_a^b |Y_t(\gamma)|\mathrm{d}t, \qquad \forall \gamma \in \Gamma,$$

where K_γ is the Lipschitz constant of the sample path $C_t(\gamma)$.

We will study some UDEs in which the difference of two pairs of terms, say $A - C$ and $B - D$, seem to have little or no effect on the behavior of the solutions of the equations. For example consider the UDE

$$\mathrm{d}X_t = [f(t, X_t) + A]\mathrm{d}t + [g(t, X_t) + B]\mathrm{d}C_t,$$

which is most intractable but

$$\mathrm{d}X_t = [f(t, X_t) + C]\mathrm{d}t + [g(t, X_t) + D]\mathrm{d}C_t, \tag{2.1}$$

would be easy to analyze. The real advantage of identification of such two pairs is the idea of using the technique of adding and subtracting the same thing with the hope that

$$\mathrm{d}X_t = [f(t, X_t) + (A - C) + C]\mathrm{d}t + [g(t, X_t) + (B - D) + D]\mathrm{d}C_t,$$

will have the same behavior of (2.1).

For example, let uncertain process X_t denotes the number of individuals in a population at time t. Assume also that, $A(t) = h(X_t)$ and $B(t) = k(X_t)$ be the number of insure and uncertain births. If every individual has lifespan L and dies at age L, then the population growth is governed by equation

$$\mathrm{d}X_t = [h(X_t) - h(X_{t-L})]\mathrm{d}t + k(X_t)\mathrm{d}C_t - k(X_{t-L})\mathrm{d}C_{t-L}. \tag{2.2}$$

Note that h and k can be any Lipschitz functions and $h(X_t) - h(X_{t-L})$ and $k(X_t) - k(X_{t-L})$ are the net change in the population and the fluctuation per unite of time, respectively. In real-word problems, because of uncertainties, the freedom of taking h and k as two arbitrary Lipschitz functions is critical. Equation (2.2) has the property that every constant process is a solution and every solution (under certain regularity conditions) approaches to a constant. If the regularity conditions fail, then a solution may tend to $\pm\infty$.

If

$$\mathrm{d}X_t = f(t, X_t)\mathrm{d}t + g(t, X_t)\mathrm{d}C_t,$$

enjoys the stability properties, then those properties should be shared with

$$\mathrm{d}X_t = [f(t, X_t) + h(X_t) - h(X_{t-L})]\mathrm{d}t + [g(t, X_t) + k(X_t)]\mathrm{d}C_t - k(X_{t-L})\mathrm{d}C_{t-L},$$

under mild conditions on h, k and L and $h(X_t) - h(X_{t-L})$ and $k(X_t)\mathrm{d}C_t - k(X_{t-L})\mathrm{d}C_{t-L}$ should be considered as harmless perturbations. Our goal is to construct a set of harmless perturbations.

To specify a solution of (2.2), an initial function on an initial interval is needed. Typically, we need a continuous function $\psi : [-L, 0] \to \mathbb{R}$ and obtain a continuous function $X_t(0, \psi)$ with $X_t(0, \psi) = \psi(t)$ on $[-L, 0]$, while X_t satisfies (2.2) for $t > 0$. For convenience, consider the solution starting at $t_0 = 0$.

Considering the initial condition, equation (2.2) can be written as

$$X_t = \psi(0) - \int_{-L}^{0} h(\psi(s))\mathrm{d}s + \int_{t-L}^{t} h(X_s)\mathrm{d}s - \int_{-L}^{0} k(\psi(s))\mathrm{d}C_s + \int_{t-L}^{t} k(X_s)\mathrm{d}C_s. \tag{2.3}$$

To solve our problem, let $L > 0$ be constant and h and k satisfy

$$\begin{aligned} |h(X_t) - h(Y_t)| &\leqslant K_1|X_t - Y_t|, \\ |k(X_t) - k(Y_t)| &\leqslant K_2|X_t - Y_t|. \end{aligned} \tag{2.4}$$

Suppose also that there is a positive constant $\alpha < 1$ such that

$$L(K_1 + K_2K_\gamma) < \alpha, \tag{2.5}$$

where K_γ is the Lipschitz constant of sample pass $C(\gamma)$.

Theorem 2.2. *Let (2.4) and (2.5) hold and $\psi : [-L, 0] \to \mathbb{R}$ be a continuous function. Then, there is a unique constant λ satisfying*

$$\lambda = \psi(0) + h(\lambda)L - \int_{-L}^{0} h(\psi(s))\mathrm{d}s + k(\lambda)LK_\gamma - \int_{-L}^{0} k(\psi(s))\mathrm{d}C_s, \tag{2.6}$$

such that the unique solution of (2.2) with initial function ψ satisfies $X_t(0, \psi) \to \lambda$ as $t \to \infty$.

Proof. Define mapping $P : \mathbb{R} \to \mathbb{R}$ as

$$P\lambda = \psi(0) + h(\lambda)L - \int_{-L}^{0} h(\psi(s))\mathrm{d}s + k(\lambda)LK_\gamma - \int_{-L}^{0} k(\psi(s))\mathrm{d}C_s.$$

We show that P is a contraction mapping. Let $m, n \in \mathbb{R}$. Then,

$$\begin{aligned} |Pm - Pn| &\leqslant L|h(m) - h(n)| + LK_\gamma|k(m) - k(n)| \leqslant L(K_1 + K_2K_\gamma)|m - n| \\ &< \alpha|m - n|. \end{aligned}$$

Therefore, P is a contraction mapping on the complete metric space $(\mathbb{R}, |.|)$ where $|.|$ denotes the absolute value. Thus, P has a unique fixed point λ.

Now, define

$$\begin{aligned} M = &\{\phi : [-L, \infty) \to \mathbb{R} \mid \phi(t) = \psi(t) \text{ on } [-L, 0],\ \phi(t) \to \lambda \text{ as } t \to \infty,\ \phi \\ &\text{is bounded continuous function}\}. \end{aligned}$$

Then, $(M, \|.\|)$ is a complete normed space with the supremum norm. Applying (2.3), define $Q : M \to M$ on $[-L, 0]$ as

$$(Q\phi)(t) = \psi(t),$$

and for $t > 0$ let

$$(Q\phi)(t) = \psi(0) - \int_{-L}^{0} h(\psi(s))\mathrm{d}s + \int_{t-L}^{t} h(\phi(s))\mathrm{d}s - \int_{-L}^{0} k(\psi(s))\mathrm{d}C_s + \int_{t-L}^{t} k(\phi(s))\mathrm{d}C_s. \tag{2.7}$$

It is clear that a fixed point of Q will solve (2.2) and (2.3). Notice that since $\phi(t) \to \lambda$, we have $\int_{t-L}^{t} h(\phi(s))\mathrm{d}s \to h(\lambda)L$ and $\int_{t-L}^{t} k(\phi(s))\mathrm{d}C_s \to k(\lambda)(C_t - C_{t-L})$ as $t \to \infty$. Using this, (2.7) and then (2.6), we see that $(Q\phi)(t) \to \lambda$ as $t \to \infty$. Thus, $Q : M \to M$.

It will be shown that Q is a contraction mapping. Let $\phi, \eta \in M$. Then,

$$\begin{aligned} |(Q\phi)(t) - (Q\eta)(t)| &\leqslant \int_{t-L}^{t} |h(\phi(s)) - h(\eta(s))|\mathrm{d}s + \int_{t-L}^{t} |k(\phi(s)) - k(\eta(s))|\mathrm{d}C_s \\ &\leqslant K_1L\|\phi - \eta\| + LK_2K_\gamma\|\phi - \eta\| \leqslant \alpha\|\phi - \eta\|. \end{aligned}$$

Therefore, Q is a contraction mapping with unique fixed point $\phi \in M$ and the proof is complete. □

3. Stability of uncertain delay differential equations

The study of UDDEs is a new field of the theory of uncertain differential equations. In this section the fixed point method will be used to study the stability of four different types of UDDEs of Volterra-Levin type. As we see, each of these equations will be approached via the harmless perturbation idea.

The problems we consider here are:

$$
\begin{aligned}
\mathrm{d}X_t &= -\int_{t-L}^{t} p(s-t)f(X_s)\mathrm{d}t\mathrm{d}s - \int_{t-L}^{t} q(s-t)g(X_s)\mathrm{d}C_s\mathrm{d}C_{s-t}\\
\mathrm{d}X_t &= -\int_{0}^{t} e^{-a(t-s)}\sin(t-s)\mathrm{d}tf(X_s)\mathrm{d}s - \int_{0}^{t} e^{-b(t-s)}\sin(t-s)\mathrm{d}C_{t-s}g(X_s)\mathrm{d}C_s,\\
\mathrm{d}X_t &= -\int_{-\infty}^{t} p(s-t)\mathrm{d}tf(X_s)\mathrm{d}s + \int_{-\infty}^{t} q(s-t)\mathrm{d}C_{s-t}g(X_s)\mathrm{d}C_s,\\
\mathrm{d}X_t &= [-a(t)f(X_{q(t)})]\mathrm{d}t + [-b(t)g(X_{q(t)})]\mathrm{d}C_t.
\end{aligned}
$$

Each of these equations can be displayed as

$$\mathrm{d}X_t = -f(X_t)\mathrm{d}t + \text{ a harmless perturbation} - g(X_t)\mathrm{d}C_t + \text{an uncertain harmless perturbation.}$$

Then, the stability of these equations will be studied by applying suitable contraction mappings. In this way, we show that the fixed point technique can be applied on a distributed bounded delay, a distributed unbounded delay, a distributed infinite delay and a pointwise variable delay.

In all of the equations, let $f, g : \mathbb{R} \to \mathbb{R}$ be functions satisfying

$$
\begin{aligned}
|f(X_t) - f(Y_t)| &\leqslant K_1|X_t - Y_t|,\\
|g(X_t) - g(Y_t)| &\leqslant K_2|X_t - Y_t|,
\end{aligned}
\tag{3.1}
$$

for some positive K_1 and K_2 and for all X_t, $Y_t \in \mathbb{R}$. Also, assume that

$$
\begin{aligned}
&\frac{f(X_t)}{X_t} \geqslant 0 \quad \text{and} \quad \frac{g(X_t)}{X_t} \geqslant 0,\\
&\lim_{X_t\to 0}\frac{f(X_t)}{X_t} \quad \text{and} \quad \lim_{X_t\to 0}\frac{g(X_t)}{X_t} \quad \text{exist},
\end{aligned}
\tag{3.2}
$$

and sometimes

$$\frac{f(X_t)}{X_t} \geqslant \beta \quad \text{and} \quad \frac{g(X_t)}{X_t} \geqslant \beta, \tag{3.3}$$

for some $\beta > 0$.

Briefly, for the type of initial function, owing to the continuity and the Lipschitz condition, there will be a unique solution. Because of the Lipschitz growth condition, that solution can be continued for all future time. In the following four theorems, we will show that the solutions of the mentioned equations are bounded and, with an additional condition, their solutions tend to zero.

Consider the first equation which is an scalar uncertain equation of Volterra-Levin type as

$$\mathrm{d}X_t = -\int_{t-L}^{t} p(s-t)f(X_s)\mathrm{d}t\mathrm{d}s - \int_{t-L}^{t} q(s-t)g(X_s)\mathrm{d}C_s\mathrm{d}C_{s-t}, \tag{3.4}$$

where $L > 0$ and p and q are continuous functions satisfying

$$\int_{-L}^{0} p(s)\mathrm{d}s = \int_{-L}^{0} q(s)\mathrm{d}C_{s-t} = 1. \tag{3.5}$$

For K_1 and K_2 of (3.1) let

$$2K_1 \int_{-L}^{0} |p(v)v| \mathrm{d}v + 2K_2 K_\gamma \int_{-L}^{0} |q(v)v| \mathrm{d}v =: \alpha < 1, \tag{3.6}$$

where K_γ is the Lipschitz constant for sample path $C(\gamma)$.

Theorem 3.1. *Let* (3.1), (3.2), (3.5) *and* (3.6) *hold. Then, every solution of* (3.4) *is bounded. Moreover, every solution of* (3.4) *tends to zero if* (3.3) *holds.*

Proof. Let $\psi : [-L, 0] \to \mathbb{R}$ be a continuous initial function and $X_{1t} := X_t(0, \psi)$ be the unique solution. Then, (3.4) can be written as

$$\mathrm{d}X_t = -f(X_t)\mathrm{d}t + \mathrm{d}\int_{-L}^{0} p(s) \int_{t+s}^{t} f(X_u)\mathrm{d}u\mathrm{d}s - g(X_t)\mathrm{d}C_t + \mathrm{d}\int_{-L}^{0} q(s) \int_{t+s}^{t} g(X_u)\mathrm{d}C_u\mathrm{d}C_s.$$

Define non-negative continuous functions $a, b : [0, \infty) \to [0, \infty)$ as

$$a(t) := \frac{f(X_{1t})}{X_{1t}}, \qquad b(t) := \frac{g(X_{1t})}{X_{1t}}.$$

Therefore, for the fixed solution, the equation can be written as

$$\mathrm{d}X_t = -a(t)X_t\mathrm{d}t + \mathrm{d}\int_{-L}^{0} p(s) \int_{t+s}^{t} f(X_u)\mathrm{d}u\mathrm{d}s - b(t)X_t\mathrm{d}C_t + \mathrm{d}\int_{-L}^{0} q(s) \int_{t+s}^{t} g(X_u)\mathrm{d}C_u\mathrm{d}C_s.$$

Now, by the variation of parameters and integration by parts, we have

$$\begin{aligned}
X_t &= \psi(0)e^{-\int_0^t a(s)\mathrm{d}s - \int_0^t b(s)\mathrm{d}C_s} + \int_0^t e^{-\int_v^t a(s)\mathrm{d}s - \int_v^t b(s)\mathrm{d}C_s} \frac{\mathrm{d}}{\mathrm{d}v} \int_{-L}^{0} p(s) \int_{v+s}^{v} f(X_u)\mathrm{d}u\mathrm{d}s\mathrm{d}v \\
&\quad + \int_0^t e^{-\int_v^t a(s)\mathrm{d}s - \int_v^t b(s)\mathrm{d}C_s} \frac{\mathrm{d}}{\mathrm{d}v} \int_{-L}^{0} q(s) \int_{v+s}^{v} g(X_u)\mathrm{d}C_u\mathrm{d}C_s\mathrm{d}C_v \\
&= \psi(0)e^{-\int_0^t a(s)\mathrm{d}s - \int_0^t b(s)\mathrm{d}C_s} + e^{-\int_v^t a(s)\mathrm{d}s - \int_v^t b(s)\mathrm{d}C_s} \int_{-L}^{0} p(s) \int_{v+s}^{v} f(X_u)\mathrm{d}u\mathrm{d}s \,\Big|_0^t \\
&\quad - \int_0^t e^{-\int_v^t a(s)\mathrm{d}s - \int_v^t b(s)\mathrm{d}C_s} [a(v) + b(v)] \int_{-L}^{0} p(s) \int_{v+s}^{v} f(X_u)\mathrm{d}u\mathrm{d}s\mathrm{d}v \\
&\quad + e^{-\int_v^t a(s)\mathrm{d}s - \int_v^t b(s)\mathrm{d}C_s} \int_{-L}^{0} q(s) \int_{v+s}^{v} g(X_u)\mathrm{d}C_u\mathrm{d}C_s \,\Big|_0^t \\
&\quad - \int_0^t e^{-\int_v^t a(s)\mathrm{d}s - \int_v^t b(s)\mathrm{d}C_s} [a(v) + b(v)] \int_{-L}^{0} q(s) \int_{v+s}^{v} g(X_u)\mathrm{d}C_u\mathrm{d}C_s\mathrm{d}C_v \\
&= \psi(0)e^{-\int_0^t a(s)\mathrm{d}s - \int_0^t b(s)\mathrm{d}C_s} + \int_{-L}^{0} p(s) \int_{t+s}^{t} f(X_u)\mathrm{d}u\mathrm{d}s \\
&\quad - e^{-\int_0^t a(s)\mathrm{d}s - \int_0^t b(s)\mathrm{d}C_s} \int_{-L}^{0} p(s) \int_{s}^{0} f(\psi(u))\mathrm{d}u\mathrm{d}s \\
&\quad - \int_0^t e^{-\int_v^t a(s)\mathrm{d}s - \int_v^t b(s)\mathrm{d}C_s} [a(v) + b(v)] \int_{-L}^{0} p(s) \int_{v+s}^{v} f(X_u)\mathrm{d}u\mathrm{d}s\mathrm{d}v \\
&\quad + \int_{-L}^{0} q(s) \int_{t+s}^{t} g(X_u)\mathrm{d}C_u\mathrm{d}C_s - e^{-\int_0^t a(s)\mathrm{d}s - \int_0^t b(s)\mathrm{d}C_s} \int_{-L}^{0} q(s) \int_{s}^{0} g(\psi(u))\mathrm{d}C_u\mathrm{d}C_s \\
&\quad - \int_0^t e^{-\int_v^t a(s)\mathrm{d}s - \int_v^t b(s)\mathrm{d}C_s} [a(v) + b(v)] \int_{-L}^{0} q(s) \int_{v+s}^{v} g(X_u)\mathrm{d}C_u\mathrm{d}C_s\mathrm{d}C_v.
\end{aligned}$$

Let

$$M = \{\phi : [-L,\infty) \to \mathbb{R} \mid \phi_0 = \psi,\ \phi \text{ is bounded and continuous}\}.$$

Define $P : M \to M$ by $(P\phi)(t) = \psi(t)$ for $-L \leqslant t \leqslant 0$. If $t \geqslant 0$, then define

$$\begin{aligned}(P\phi)(t) = {} & \psi(0)e^{-\int_0^t a(s)\mathrm{d}s - \int_0^t b(s)\mathrm{d}C_s} + \int_{-L}^0 p(s)\int_{t+s}^t f(\phi(u))\mathrm{d}u\mathrm{d}s \\ & - e^{-\int_0^t a(s)\mathrm{d}s-\int_0^t b(s)\mathrm{d}C_s}\int_{-L}^0 p(s)\int_s^0 f(\psi(u))\mathrm{d}u\mathrm{d}s \\ & - \int_0^t e^{-\int_v^t a(s)\mathrm{d}s-\int_v^t b(s)\mathrm{d}C_s}[a(v)+b(v)]\int_{-L}^0 p(s)\int_{v+s}^v f(\phi(u))\mathrm{d}u\mathrm{d}s\mathrm{d}v \\ & + \int_{-L}^0 q(s)\int_{t+s}^t g(\phi(u))\mathrm{d}C_u\mathrm{d}C_s \\ & - e^{-\int_0^t a(s)\mathrm{d}s-\int_0^t b(s)\mathrm{d}C_s}\int_{-L}^0 q(s)\int_s^0 g(\psi(u))\mathrm{d}C_u\mathrm{d}C_s \\ & - \int_0^t e^{-\int_v^t a(s)\mathrm{d}s-\int_v^t b(s)\mathrm{d}C_s}[a(v)+b(v)]\int_{-L}^0 q(s)\int_{v+s}^v g(\phi(u))\mathrm{d}C_u\mathrm{d}C_s\mathrm{d}C_v.\end{aligned}$$

Let $\phi,\eta \in M$. Then,

$$\begin{aligned}|(P\phi)(t)-(P\eta)(t)| \leqslant {} & \int_{-L}^0 |p(s)|\int_{t+s}^t |f(\phi(u))-f(\eta(u))|\mathrm{d}u\mathrm{d}s \\ & + \int_0^t e^{-\int_v^t a(s)\mathrm{d}s+\int_v^t b(s)\mathrm{d}C_s}[a(v)+b(v)]\int_{-L}^0 |p(s)|\int_{v+s}^v |f(\phi(u))-f(\eta(u))|\mathrm{d}u\mathrm{d}s\mathrm{d}v \\ & + \int_{-L}^0 |q(s)|\int_{t+s}^t |g(\phi(u))-g(\eta(u)|\mathrm{d}C_u\mathrm{d}C_s \\ & + \int_0^t e^{-\int_v^t a(s)\mathrm{d}s-\int_v^t b(s)\mathrm{d}C_s}[a(v)+b(v)]\int_{-L}^0 |q(s)|\int_{v+s}^v |g(\phi(u))-g(\eta(u))|\mathrm{d}C_u\mathrm{d}C_s\mathrm{d}C_v \\ \leqslant {} & \left[2K_1\int_{-L}^0 |p(s)s|\mathrm{d}s + 2K_2K_\gamma\int_{-L}^0 |q(s)s|\mathrm{d}s\right]\|\phi-\eta\| \leqslant \alpha\|\phi-\eta\|.\end{aligned}$$

Hence, P is a contraction mapping on M and there is a unique fixed point, a bounded solution.

If (3.3) holds and we add to M the condition that $\phi(t) \to 0$ as $t \to \infty$, then it can be shown that $(P\phi)(t) \to 0$ whenever $\phi(t) \to 0$ and the solution of (3.4) tends to zero. □

Consider the second equation as

$$\mathrm{d}X_t = -\int_0^t e^{-a(t-s)}\sin(t-s)\mathrm{d}tf(X_s)\mathrm{d}s - \int_0^t e^{-b(t-s)}\sin(t-s)\mathrm{d}C_{t-s}g(X_s)\mathrm{d}C_s, \tag{3.7}$$

where $a, b > 0$. For K_1 and K_2 of (3.1) suppose

$$2K_1\sup_{t\geqslant 0}\int_0^t\int_{t-u}^\infty e^{-av}|\sin(v)|\mathrm{d}v\mathrm{d}u + 2K_2K_\gamma\sup_{t\geqslant 0}\int_0^t\int_{t-u}^\infty e^{-bv}|\sin(v)|\mathrm{d}v\mathrm{d}u =: \alpha < 1. \tag{3.8}$$

Define

$$l_1 := \int_0^\infty e^{-av}\sin(v)\mathrm{d}v,\ l_2 := \int_0^\infty e^{-bv}\sin(v)\mathrm{d}C_v.$$

Because of the Lipschitz condition on f and g, it can be shown that for each X_0 there is a unique solution $X_t(0, X_0)$.

Theorem 3.2. *Let* (3.1), (3.2), *and* (3.8) *hold. Then, every solution of* (3.7) *is bounded. If, in addition,* (3.3) *holds, then every solution tends to zero as* $t \to \infty$.

Proof. Let $X_{t_0} = X_0$ and X_{1t} be the unique solution. Hence, (3.7) can be written as

$$\mathrm{d}X_t = -l_1 f(X_t)\mathrm{d}t + \mathrm{d}\int_0^t \int_{t-s}^{\infty} e^{-av}\sin(v)\mathrm{d}v f(X_s)\mathrm{d}s - l_2 g(X_t)\mathrm{d}C_t + \mathrm{d}\int_0^t \int_{t-s}^{\infty} e^{-bv}\sin(v)\mathrm{d}C_v g(X_s)\mathrm{d}C_s.$$

Define $c_1(t)$ and $c_2(t)$ as

$$c_1(t) := \frac{f(X_{1t})}{X_{1t}},\ c_2(t) := \frac{g(X_{1t})}{X_{1t}}.$$

Therefore, the equation can be written as

$$\mathrm{d}X_t = -l_1 c_1(t) X_t \mathrm{d}t + \mathrm{d}\int_0^t \int_{t-s}^{\infty} e^{-av}\sin(v)\mathrm{d}v f(X_s)\mathrm{d}s - l_2 c_2(t) X_t \mathrm{d}C_t + \mathrm{d}\int_0^t \int_{t-s}^{\infty} e^{-bv}\sin(v)\mathrm{d}C_v g(X_s)\mathrm{d}C_s.$$

Use the variation of parameters formula to write the solution as

$$\begin{aligned}
X_t &= X_0 e^{-l_1\int_0^t c_1(s)\mathrm{d}s - l_2\int_0^t c_2(s)\mathrm{d}C_s} + \int_0^t e^{-l_1\int_u^t c_1(s)\mathrm{d}s - l_2\int_u^t c_2(s)\mathrm{d}C_s}\frac{\mathrm{d}}{\mathrm{d}u}\int_0^u \int_{u-s}^{\infty} e^{-av}\sin(v)\mathrm{d}v f(X_s)\mathrm{d}s\mathrm{d}u \\
&\quad + \int_0^t e^{-l_1\int_u^t c_1(s)\mathrm{d}s - l_2\int_u^t c_2(s)\mathrm{d}C_s}\frac{\mathrm{d}}{\mathrm{d}u}\int_0^u \int_{u-s}^{\infty} e^{-bv}\sin(v)\mathrm{d}C_v g(X_s)\mathrm{d}C_s\mathrm{d}C_u \\
&= X_0 e^{-l_1\int_0^t c_1(s)\mathrm{d}s - l_2\int_0^t c_2(s)\mathrm{d}C_s} + e^{-l_1\int_u^t c_1(s)\mathrm{d}s - l_2\int_u^t c_2(s)\mathrm{d}C_s}\int_0^u \int_{u-s}^{\infty} e^{-av}\sin(v)\mathrm{d}v f(X_s)\mathrm{d}s\,\Big|_0^t \\
&\quad - \int_0^t e^{-l_1\int_u^t c_1(s)\mathrm{d}s - l_2\int_u^t c_2(t)\mathrm{d}C_s}[l_1c_1(u) + l_2c_2(u)]\int_0^u \int_{u-s}^{\infty} e^{-av}\sin(v)\mathrm{d}v f(X_s)\mathrm{d}s\mathrm{d}u \\
&\quad + e^{-l_1\int_u^t c_1(s)\mathrm{d}s - l_2\int_u^t c_2(s)\mathrm{d}C_s}\int_0^u \int_{u-s}^{\infty} e^{-bv}\sin(v)\mathrm{d}C_v g(X_s)\mathrm{d}C_s\Big|_0^t \\
&\quad - \int_0^t e^{-l_1\int_u^t c_1(s)\mathrm{d}s - l_2\int_u^t c_2(t)\mathrm{d}C_s}[l_1c_1(u) + l_2c_2(u)]\int_0^u \int_{u-s}^{\infty} e^{-bv}\sin(v)\mathrm{d}C_v g(X_s)\mathrm{d}C_s\mathrm{d}C_u \\
&= X_0 e^{-l_1\int_0^t c_1(s)\mathrm{d}s - l_2\int_0^t c_2(s)\mathrm{d}C_s} + \int_0^t \int_{t-s}^{\infty} e^{-av}\sin(v)\mathrm{d}v f(X_s)\mathrm{d}s + \int_0^t \int_{t-s}^{\infty} e^{-bv}\sin(v)\mathrm{d}C_v g(X_s)\mathrm{d}C_s \\
&\quad - \int_0^t e^{-l_1\int_u^t c_1(s)\mathrm{d}s - l_2\int_u^t c_2(t)\mathrm{d}C_s}[l_1c_1(u) + l_2c_2(u)]\int_0^u \int_{u-s}^{\infty} e^{-av}\sin(v)\mathrm{d}v f(X_s)\mathrm{d}s\mathrm{d}u \\
&\quad - \int_0^t e^{-l_1\int_u^t c_1(s)\mathrm{d}s - l_2\int_u^t c_2(t)\mathrm{d}C_s}[l_1c_1(u) + l_2c_2(u)]\int_0^u \int_{u-s}^{\infty} e^{-bv}\sin(v)\mathrm{d}C_v g(X_s)\mathrm{d}C_s\mathrm{d}C_u.
\end{aligned}$$

Now, define

$$M = \{\phi : [0,\infty) \to \mathbb{R} \mid \phi \text{ is bounded and continuous, } \phi(0) = X_0\}.$$

Also, using the equation above, define $P : M \to M$ like in the proof of the Theorem 3.1. Let $\phi, \eta \in M$. Then,

$$\begin{aligned}
|(P\phi)(t) - (P\eta)(t)| &\leqslant \left(2K_1 \sup_{t\geqslant 0}\int_0^t \int_{t-u}^{\infty} e^{-av}|\sin(v)|\mathrm{d}v\mathrm{d}u + 2K_2K_\gamma \sup_{t\geqslant 0}\int_0^t \int_{t-u}^{\infty} e^{-bv}|\sin(v)|\mathrm{d}v\mathrm{d}u\right)\|\phi - \eta\| \\
&< \alpha\|\phi - \eta\|.
\end{aligned}$$

Thus, P is a contraction mapping and we have a unique fixed point, a bounded function satisfying the differential equation. If $\frac{f(X_t)}{X_t}, \frac{g(X_t)}{X_t} \geqslant \beta > 0$, then it can be shown that $(P\phi)(t) \to 0$ whenever $\phi(t) \to 0$. Thus, all solutions tend to zero. □

We next consider the third equation as

$$\mathrm{d}X_t = -\int_{-\infty}^{t} p(s-t)\mathrm{d}tf(X_s)\mathrm{d}s + \int_{-\infty}^{t} q(s-t)\mathrm{d}C_{s-t}g(X_s)\mathrm{d}C_s, \tag{3.9}$$

in which

$$\begin{aligned} &\int_{-\infty}^{0} p(s)\mathrm{d}s = \int_{-\infty}^{0} q(s)\mathrm{d}C_s = 1 \quad \text{and} \\ &\int_{-\infty}^{0}\int_{-\infty}^{v} |p(u)|\mathrm{d}u\mathrm{d}v \quad \text{and} \quad \int_{-\infty}^{0}\int_{-\infty}^{v} |q(u)|\mathrm{d}C_u\mathrm{d}C_v \quad \text{exist.} \end{aligned} \tag{3.10}$$

Assume also that for K_1 and K_2 of (3.1), there exists a positive $\alpha < 1$ such that

$$\left(2K_1 \sup_{t\geqslant 0}\int_0^t \int_{-\infty}^{s-t} |p(u)|\mathrm{d}u\mathrm{d}s + 2K_2K_\gamma \sup_{t\geqslant 0}\int_0^t \int_{-\infty}^{s-t} |q(u)|\mathrm{d}u\mathrm{d}s\right) \leqslant \alpha. \tag{3.11}$$

Theorem 3.3. *Let* (3.1), (3.2), (3.10) *and* (3.11) *hold. Then, every solution of* (3.9) *with bounded continuous initial function* $\psi : (-\infty, 0] \to \mathbb{R}$ *is bounded. If in addition* (3.3) *holds, then those solutions tend to zero as* $t \to \infty$*.*

Proof. We can write (3.9) as

$$\mathrm{d}X_t = -f(X_t)\mathrm{d}t + \mathrm{d}\int_{-\infty}^{t}\int_{-\infty}^{s-t} p(u)\mathrm{d}uf(X_s)\mathrm{d}s - g(X_t)\mathrm{d}C_t + \mathrm{d}\int_{-\infty}^{t}\int_{-\infty}^{s-t} q(u)\mathrm{d}C_ug(X_s)\mathrm{d}C_s.$$

For a given bounded continuous initial function ψ, let X_{1t} be the unique solution which is defined on $[0, \infty)$. Define continuous functions $a(t)$ and $b(t)$ as

$$a(t) := \frac{f(X_{1t})}{X_{1t}}, \qquad b(t) := \frac{g(X_{1t})}{X_{1t}},$$

and write the equation as

$$\mathrm{d}X_t = -a(t)X_t\mathrm{d}t + \mathrm{d}\int_{-\infty}^{t}\int_{-\infty}^{s-t} p(u)\mathrm{d}uf(X_s)\mathrm{d}s - b(t)X_t\mathrm{d}C_t + \mathrm{d}\int_{-\infty}^{t}\int_{-\infty}^{s-t} q(u)\mathrm{d}C_ug(X_s)\mathrm{d}C_s.$$

Now, use the variation of parameters formula to write the solution as

$$\begin{aligned} X_t = {}& \psi(0)e^{-\int_0^t a(s)\mathrm{d}s - \int_0^t b(s)\mathrm{d}C_s} + \int_0^t e^{-\int_v^t a(u)\mathrm{d}u - \int_v^t b(u)\mathrm{d}C_u} \frac{\mathrm{d}}{\mathrm{d}v}\int_{-\infty}^{v}\int_{-\infty}^{s-v} p(u)\mathrm{d}uf(X_s)\mathrm{d}s\mathrm{d}v \\ &+ \int_v^t e^{-\int_v^t a(u)\mathrm{d}u - \int_v^t b(u)\mathrm{d}C_u} \frac{\mathrm{d}}{\mathrm{d}v}\int_{-\infty}^{v}\int_{-\infty}^{s-v} q(u)\mathrm{d}C_ug(X_s)\mathrm{d}C_s\mathrm{d}C_v \\ = {}& \psi(0)e^{-\int_0^t a(s)\mathrm{d}s - \int_0^t b(s)\mathrm{d}C_s} + e^{-\int_v^t a(u)\mathrm{d}u - \int_v^t b(u)\mathrm{d}C_u} \int_{-\infty}^{v}\int_{-\infty}^{s-v} p(u)\mathrm{d}uf(X_s)\mathrm{d}s|_0^t \\ &- \int_0^t [a(v)+b(v)]e^{-\int_v^t a(s)\mathrm{d}s - \int_v^t b(s)\mathrm{d}C_s} \int_{-\infty}^{v}\int_{-\infty}^{s-v} p(u)\mathrm{d}uf(X_s)\mathrm{d}s\mathrm{d}v \\ &+ e^{-\int_v^t a(u)\mathrm{d}u - \int_v^t b(u)\mathrm{d}C_u} \int_{-\infty}^{v}\int_{-\infty}^{s-v} q(u)\mathrm{d}C_ug(X_s)\mathrm{d}C_s|_0^t \\ &- \int_0^t (a(v)+b(v))e^{-\int_v^t a(s)\mathrm{d}s - \int_v^t b(s)\mathrm{d}C_s} \int_{-\infty}^{v}\int_{-\infty}^{s-v} q(u)\mathrm{d}C_ug(X_s)\mathrm{d}C_s\mathrm{d}C_v \end{aligned}$$

$$
\begin{aligned}
&= \psi(0)e^{-\int_0^t a(s)\mathrm{d}s-\int_0^t b(s)\mathrm{d}C_s} + \int_{-\infty}^{t}\int_{-\infty}^{s-t} p(u)\mathrm{d}u f(X_s)\mathrm{d}s \\
&\quad - e^{-\int_0^t a(u)\mathrm{d}u-\int_0^t b(u)\mathrm{d}C_u}\int_{-\infty}^{0}\int_{-\infty}^{s} p(u)\mathrm{d}u f(\psi_s)\mathrm{d}s \\
&\quad - \int_0^t [a(v)+b(v)]e^{-\int_v^t a(s)\mathrm{d}s-\int_v^t b(s)\mathrm{d}C_s}\int_{-\infty}^{v}\int_{-\infty}^{s-v} p(u)\mathrm{d}u f(X_s)\mathrm{d}s\mathrm{d}v \\
&\quad + \int_{-\infty}^{t}\int_{-\infty}^{s-t} q(u)\mathrm{d}C_u g(X_s)\mathrm{d}C_s - e^{-\int_0^t a(u)\mathrm{d}u-\int_0^t b(u)\mathrm{d}C_u}\int_{-\infty}^{0}\int_{-\infty}^{s} q(u)\mathrm{d}C_u g(\psi_s)\mathrm{d}C_s \\
&\quad - \int_0^t [a(v)+b(v)]e^{-\int_v^t a(s)\mathrm{d}s-\int_v^t b(s)\mathrm{d}C_s}\int_{-\infty}^{v}\int_{-\infty}^{s-v} q(u)\mathrm{d}C_u g(X_s)\mathrm{d}C_s\mathrm{d}C_v.
\end{aligned}
$$

Let

$$M = \{\phi : [0,\infty) \to \mathbb{R} \mid \phi \text{ is bounded and continuous, } \phi(t) = \psi(t) \text{ for } t \leqslant 0\}.$$

For $t \leqslant 0$ define $P : M \to M$ as $(P\phi)(t) = \psi(t)$ and for $t \geq 0$ define

$$
\begin{aligned}
(P\phi)(t) &= \psi(0)e^{-\int_0^t a(s)\mathrm{d}s-\int_0^t b(s)\mathrm{d}C_s} + \int_{-\infty}^{t}\int_{-\infty}^{s-t} p(u)\mathrm{d}u f(\phi(s))\mathrm{d}s \\
&\quad - e^{-\int_0^t a(u)\mathrm{d}u-\int_0^t b(u)\mathrm{d}C_u}\int_{-\infty}^{0}\int_{-\infty}^{s} p(u)\mathrm{d}u f(\psi_s)\mathrm{d}s \\
&\quad - \int_0^t [a(v)+b(v)]e^{-\int_v^t a(s)\mathrm{d}s-\int_v^t b(s)\mathrm{d}C_s}\int_{-\infty}^{v}\int_{-\infty}^{s-v} p(u)\mathrm{d}u f(\phi(s))\mathrm{d}s\mathrm{d}v \\
&\quad + \int_{-\infty}^{t}\int_{-\infty}^{s-t} q(u)\mathrm{d}C_u g(\phi(s))\mathrm{d}C_s - e^{-\int_0^t a(u)\mathrm{d}u-\int_0^t b(u)\mathrm{d}C_u}\int_{-\infty}^{0}\int_{-\infty}^{s} q(u)\mathrm{d}C_u g(\psi_s)\mathrm{d}C_s \\
&\quad - \int_0^t [a(v)+b(v)]e^{-\int_v^t a(s)\mathrm{d}s-\int_v^t b(s)\mathrm{d}C_s}\int_{-\infty}^{v}\int_{-\infty}^{s-v} q(u)\mathrm{d}C_u g(\phi(s))\mathrm{d}C_s\mathrm{d}C_v.
\end{aligned}
$$

Let $\xi, \eta \in M$. Then,

$$
\begin{aligned}
|(P\xi)(t) - (P\eta)(t)| &\leqslant \int_{-\infty}^{v}\int_{-\infty}^{s-t} |p(u)|\mathrm{d}u|f(\xi(s)) - f(\eta(s))|\mathrm{d}s \\
&\quad - \int_0^t [a(v)+b(v)]e^{-\int_v^t a(s)\mathrm{d}s-\int_v^t b(s)\mathrm{d}C_s}\int_{-\infty}^{v}\int_{-\infty}^{s-v} |p(u)|\mathrm{d}u|f(\xi(s)) - f(\eta(s))|\mathrm{d}s\mathrm{d}v \\
&\quad + \int_{-\infty}^{v}\int_{-\infty}^{s-t} |q(u)|\mathrm{d}u|g(\xi(s)) - g(\eta(s))|\mathrm{d}C_s \\
&\quad - \int_0^t [a(v)+b(v)]e^{-\int_v^t a(s)\mathrm{d}s-\int_v^t b(s)\mathrm{d}C_s}\int_{-\infty}^{v}\int_{-\infty}^{s-v} |q(u)|\mathrm{d}C_u|g(\xi(s)) - g(\eta(s))|\mathrm{d}C_s\mathrm{d}C_v \\
&\leqslant \left(2K_1\sup_{t\geqslant 0}\int_0^t\int_{-\infty}^{s-t}|p(u)|\mathrm{d}u\mathrm{d}s + 2K_2K_\gamma\sup_{t\geqslant 0}\int_0^t\int_{-\infty}^{s-t}|q(u)|\mathrm{d}u\mathrm{d}s\right)\|\xi - \eta\| \\
&< \alpha\|\xi - \eta\|.
\end{aligned}
$$

Therefore, P is a contraction and there is a unique fixed point.

If in addition (3.3) holds, then modify M to include the condition $\phi(t) \to 0$. Then, it can be shown that $(P\phi)(t) \to 0$ whenever $\phi(t) \to 0$ and the solution of (3.4) tends to zero. □

Our final equation is as follows.

$$\mathrm{d}X_t = [-a(t)f(X_{q(t)})]\mathrm{d}t + [-b(t)g(X_{q(t)})]\mathrm{d}C_t, \tag{3.12}$$

where $q : \mathbb{R} \to \mathbb{R}$ is continuous and strictly increasing to ∞, $q(t) < t$, q has the inverse function $h(t)$ and a and b are continuous from $[0,\infty)$ to $[0,\infty)$. Suppose that for K_1 and K_2 of (3.1), there is an α such that

$$\sup_{t\geqslant 0}\int_t^{h(t)} [K_1a(u) + K_2K_\gamma b(u)]\mathrm{d}u \leqslant \alpha < 1. \tag{3.13}$$

Theorem 3.4. *Let* (3.1), (3.2) *and* (3.13) *hold. Then, every solution of* (3.12) *is bounded. If, in addition,* (3.3) *holds and*

$$\left(\int_0^t a(s)\mathrm{d}s + \int_0^t b(s)\mathrm{d}Cs\right) \to \infty \quad as \quad t \to \infty, \tag{3.14}$$

then every solution of (3.12) *tends to zero as* $t \to \infty$.

Proof. Write (3.12) as

$$\mathrm{d}X_t = -a(h(t))h'(t)f(X_t)\mathrm{d}t - b(h(t))h'(t)g(X_t)\mathrm{d}C_t - \mathrm{d}\int_{h(t)}^t a(s)f(X_{q(s)})\mathrm{d}s - \mathrm{d}\int_{h(t)}^t b(s)g(X_{q(s)})\mathrm{d}C_s.$$

Now, considering a continuous initial function $\psi : [q(0), 0] \to \mathbb{R}$, let X_{1t} denotes the unique solution and define the following continuous functions.

$$L(t)X_t := a(h(t))h'(t)\frac{f(X_{1t})}{X_{1t}}, \qquad D(t)X_t := b(h(t))h'(t)\frac{g(X_{1t})}{X_{1t}}.$$

Therefore, the equation can be written as

$$\mathrm{d}X_t = -L(t)X_t\mathrm{d}t - \mathrm{d}\int_{h(t)}^t a(s)f(X_{q(s)})\mathrm{d}s - D(t)X_t\mathrm{d}C_t\mathrm{d}\int_{h(t)}^t b(s)g(X_{q(s)})\mathrm{d}C_s.$$

Like the method used in the previous theorems, we have

$$\begin{aligned}
X_t &= \psi(0)e^{-\int_0^t L(s)\mathrm{d}s - \int_0^t D(s)\mathrm{d}C_s} - \int_0^t e^{-\int_s^t L(u)\mathrm{d}u - \int_s^t D(u)\mathrm{d}C_u}\frac{\mathrm{d}}{\mathrm{d}s}\int_{h(t)}^t a(u)f(X_{q(u)})\mathrm{d}u\mathrm{d}s \\
&\quad - \int_0^t e^{-\int_s^t L(u)\mathrm{d}u - \int_s^t D(u)\mathrm{d}C_u}\frac{\mathrm{d}}{\mathrm{d}s}\int_{h(t)}^t b(u)g(X_{q(u)})\mathrm{d}C_u\mathrm{d}C_s \\
&= \psi(0)e^{-\int_0^t L(s)\mathrm{d}s - \int_0^t D(s)\mathrm{d}C_s} - e^{-\int_s^t L(u)\mathrm{d}u - \int_s^t D(u)\mathrm{d}C_u}\int_{h(s)}^s a(u)f(X_{q(u)})\mathrm{d}u|_0^t \\
&\quad + \int_0^t [L(s)+D(s)]e^{-\int_s^t L(u)\mathrm{d}u - \int_s^t D(u)\mathrm{d}C_u}\int_{h(s)}^s a(u)f(X_{q(u)})\mathrm{d}u\mathrm{d}s \\
&\quad - e^{-\int_s^t L(u)\mathrm{d}u - \int_s^t D(u)\mathrm{d}C_u}\int_{h(s)}^s b(u)g(X_{q(u)})\mathrm{d}C_u|_0^t \\
&\quad + \int_0^t [L(s)+D(s)]e^{-\int_s^t L(u)\mathrm{d}u - \int_s^t D(u)\mathrm{d}C_u}\int_{h(s)}^s b(u)g(X_{q(u)})\mathrm{d}C_u\mathrm{d}C_s \\
&= \psi(0)e^{-\int_0^t L(s)\mathrm{d}s - \int_0^t D(s)\mathrm{d}C_s} - \int_{h(t)}^t a(u)f(X_{q(u)})\mathrm{d}u + e^{-\int_0^t L(u)\mathrm{d}u - \int_0^t D(u)\mathrm{d}C_u}\int_{h(0)}^0 a(u)f(X_{q(u)})\mathrm{d}u \\
&\quad + \int_0^t [L(s)+D(s)]e^{-\int_s^t L(u)\mathrm{d}u - \int_s^t D(u)\mathrm{d}C_u}\int_{h(s)}^s a(u)f(X_{q(u)})\mathrm{d}u\mathrm{d}s \\
&\quad - \int_{h(t)}^t b(u)g(X_{q(u)})\mathrm{d}u + e^{-\int_0^t L(u)\mathrm{d}u - \int_0^t D(u)\mathrm{d}C_u}\int_{h(0)}^0 b(u)g(X_{q(u)})\mathrm{d}C_u \\
&\quad + \int_0^t [L(s)+D(s)]e^{-\int_s^t L(u)\mathrm{d}u - \int_s^t D(u)\mathrm{d}C_u}\int_{h(s)}^s b(u)g(X_{q(u)})\mathrm{d}C_u\mathrm{d}C_s.
\end{aligned}$$

Let

$$M = \{\phi : [0,\infty) \to \mathbb{R} \mid \phi \text{ is bounded and continuous, } \phi(0) = X_0\},$$

and define $P : M \to M$ for $t \geqslant 0$ by

$$
\begin{aligned}
(P\phi)(t) = &\psi(0)e^{-\int_0^t L(s)\mathrm{d}s-\int_0^t D(s)\mathrm{d}C_s} - \int_{h(t)}^t a(u)f(\phi(u))\mathrm{d}u + e^{-\int_0^t L(u)\mathrm{d}u-\int_0^t D(u)\mathrm{d}C_u}\int_{h(0)}^0 a(u)f(\phi(u))\mathrm{d}u \\
&+ \int_0^t [L(s)+D(s)]e^{-\int_s^t L(u)\mathrm{d}u-\int_s^t D(u)\mathrm{d}C_u}\int_{h(s)}^s a(u)f(\phi(u))\mathrm{d}u\mathrm{d}s \\
&- \int_{h(t)}^t b(u)g(\phi(u))\mathrm{d}u + e^{-\int_0^t L(u)\mathrm{d}u-\int_0^t D(u)\mathrm{d}C_u}\int_{h(0)}^0 b(u)g(\phi(u))\mathrm{d}C_u \\
&+ \int_0^t [L(s)+D(s)]e^{-\int_s^t L(u)\mathrm{d}u-\int_s^t D(u)\mathrm{d}C_u}\int_{h(s)}^s b(u)g(\phi(u))\mathrm{d}C_u\mathrm{d}C_s.
\end{aligned}
$$

For $\phi, \eta \in M$, we have

$$
|(P\phi)(t)-(P\eta)(t)| \leqslant \int_{h(t)}^t a(u)|(f\phi)(u)-(f\eta)(u)|\mathrm{d}u + \int_{h(t)}^t b(u)|(g\phi)(u)-(g\eta)(u)|\mathrm{d}C_u \leqslant \alpha\|\phi-\eta\|.
$$

Thus, according to (3.13), P is a contraction mapping on M, and we have a bounded solution. If we have (3.3), (3.14) and $\phi(t) \to 0$ as $t \to \infty$, then the solution (3.12) tends to zero. □

Example 3.5. Consider the equation

$$
\mathrm{d}X_t = \left[-\big(1.1+\frac{\sin t}{A}\big)X_{t-r}\right]\mathrm{d}t + \left[-\big(1.1+\frac{\cos t}{B}\big)X_{t-r}\right]\mathrm{d}C_t, \tag{3.15}
$$

where A, B and r are positive real numbers. It is easy to see that for r small enough and A and B large enough, all conditions of Theorem 3.4 are satisfied. Thus, every solution of (3.15) tend to zero as $t \to \infty$.

4. A necessary and sufficient condition

In this section, a necessary and sufficient condition for the asymptotic stability of an UDDE will be presented.

Consider the UDDE

$$
\mathrm{d}X_t = [-a(t)X_t + f(t, X_{t-r})]\mathrm{d}t + [-m(t)X_t + g(t, X_{t-r})]\mathrm{d}C_t, \tag{4.1}
$$

in which $a,m : [0,\infty) \to \mathbb{R}$ and $f, g : [0,\infty)\times G \to \mathbb{R}$ are continuous where $G = \{\phi : (-\infty, 0] \to \mathbb{R},\ \phi \text{ is bounded continuous}\}$ which is a Banach space with the supremum norm $\|.\|$.

For each $\beta > 0$, define $G(\beta) = \{\phi \in G : \|\phi\| \leqslant \beta\}$. Also, given a function $\psi : \mathbb{R} \to \mathbb{R}$, define $\|\psi\|_{[s,t]} = \sup\{|\psi(u)| : s \leqslant u \leqslant t\}$. For $A > 0$, a continuous function $X_t : (-\infty, A) \to \mathbb{R}$ is called a solution of (4.1) through $(t_0, \phi) \in [0,\infty)\times G$ if $X_{t_0} = \phi$ and X_t satisfies (4.1) on $[t_0, A)$.

If $f(t, \phi)$ and $g(t, \phi)$ are not linear functionals, we may find many fundamental difficulties in the process of constructing a Lyapunov function. Therefore, fixed point method will be used to present a necessary and sufficient condition about the stationary of (4.1).

Theorem 4.1. *Suppose that there exist positive constants α and L and continuous functions $b,d : [0,\infty) \to [0,\infty)$ such that the following conditions hold.*

(I) $\liminf_{t\to\infty}\int_0^t a(s)\mathrm{d}s > -\infty$ *and* $\liminf_{t\to\infty}\int_0^t m(s)\mathrm{d}C_s > -\infty$.

(II) $\int_0^t e^{-\int_s^t a(u)\mathrm{d}u-\int_s^t m(u)\mathrm{d}C_u}h(s)\mathrm{d}s \leqslant \alpha < 1$ *where* $h(s) = \max\{b(s), d(s)\}$ *for* $0 \leqslant s \leqslant t$ *and* $t \geqslant 0$.

(III) $|f(t,\phi)-f(t,\psi)| \leqslant b(t)\|\phi-\psi\|$ *and* $|g(t,\phi)-g(t,\psi)| \leqslant d(t)\|\phi-\psi\|$ *for all* $\phi,\psi \in G(L)$ *and* $f(t,0) = g(t,0) = 0$.

(IV) For each $\epsilon > 0$ and $t_1 > 0$, there exists $t_2 > t_1$ such that $[t \geqslant t_2, X_t \in G(L)]$ implies $|f(t, X_t)| \leqslant b(t)(\epsilon + \|X\|_{[t_1,t]})$ and $|g(t, X_t)| \leqslant d(t)(\epsilon + \|X\|_{[t_1,t]})$.

Then, the zero solution of (4.1) is asymptotically stable if and only if

(V) $\int_0^t a(s)\mathrm{d}s \to \infty$ as $t \to \infty$ and $\int_0^t m(s)\mathrm{d}C_s \to \infty$ as $t \to \infty$.

Proof. First, suppose that (V) holds. Let $t_0 \geqslant 0$ and find $\delta_0 > 0$ such that $\delta_0 K + (1 + K_\gamma)\alpha L \leqslant L$ where

$$K = \sup_{t \geqslant t_0} \left\{ e^{-\int_{t_0}^t a(s)\mathrm{d}s - \int_{t_0}^t m(s)\mathrm{d}C_s} \right\}. \tag{4.2}$$

Let $\phi \in G(\delta_0)$ be fixed and define

$$S = \{X : \mathbb{R} \to \mathbb{R},\ X_{t_0} = \phi, X_t \in G(L) \text{ for } t \geqslant t_0,\ X_t \to 0 \ as\ t \to \infty\}.$$

Then, S is a complete metric space with metric $\rho(X, Y) = \sup_{t \geqslant t_0}\{|X_t - Y_t|\}$. Define $P : S \to S$ by $(PX)(t) = \phi(t)$ for $t \leqslant t_0$ and

$$(PX)(t) = \phi(t_0)e^{-\int_{t_0}^t a(s)\mathrm{d}s - \int_{t_0}^t m(s)\mathrm{d}C_s} + \int_{t_0}^t e^{-\int_s^t a(u)\mathrm{d}u - \int_s^t m(u)\mathrm{d}C_u} f(s, X_s)\mathrm{d}s$$
$$+ \int_{t_0}^t e^{-\int_s^t a(u)\mathrm{d}u - \int_s^t m(u)\mathrm{d}C_u} g(s, X_s)\mathrm{d}C_s,$$

for $t \geqslant t_0$. Clearly $(PX) : \mathbb{R} \to \mathbb{R}$ is continuous with $(PX)(t_0) = \phi$ and

$$\begin{aligned}|(PX)(t)| &\leqslant |\phi(t_0)|e^{-\int_{t_0}^t a(s)\mathrm{d}s - \int_{t_0}^t m(s)\mathrm{d}C_s} + \int_{t_0}^t e^{-\int_s^t a(u)\mathrm{d}s - \int_s^t m(u)\mathrm{d}C_u} b(s)\|X_s\|\mathrm{d}s \\ &+ \int_{t_0}^t e^{-\int_s^t a(u)\mathrm{d}s - \int_s^t m(u)\mathrm{d}C_u} d(s)\|X_s\|\mathrm{d}C_s \\ &\leqslant |\phi(t_0)|e^{-\int_{t_0}^t a(s)\mathrm{d}s - \int_{t_0}^t m(s)\mathrm{d}C_s} + \int_{t_0}^t e^{-\int_s^t a(u)\mathrm{d}s - \int_s^t m(u)\mathrm{d}C_u} h(s)\|X_s\|\mathrm{d}s \\ &+ K_\gamma \int_{t_0}^t e^{-\int_s^t a(u)\mathrm{d}s - \int_s^t m(u)\mathrm{d}C_u} h(s)\|X_s\|\mathrm{d}s \leqslant K\delta_0 + \alpha L + K_\gamma \alpha L \leqslant L.\end{aligned}$$

Thus, $(PX)(t) \in G(L)$ for $t \geqslant t_0$.

Now, we show that $(PX)(t) \to 0$ as $t \to \infty$. Let $X \in S$ and $\epsilon > 0$ be given. Since $X_t \to 0$ as $t \to \infty$, there exists $t_1 > t_0$ such that $|X_t| < \epsilon$ for all $t > t_1$. Since $|X_t| \leqslant L$ for all $t \in \mathbb{R}$, by (IV) there is $t_2 > t_1$ such that $t > t_2$ implies

$$|f(t, X_t)| \leqslant b(t)(\epsilon + \|X\|_{[t_1,t]}),\ |g(t, X_t)| \leqslant d(t)(\epsilon + \|X\|_{[t_1,t]}).$$

For $t \geqslant t_2$, we have

$$\begin{aligned}&\left| \int_{t_0}^t e^{-\int_s^t a(u)\mathrm{d}u - \int_s^t m(u)\mathrm{d}C_u} f(s, X_s)\mathrm{d}s + \int_{t_0}^t e^{-\int_s^t a(u)\mathrm{d}u - \int_s^t m(u)\mathrm{d}C_u} g(s, X_s)\mathrm{d}C_s \right| \\ &\leqslant \int_{t_0}^{t_2} e^{-\int_s^t a(u)\mathrm{d}u - \int_s^t m(u)\mathrm{d}C_u} |f(s, X_s)|\mathrm{d}s + \int_{t_2}^t e^{-\int_s^t a(u)\mathrm{d}u - \int_s^t m(u)\mathrm{d}C_u} |f(s, X_s)|\mathrm{d}s \\ &+ \int_{t_0}^{t_2} e^{-\int_s^t a(u)\mathrm{d}u - \int_s^t m(u)\mathrm{d}C_u} |g(s, X_s)|\mathrm{d}C_s + \int_{t_2}^t e^{-\int_s^t a(u)\mathrm{d}u - \int_s^t m(u)\mathrm{d}C_u} |g(s, X_s)|\mathrm{d}C_s \\ &\leqslant \int_{t_0}^{t_2} e^{-\int_s^t a(u)\mathrm{d}u - \int_s^t m(u)\mathrm{d}C_u} b(s)\|X_s\|\mathrm{d}s + \int_{t_2}^t e^{-\int_s^t a(u)\mathrm{d}u - \int_s^t m(u)\mathrm{d}C_u} b(s)(\epsilon + \|X\|_{[t_1,s]})|\mathrm{d}s \\ &+ K_\gamma \int_{t_0}^{t_2} e^{-\int_s^t a(u)\mathrm{d}u - \int_s^t m(u)\mathrm{d}C_u} d(s)\|X_s\|\mathrm{d}s + K_\gamma \int_{t_2}^t e^{-\int_s^t a(u)\mathrm{d}u - \int_s^t m(u)\mathrm{d}C_u} d(s)(\epsilon + \|X\|_{[t_1,s]})|\mathrm{d}s \\ &\leqslant \alpha L e^{-\int_{t_2}^t a(u)\mathrm{d}u - \int_{t_2}^t m(u)\mathrm{d}C_u} + 2\alpha\epsilon + K_\gamma \alpha L e^{-\int_{t_2}^t a(u)\mathrm{d}u - \int_{t_2}^t m(u)\mathrm{d}C_u} + 2K_\gamma \alpha\epsilon.\end{aligned}$$

By (V), there exists $t_3 > t_2$ such that

$$\delta_0 e^{-\int_{t_0}^t a(s)\mathrm{d}s - \int_{t_0}^t m(s)\mathrm{d}C_s} + Le^{-\int_{t_2}^t a(s)\mathrm{d}s - \int_{t_2}^t m(s)\mathrm{d}C_s} + K_\gamma L e^{-\int_{t_2}^t a(s)\mathrm{d}s - \int_{t_2}^t m(s)\mathrm{d}C_s} < \epsilon.$$

Thus, for $t \geqslant t_3$, we have

$$\begin{aligned}|(PX)(t)| &\leqslant \delta_0 e^{-\int_{t_0}^t a(s)\mathrm{d}s - \int_{t_0}^t m(s)\mathrm{d}C_s} + \alpha L e^{-\int_{t_2}^t a(s)\mathrm{d}s - \int_{t_2}^t m(s)\mathrm{d}C_s} \\ &\quad + 2\alpha\epsilon + K_\gamma \alpha L e^{-\int_{t_2}^t a(s)\mathrm{d}s - \int_{t_2}^t m(s)\mathrm{d}C_s} + 2K_\gamma\alpha\epsilon \leqslant 2(1+K_\gamma)\alpha\epsilon.\end{aligned}$$

Therefore, $(PX)(t) \to 0$ as $t \to \infty$ and hence $(PX) \in S$.

To show that P is a contraction mapping, observer that for $t \geqslant t_0$

$$\begin{aligned}|(PX)(t) - (PY)(t)| &\leqslant \int_{t_0}^t e^{-\int_s^t a(u)\mathrm{d}u - \int_s^t m(u)\mathrm{d}C_u} |f(s,X_s) - f(s,Y_s)|\mathrm{d}s \\ &\quad + \int_{t_0}^t e^{-\int_s^t a(u)\mathrm{d}u - \int_s^t m(u)\mathrm{d}C_u} |g(s,X_s) - g(s,Y_s)|\mathrm{d}C_s \\ &\leqslant \int_{t_0}^t e^{-\int_s^t a(u)\mathrm{d}u - \int_s^t m(u)\mathrm{d}C_u} b(s)|X_s - Y_s|\mathrm{d}s \\ &\quad + K_\gamma \int_{t_0}^t e^{-\int_s^t a(u)\mathrm{d}u - \int_s^t m(u)\mathrm{d}C_u} d(s)|X_s - Y_s|\mathrm{d}s \\ &\leqslant \alpha|X_t - Y_t| + K_\gamma|X_t - Y_t| = \alpha(1+K_\gamma)|X_t - Y_t|.\end{aligned}$$

Thus, P has a unique fixed point X in S which is a solution of (4.1) with $\phi \in G(\delta_0)$ and $X_t = X_t(t_0,\phi) \to 0$ as $t \to \infty$.

To prove the asymptotic stability, we need to show that the zero solution of (4.1) is stable. Let $\epsilon > 0$ ($\epsilon < L$) be given. Choose $\delta > 0$ ($\delta < \epsilon$) such that $\delta K + (1+K_\gamma)\alpha\epsilon < \epsilon$. If $X_t = X_t(t_0,\phi)$ be a solution of (4.1) with $\|\phi\| < \delta$, then

$$\begin{aligned}X_t &= \phi(t_0) e^{-\int_{t_0}^t a(u)\mathrm{d}u - \int_{t_0}^t m(u)\mathrm{d}C_u} + \int_{t_0}^t e^{-\int_s^t a(u)\mathrm{d}u - \int_s^t m(u)\mathrm{d}C_u} f(s,X_s)\mathrm{d}s \\ &\quad + \int_{t_0}^t e^{-\int_s^t a(u)\mathrm{d}u - \int_s^t m(u)\mathrm{d}C_u} g(s,X_s)\mathrm{d}C_s.\end{aligned}$$

We claim that $|X_t| < \epsilon$ for all $t \geqslant t_0$. Notice that $|X_{t_0}| < \epsilon$. If there exists $t^* > t_0$ such that $|X_{t^*}| = \epsilon$ and $|X_s| < \epsilon$ for $t_0 \leqslant s < t^*$, then

$$\begin{aligned}|X_{t^*}| &\leqslant \delta e^{-\int_{t_0}^{t^*} a(s)\mathrm{d}s - \int_{t_0}^{t^*} m(s)\mathrm{d}C_s} + \int_{t_0}^{t^*} e^{-\int_s^{t^*} a(u)\mathrm{d}u - \int_s^{t^*} m(u)\mathrm{d}C_u} b(s)\|X_s\|\mathrm{d}s \\ &\quad + \int_{t_0}^{t^*} e^{-\int_s^{t^*} a(u)\mathrm{d}u - \int_s^{t^*} m(u)\mathrm{d}C_u} d(s)\|X_s\|\mathrm{d}C_s \\ &\leqslant \delta e^{-\int_{t_0}^{t^*} a(s)\mathrm{d}s - \int_{t_0}^{t^*} m(s)\mathrm{d}C_s} + \int_{t_0}^{t^*} e^{-\int_s^{t^*} a(u)\mathrm{d}u - \int_s^{t^*} m(u)\mathrm{d}C_u} b(s)\|X_s\|\mathrm{d}s \\ &\quad + K_\gamma \int_{t_0}^{t^*} e^{-\int_s^{t^*} a(u)\mathrm{d}u - \int_s^{t^*} m(u)\mathrm{d}C_u} d(s)\|X_s\|\mathrm{d}s \\ &\leqslant \delta K + (1+K_\gamma)\alpha\epsilon < \epsilon,\end{aligned}$$

which contradicts the definition of t^*. Thus, $|X_t| < \epsilon$ for $t \geqslant t_0$ and the zero solution of (4.1) is stable. This shows that the zero solution of (4.1) is asymptotically stable if (V) holds.

Conversely, assume that (V) does not satisfied. Then, by (I) there exists a sequence $\{t_n\}$, $t_n \to \infty$ as $n \to \infty$, such that

$$\lim_{n\to\infty}\int_0^{t_n} a(s)\mathrm{d}s + \lim_{n\to\infty}\int_0^{t_n} m(s)\mathrm{d}C_s = l,$$

for some $l \in \mathbb{R}$. We may choose a positive constant Q satisfying

$$-Q < \int_0^{t_n} a(s)\mathrm{d}s + \int_0^{t_n} m(s)\mathrm{d}C_s < Q,$$

for all $n = 1, 2, \ldots$. By (II), we have

$$\int_0^{t_n} e^{-\int_s^{t_n} a(u)\mathrm{d}u - \int_s^{t_n} m(u)\mathrm{d}C_u} h(s)\mathrm{d}s < \alpha.$$

This yields

$$\begin{aligned}
&\int_0^{t_n} e^{\int_0^s a(u)\mathrm{d}u + \int_0^s m(u)\mathrm{d}C_u} b(s)\mathrm{d}s + \int_0^{t_n} e^{\int_0^s a(u)\mathrm{d}u + \int_0^s m(u)\mathrm{d}C_u} d(s)\mathrm{d}C_s \\
&\leqslant \alpha e^{\int_0^{t_n} a(u)\mathrm{d}u + \int_0^{t_n} m(u)\mathrm{d}C_u} + \alpha K_\gamma e^{\int_0^{t_n} a(u)\mathrm{d}u + \int_0^{t_n} m(u)\mathrm{d}C_u} \\
&< \alpha(1+K_\gamma) e^{\int_0^{t_n} a(u)\mathrm{d}u + \int_0^{t_n} m(u)\mathrm{d}C_u} < \alpha(1+K_\gamma)e^Q.
\end{aligned}$$

Therefore, the sequence

$$\left\{\int_0^{t_n} e^{\int_0^s a(u)\mathrm{d}u + \int_0^s m(u)\mathrm{d}C_u} b(s)\mathrm{d}s + \int_0^{t_n} e^{\int_0^s a(u)\mathrm{d}u + \int_0^s m(u)\mathrm{d}C_u} d(s)\mathrm{d}C_t\right\}_{n=0}^{\infty}$$

is bounded. Hence, there exists a convergent subsequence. We may assume

$$\lim_{n\to\infty}\int_0^{t_n} e^{\int_0^s a(u)\mathrm{d}u + \int_0^s m(u)\mathrm{d}C_u}[b(s)\mathrm{d}s + d(s)\mathrm{d}C_s] = \gamma,$$

for some $\gamma \in [0, \infty)$ and choose a large enough positive integer $\bar{k}$ such that

$$\int_{t_{\bar{k}}}^{t_n} e^{\int_0^s a(u)\mathrm{d}u + \int_0^s m(u)\mathrm{d}C_u}[b(s)\mathrm{d}s + d(s)\mathrm{d}C_s] < \frac{1-\alpha-\alpha K_\gamma}{2K^2},$$

for all $n \geqslant \bar{k}$. By (I), K in (4.2) is well-defined. Now, consider the solution $X_t = X_t(t_{\bar{k}}, \phi)$ with $\phi(s) = \delta_0$ for $s \leqslant t_{\bar{k}}$. Then, $|X_t| \leqslant L$ for all $t \geqslant t_{\bar{k}}$ and

$$\begin{aligned}
|X_t| \leqslant \delta_0 e^{-\int_{t_{\bar{k}}}^t a(s)\mathrm{d}s - \int_{t_{\bar{k}}}^t m(s)\mathrm{d}C_s} &+ \int_{t_{\bar{k}}}^t e^{-\int_s^t a(u)\mathrm{d}u - \int_s^t m(u)\mathrm{d}C_u} b(s)\|X_s\|\mathrm{d}s \\
&+ \int_{t_{\bar{k}}}^t e^{-\int_s^t a(u)\mathrm{d}u - \int_s^t m(u)\mathrm{d}C_u} d(s)\|X_s\|\mathrm{d}C_s \leqslant \delta_0 K + (1+K_\gamma)\alpha\|X_t\|.
\end{aligned}$$

This implies that

$$\|X_t\| \leqslant \frac{\delta_0 K}{[1-(1+K_\gamma)\alpha]} =: \lambda,$$

for all $t \geqslant t_k$. On the other hand, for $n \geqslant \bar{k}$, we have

$$\begin{aligned}
|X_{t_n}| &\geqslant \delta_0 e^{-\int_{t_{\bar{k}}}^{t_n} a(s)\mathrm{d}s - \int_{t_{\bar{k}}}^{t_n} m(s)\mathrm{d}C_s} - \int_{t_{\bar{k}}}^{t_n} e^{-\int_s^{t_n} a(u)\mathrm{d}u - \int_s^{t_n} m(u)\mathrm{d}C_u} b(s)\|X_s\|\mathrm{d}s \\
&\quad - \int_{t_{\bar{k}}}^{t_n} e^{-\int_s^{t_n} a(u)\mathrm{d}u - \int_s^{t_n} m(u)\mathrm{d}C_u} d(s)\|X_s\|\mathrm{d}C_s \\
&\geqslant \delta_0 e^{-\int_{t_{\bar{k}}}^{t_n} a(s)\mathrm{d}s - \int_{t_{\bar{k}}}^{t_n} m(s)\mathrm{d}C_s} - \lambda e^{-\int_0^{t_n} a(u)\mathrm{d}u - \int_0^{t_n} m(u)\mathrm{d}C_u} \int_{t_{\bar{k}}}^{t_n} e^{\int_0^s a(u)\mathrm{d}u + \int_0^s m(u)\mathrm{d}C_u} b(s)\mathrm{d}s \\
&\quad - \lambda e^{-\int_0^{t_n} a(u)\mathrm{d}u - \int_0^{t_n} m(u)\mathrm{d}C_u} \int_{t_{\bar{k}}}^{t_n} e^{\int_0^s a(u)\mathrm{d}u + \int_0^s m(u)\mathrm{d}C_u} d(s)\mathrm{d}C_s \\
&\geqslant e^{-\int_{t_{\bar{k}}}^{t_n} a(s)\mathrm{d}s - \int_{t_{\bar{k}}}^{t_n} m(s)\mathrm{d}C_s} \left[\delta_0 - \lambda K \int_{t_{\bar{k}}}^{t_n} e^{\int_0^s a(u)\mathrm{d}u + \int_0^s m(u)\mathrm{d}u} b(s)\mathrm{d}s - \lambda K \int_{t_{\bar{k}}}^{t_n} e^{\int_0^s a(u)\mathrm{d}u + \int_0^s m(u)\mathrm{d}u} d(s)\mathrm{d}C_s\right] \\
&\geqslant \frac{1}{2}\delta_0 e^{-\int_{t_{\bar{k}}}^{t_n} a(s)\mathrm{d}s - \int_{t_{\bar{k}}}^{t_n} m(s)\mathrm{d}C_s} \geqslant \frac{1}{2}\delta_0 e^{-2Q}.
\end{aligned}$$

This implies $X_t \nrightarrow 0$ as $t \to \infty$. Thus, condition (V) is necessary for the asymptotic stability of the zero solution of (4.1) and the proof is complete. □

Example 4.2. Consider the UDDE

$$\mathrm{d}X_t = \left[-\sqrt{t}X_t + \frac{1}{A}e^{\frac{-2}{3}t^{\frac{3}{2}}} X_{t-r}\right]\mathrm{d}t + \left[-\sqrt[3]{t}X_t + \frac{1}{B}\sqrt{t}X_{t-r}\right]\mathrm{d}C_t, \tag{4.3}$$

where A and B are positive real numbers. It is easy to see that for A and B large enough and by choosing $b(s) = \frac{1}{A}e^{\frac{-2}{3}s^{\frac{3}{2}}}$ and $d(s) = \frac{1}{B}\sqrt{s}$, there exist $\alpha \in (0,1)$ and L such that all conditions of Theorem 4.1 are satisfied. Therefore, the zero solution of (4.3) is asymptotically stable.

5. Conclusion

In this work, we have studied the stability of four types of uncertain delay differential equations of Volterra-Levin type. We have showed that the solutions were bounded and, with an additional condition, the solutions tend to zero. Also, we have presented a necessary and sufficient condition for the stability of an uncertain delay differential equation with the mean of the fixed point method.

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