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Research Article

A note on CSP rings

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Abstract

A ring R is called right CSP if the sum of any two closed right ideals of R is also a closed right ideal of R. Left CSP rings can be defined similarly. An example is given to show that a left CSP ring may not be right CSP. It is shown that a matrix ring over a right CSP ring may not be right CSP. It is proved that $\mathbb{M}_2(R)$ is right CSP if and only if R is right self-injective and von Neumann regular. The equivalent characterization is given for the trivial extension $R \propto R$ of R to be right CSP.

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1. Introduction

Throughout this paper, all rings are associative with identity and all modules are unitary modules. Let R be a ring and M an R-module. Recall that a submodule N of M is *essential* in M [1], in case for every submodule $L \leq M$, $N \cap L = 0$ implies $L = 0$. We use $N \leq_{ess} M$ to show that N is an essential submodule of M. $\mathbb{M}_n(R)$ denotes the ring of $n \times n$ matrices over R. Let Λ be an infinite set. $\mathbb{CFM}_\Lambda(R)$ means the column finite $card(\Lambda) \times card(\Lambda)$ matrix ring over R, where $card(\Lambda)$ is the cardinality of Λ.

Recall that an R-module M_R is called an *SSP module* if the sum of any two direct summands of M_R is also a direct summand of M_R [3]. And a ring R is called *right SSP* if R_R is an SSP module. Left SSP rings can be defined similarly. It is known that a ring R is right SSP if and only if R is left SSP (see [10, Theorem 2.4]). Recall that, a submodule N_R of M_R is called *closed* if it has no proper essential extensions in M_R, that is, for any submodule K_R of M_R such that $N_R \leq_{ess} K_R$, then $N = K$ [6]. It is well known that a closed submodule of M_R is equivalent to a relative complement for some submodule of M [1]. It is known that any nonzero submodule of M_R must be essential in some closed submodules of M_R. A right ideal I of a ring R is called a *closed right ideal* of R if it is a closed submodule of R_R. Closed left ideals of R can be defined analogously. Since a direct summand of R_R is a closed right ideal of R, inspired by the definitions of SSP rings, we introduce the rings such that the sum of any two closed right ideals of R is also a closed right ideal of R. These rings are called *right CSP* rings. Left CSP rings are defined similarly. We would like to mention that the definition of CSP modules was also introduced and discussed in [5]. Recall that a module M_R is called a *CS* (*C1*) module if

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every closed submodule of M_R is a direct summand of M_R [7]. A module M_R is called a *C2* module if every submodule K of M_R such that K is isomorphic to a direct summand of M_R, then K is also a direct summand of M_R. And a module M_R is called a *C3* module if for any direct summands N_1 and N_2 of M_R such that $N_1 \cap N_2 = 0$, then $N_1 + N_2$ is also a direct summand of M_R. If a module M_R is both C1 and C2 (C3), then M_R is called a *continuous* (*quasi-continuous*) module. And a ring is called *right CS* (*C2, C3, continuous, qusi-continuous*) if the right R-module R_R is a CS (C2, C3, continuous, qusi-continuous) module [8]. The left sides of the definitions can be given analogously.

Generalizing direct summands of rings, closed ideals of rings are important in studying the properties of rings such as SSP, CS, continuous and so on. And these properties of rings are all closely connected to von Neumann's research on continuous geometries [2,4,7]. In this short note, it is shown in Proposition 2.6 that a ring R is right CSP if and only if R is right CS and SSP. We also discuss the CSP property of matrix rings and the trivial extension $R \propto R$ of R, which can be considered as a subring of $\mathbb{M}_2(R)$. It is shown in Theorem 2.12 that $\mathbb{M}_2(R)$ is right CSP if and only if R is right self-injective and von Neumann regular. And we call such rings *strongly right CSP* rings. Strongly left CSP rings can be defined similarly. Recall that a ring R is called *right self-injective* if every homomorphism f from a right ideal I of R to R_R can be extended to an endomorphism of R_R [6]. And a ring R is called (*von Neumann*) *regular* if for any element $a \in R$, there is an element $b \in R$ such that $a = aba$ [4]. By this result, it can be seen in Example 2.2 that a left CSP ring may not be right CSP. This example also shows the strongly CSP property of rings is not left-right symmetric. At last, in Theorem 3.4, an equivalent characterization is given for the trivial extension $R \propto R$ of R to be right CSP.

2. CSP rings

Definition 2.1. Let R be a ring. R is called a *right CSP ring* if the sum of any two closed right ideals of R is also a closed right ideal of R. Left CSP rings can be defined similarly. And R is called a *CSP ring* if it is both left and right CSP.

Unlike SSP property of rings, next example shows that the CSP property of rings is not left-right symmetric.

Example 2.2. Let k be a division ring and ${}_kV$ be a left k-vector space of infinite dimension. Let $E = End({}_kV)$, defined as a ring of right opertiors on V. Set $R = \mathbb{M}_2(E)$. Then R is left CSP but not right CSP.

Proof. According to [6, Example 3.74B], E is a regular and left self-injective ring, but it is not right self-injective. By following Theorem 2.12, R is left CSP but not right CSP. □

Recall that the *Goldie dimension* of a module M_R is the infimum of those cardinal numbers c such that *card* $A \leq c$ for every independent set $(M_\alpha)_{\alpha \in A}$ of non-zero submodules of M [6]. We use G.dim(M_R) to denote the Goldie dimension of M_R. If G.dim(M_R)=n, it is also equivalent to saying that there are n independent submodules M_i of M such that $\oplus_{i=1}^n M_i \leq_{ess} M$.

Example 2.3. Any ring with G.dim(R_R) =1 is right CSP. In particular, any integral domain is a CSP ring.

Proof. If G.dim(R_R) =1, by the fact that any nonzero right ideal of R must be essential in a closed right ideal of R, then R has only two trivial closed right ideals 0 and R_R. So it is clear that R is right CSP. And it is easy to see that the Goldie dimension of an integral domain is 1. □

Proposition 2.4. A direct product of rings R=$\prod_{i \in I} R_i$ is right CSP if and only if R_i is right CSP, for all $i \in I$.

Proof. For $i \in I$, let π_i and ι_i be the ith projection map and the ith inclusion map respectively. If R is right CSP, for each i, assume that T_i and T_i' are two closed right ideals of R_i. It is easy to see that $0 \times \cdots \times T_i \times \cdots \times 0$ and $0 \times \cdots \times T_i' \times \cdots \times 0$ are two closed right ideals of R. Since R is right CSP, $0 \times \cdots \times (T_i + T_i') \times \cdots \times 0$ is a closed right ideal of R. Therefore, it is not difficult to see that $T_i + T_i'$ is a closed right ideal of R_i. So R_i is right CSP, for all $i \in I$.

Conversely, let T and T' be two closed right ideals of R. For each $i \in I$, let $T_i = \{x \in R_i \mid \iota_i(x) \in T\}$ and $T_i' = \{x \in R_i \mid \iota_i(x) \in T'\}$. Since T and T' are closed, it can be seen that T_i and T_i' are closed right ideals of R_i, for all $i \in I$. As R_i is right CSP, $T_i + T_i'$ is a closed right ideal of R_i. Since T and T' are right ideals of R,

$$T + T' = \Pi_{i \in I}(T_i + T_i').$$

This shows that R is right CSP. □

Lemma 2.5. [9, Theorem 1.4.1(i)(vi)] Let R be a ring and M be a right R-module. Then M_R is quasi-continuous if and only if whenever L_1 and L_2 are two closed submodules of M with $L_1 \cap L_2 = 0$, then $L_1 \oplus L_2$ is also a closed submodule of M_R.

The necessity of the following proposition can also be obtained from [5, Proposition 1.8] and [10, Theorem 2.4], to be self contained, we write down the complete proof.

Proposition 2.6. Let R be a ring. Then R is right CSP if and only if R is right CS and SSP.

Proof. If R is right CSP, by Lemma 2.5, R is right quasi-continuous, so it is right CS. Hence every closed right ideal of R is a direct summand of R_R. Since any direct summand of R_R is a closed right ideal of R and R is right CSP, the sum of any two direct summands of R_R is a closed right ideal of R. Again since R is right CS, the sum of any two direct summands of R_R is also a direct summand of R_R. Therefore, R is right SSP. By [10, Theorem 2.4], R is SSP. Conversely, if R is right CS, then every closed right ideal of R is a direct summand of R_R. So any sum of two closed right ideals of R is a sum of two direct summands of R_R. Since R is SSP, any sum of two closed right ideals of R is a direct summand of R_R, which is a closed right ideal of R. Thus, R is right CSP. □

Lemma 2.7. [12, Lemma 5] Let R be a ring and $e^2 = e \in R$ such that $ReR = R$. If T is a nonzero closed right ideal of eRe, then TR is also a nonzero closed right ideal of R.

Theorem 2.8. Let R be a ring and $e^2 = e \in R$ such that $ReR = R$. If R is right CSP, then eRe is also right CSP.

Proof. Assume that T and S are two nonzero closed right ideals of eRe. By the fact that any nonzero right ideal of a ring must be essential in a closed right ideal of the ring, there exists a closed right ideal L of eRe such that

$$T + S \leq_{ess} L.$$

Next we only need to show that $T + S = L$. Since T and S are closed right ideals of eRe, by Lemma 2.7, TR and SR are two closed right ideals of R. As R is right CSP, $TR + SR = (T + S)R$ is a closed right ideal of R. Then $(T + S)R \subseteq LR$, which is a right ideal of R. Next we show that $(T + S)R \leq_{ess} (LR)_R$. Since $ReR = R$, there exist $a_j, b_j \in R$, $j = 1, 2, \ldots, n$, such that

$$1 = \sum_{j=1}^{n} a_j e b_j.$$

Then for any $0 \neq x \in LR$, there must exist some $j \in \{1, 2, \ldots, n\}$ such that

$$0 \neq x a_j e \in LRe = LeRe = L.$$

Since $T+S \leq_{ess} L_{eRe}$, there exists $ere \in eRe$ such that

$$0 \neq xa_j eere = xa_j ere \in T+S \subseteq (T+S)R.$$

So $(T+S)R \leq_{ess} LR_R$. Since $(T+S)R$ is a closed right ideal of R, $(T+S)R = LR$. Thus,

$$T+S = (T+S)eRe = (T+S)Re = LRe = LeRe = L.$$

□

Question 2.9. Let R be a right CSP ring and $e = e^2 \in R$. Is eRe a right CSP ring?

Lemma 2.10. [8, Theorem 1.35] The following are equivalent for a ring R.

(1) R is right self-injective;
(2) $\mathbb{M}_2(R)$ is right continuous (quasi-continuous);
(3) $\mathbb{M}_n(R)$ is right continuous (quasi-continuous) for all $n \geq 1$;
(4) $\mathbb{M}_n(R)$ is right self-injective for all $n \geq 1$.

Lemma 2.11. [10, Theorem 2.15] The following are equivalent for a ring R.

(1) R is regular.
(2) $\mathbb{M}_2(R)$ is SSP.
(3) $\mathbb{M}_n(R)$ is SSP for some $n > 1$.
(4) $\mathbb{M}_n(R)$ is SSP for every $n > 1$.

Theorem 2.12. Let R be a ring. The following are equivalent.

(1) $\mathbb{M}_2(R)$ is right CSP;
(2) $\mathbb{M}_n(R)$ is right CSP for some $n > 1$;
(3) $\mathbb{M}_n(R)$ is right CSP for each $n \geq 1$;
(4) R is right self-injective and regular.

Proof. (1)⇔(4). If $\mathbb{M}_2(R)$ is right CSP, by Proposition 2.6, $\mathbb{M}_2(R)$ is right CS and SSP, hence it is right quasi-continuous. Then by Lemma 2.10 and Lemma 2.11, R is right self-injective and regular. Conversely, assume that R is right self-injective and regular. By Lemma 2.10 and Lemma 2.11, $\mathbb{M}_2(R)$ is right CS and SSP. By Proposition 2.6, $\mathbb{M}_2(R)$ is right CSP.

(2)⇔(4) and (3)⇔(4) can be proved in a similar way. □

Remark 2.13. The right CSP property is not a Morita invariant for rings. For example, by Example 2.3, the ring $\mathbb{Z}$ of integers is CSP, but $\mathbb{M}_2(\mathbb{Z})$ is not right CSP. Because if $\mathbb{M}_2(\mathbb{Z})$ is right CSP, according to Theorem 2.12, $\mathbb{Z}$ is regular, this is impossible.

Definition 2.14. A ring R is called *strongly right CSP* if it satisfies any one of the conditions in Theorem 2.12. It is clear that a strongly right CSP ring must be right CSP. By Remark 2.13, a right CSP ring may not be strongly right CSP. Since right self-injectivity and regularity are both Morita invariants, according to Theorem 2.12, strongly right CSP property is also a Morita invariant. According to Theorem 2.12, Example 2.2 also shows that the strongly CSP property of rings is not left-right symmetric.

Lemma 2.15. [10, Theorem 2.18] The following are equivalent for a ring R.
(1) R is semisimple.
(2) $\mathbb{CFM}_{\mathbb{N}}(R)$ is SSP.
(3) $\mathbb{CFM}_{\Lambda}(R)$ is SSP for some infinite set Λ.
(4) $\mathbb{CFM}_{\Lambda}(R)$ is SSP for every infinite set Λ.

Let R be a ring and Λ an infinite set. We write $R_R^{(\Lambda)}$ as the set of all $card(\Lambda) \times 1$ column matrices with finite nonzero entries in R. For any element A in $\mathbb{CFM}_{\Lambda}(R)$, we write $A := \langle \alpha_\lambda \rangle$, $\lambda \in \Lambda$, where $\alpha_\lambda \in R_R^{(\Lambda)}$ is the λth column of A, and we denote the zero $card(\Lambda) \times 1$ column matrix by θ.

Lemma 2.16. [11, Lemma 2.11] Let R be a ring and Λ an infinite set. If I is a right ideal of $\mathbb{CFM}_\Lambda(R)$, then

(1) I is a closed right ideal of $\mathbb{CFM}_\Lambda(R)$ if and only if $I=\{\langle\alpha_\lambda\rangle \mid \alpha_\lambda \in T\}$, where T is a closed submodule of $R_R^{(\Lambda)}$.

(2) I is a direct summand of $\mathbb{CFM}_\Lambda(R)$ if and only if $I=\{\langle\alpha_\lambda\rangle \mid \alpha_\lambda \in T\}$, where T is a direct summand of $R_R^{(\Lambda)}$.

Theorem 2.17. Let Λ be an infinite set. Then $\mathbb{CFM}_\Lambda(R)$ is right CSP if and only if R is semisimple.

Proof. By Proposition 2.6, a right CSP ring is SSP. According to Lemma 2.15, R is semisimple. Conversely, let I_1 and I_2 be two closed right ideals of $\mathbb{CFM}_\Lambda(R)$, by Lemma 2.16, there are two closed submodules T_1 and T_2 of $R_R^{(\Lambda)}$ such that $I_1=\{\langle\alpha_\lambda\rangle \mid \alpha_\lambda \in T_1\}$ and $I_2=\{\langle\beta_\lambda\rangle \mid \beta_\lambda \in T_2\}$. Then

$$I_1 + I_2=\{\langle\gamma_\lambda\rangle \mid \gamma_\lambda \in T_1 + T_2\}.$$

Since R is semisimple, $T_1 + T_2$ is a direct summand of $R_R^{(\Lambda)}$. By Lemma 2.16, $I_1 + I_2$ is a direct summand of $\mathbb{CFM}_\Lambda(R)_{\mathbb{CFM}_\Lambda(R)}$, which is clearly a closed right ideal of $\mathbb{CFM}_\Lambda(R)$. So $\mathbb{CFM}_\Lambda(R)$ is right CSP. □

3. Trivial extension of right CSP rings

Let R be a ring. The *trivial extension of* R is the set

$$R \propto R = \left\{ \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} : a, b \in R \right\}.$$

With the usual addition and multiplication of 2×2 matrices, $R \propto R$ becomes a ring. To be convenient, for any pair $a, b \in R$, we use (a, b) to denote $\begin{pmatrix} a & b \\ 0 & a \end{pmatrix}$. And for any two subsets A, B of R, we set

$$A \propto B = \{(a, b) \mid a \in A, b \in B\}.$$

Lemma 3.1. Let R be a ring and $S = R \propto R$. If I and J are two right ideals of R such that $I \subseteq J$. Then $I \propto J$ is a closed right ideal of S if and only if $I = J$ is a closed right ideal of R.

Proof. The sufficiency is obtained by [11, Lemma 4.2].

For the necessity, if $J = 0$, then $I \propto J = 0 \propto 0$, which is a trivial closed right ideal of S. If $J \neq 0$, firstly, we show that $(I \propto J)_S \leq_{ess} (J \propto J)_S$. Let $0 \neq (i, j) \in (J, J)$. If $i \neq 0$, then $(i, j)(0, 1) = (0, i) \in I \propto J$. If $i = 0$, then $j \neq 0$. So $(i, j)(1, 0) = (i, j) \in I \propto J$. Therefore, $(I \propto J)_S \leq_{ess} (J \propto J)_S$. Since $I \propto J$ is a closed right ideal, $I \propto J = J \propto J$. So $I = J$. Next we show that J is a closed right ideal of R. If not, there exists a right ideal K of R such that $J \neq K$ and $J_R \leq_{ess} K_R$. Now let $A = J \propto J$ and $B = K \propto K$. Then A is a closed right ideal of $R \propto R$. Next we show that $A_S \leq_{ess} B_S$. That is, if $0 \neq (j_1, j_2) \in B$, we want to find $0 \neq (r_1, r_2) \in S$ such that $0 \neq (j_1, j_2)(r_1, r_2) \in A$.

Case (i): If $j_1 \neq 0$, as $J_R \leq_{ess} K_R$, there exists $0 \neq r \in R$ such that $0 \neq j_1 r \in K$. Now take $(r_1, r_2) = (0, r)$, then

$$0 \neq (0, j_1 r) = (j_1, j_2)(r_1, r_2) \in A.$$

Case (ii): If $j_1 = 0$, then $j_2 \neq 0$. So there exists $0 \neq r \in R$ such that $0 \neq j_2 r \in K$. Now take $(r_1, r_2) = (r, 0)$, then

$$0 \neq (0, j_2 r) = (j_1, j_2)(r_1, r_2) \in A.$$

Thus, $A_S \leq_{ess} B_S$. Since A is a closed right ideal of S, $A_S = B_S$. So $J = K$, this is a contradicition to the assumption that $J \neq K$. So J is a closed right ideal of R. □

The following lemma can be obtained from [3, Proposition 4.5] and [10, Theorem 2.4].

Lemma 3.2. Let R be a ring. Then $R \propto R$ is SSP if and only if R is SSP and for any idempotent e of R, $eR(1-e) = 0$.

Recall that a ring R is *abelian* if all its idempotents are contained in the center of R.

Lemma 3.3. Let R be a ring. The following are equivalent.

(1) $R \propto R$ is SSP;
(2) R is abelian.
(3) Every idempotent of $R \propto R$ has the form $(e, 0)$, where e is an idempotent of R.

Proof. (1) $\Leftrightarrow$ (2). If $R \propto R$ is SSP, according to Lemma 3.2, for any idempotent e of R, $eR(1-e) = 0$. Taking the idempotent $f = 1-e$, we also have $fR(1-f) = (1-e)Re = 0$. These mean that for each $r \in R$, $er = ere = re$. So R is abelian. Conversely, if R is abelian, [10, Theorem 2.4] implies R has SSP. And for any idempotent $e \in R$, it is clear that $eR(1-e) = e(1-e)R = 0$. Again by Lemma 3.2, $R \propto R$ is SSP.

(2) $\Leftrightarrow$ (3). Assume (2). Let (a, b) be an idempotent of $R \propto R$. Then

$$(a, b)(a, b) = (a^2, ab + ba) = (a, b).$$

So $a^2 = a$ is an idempotent and $ab + ba = b$. Since R is abelian, we have $(1-2a)b = 0$. Since $(1-2a)^2 = 1$, $b = 0$. Thus $(a, b) = (a, 0)$, where a is an idempotent. Conversely, assume (3). If R is not abelian, there is an idempotent $e \in R$ such that $eR(1-e) \neq 0$. So there exists $r \in R$ such that $er \neq ere$. Let $b = er(1-e)$. Then $b \neq 0$ and

$$(e, b)(e, b) = (e, eb + be) = (e, b)$$

is an idempotent of $R \propto R$. This is a contradiction. □

At last, we obtain an equivalent characterization for $R \propto R$ to be right CSP.

Theorem 3.4. Let R be a ring. Then $R \propto R$ is right CSP if and only if the following conditions are satisfied:

(1) R is right CSP.
(2) R is abelian.
(3) Every closed right ideal of $R \propto R$ has the form $I \propto I$, where I is a closed right ideal of R.

Proof. For the necessity, assume that I_1 and I_2 are two closed right ideals of R. We need to show that $I = I_1 + I_2$ is also a closed right ideal of R. Take $A_1 = I_1 \propto I_1$ and $A_2 = I_2 \propto I_2$. By Lemma 3.1, A_1 and A_2 are two closed right ideals of S. Now let $A = A_1 + A_2$. It is clear that $A = I \propto I$. Since S is right CSP, A is a closed right ideal of $S = R \propto R$. Again by Lemma 3.1, I is a closed right ideal of R. Therefore, R is right CSP. So we have (1). Since a right CSP ring is SSP, by Lemma 3.3, R is abelian. Then we obtain (2). As $R \propto R$ is right CSP, every closed right ideal A of $R \propto R$ is generated by an idempotent e. By Lemma 3.3, we have $A = (e, 0)S = I \propto I$, where $e^2 = e \in R$. Then $I = eR$ is clearly a closed right ideal of R.

Conversely, by Lemma 3.3, $S = R \propto R$ is SSP. Next we only need to show that S is right CS. By (3), every closed right ideal A of S has the form $I \propto I$, where I is a closed right ideal of R. According to (1), R is right CSP, so R is right CS. Thus, $I = eR$, where e is an idempotent of R. So

$$A = I \propto I = eR \propto eR = (e, 0)S.$$

This shows that A is a direct summand of S_S. Hence S is right CSP. □

Remark 3.5. According to the above theorem, we have the following notes:

(i) If R is right CSP, $R \propto R$ may not be right CSP, even R is strongly right CSP. For example, let $R = \mathrm{M}_2(k)$, where k is a division ring. By Theorem 2.12, R is a strongly right CSP ring. But $R \propto R$ is not right CSP. Because if $R \propto R$ is right CSP, by Proposition 2.6, $R \propto R$ is SSP. Then according to Lemma 3.3, R is abelian. This is impossible. Because $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}$ is an idempotent of R which is not in the center of R.

(ii) Not for any ring R, $R \propto R$ satisfies the condition (3) in Theorem 3.4. If not, by Lemma 3.1, it is not difficult to prove that the trivial extension of a right CSP ring is right CSP. But by (i), this is impossible.

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