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Research Article

On an inverse boundary-value problem for the pseudohyperbolic equation with nonclassical boundary conditions

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Abstract

In this paper, we consider an inverse boundary-value problem for a fourth-order pseudohyperbolic equation with nonclassical boundary conditions. The primary purpose of the work is to study the existence and uniqueness of the classical solution of the considered inverse boundary-value problem. To investigate the solvability of the considered problem, we carried out a transformation from the original problem to some auxiliary equivalent problem with trivial boundary conditions. Furthermore, we prove the existence and uniqueness theorem for the auxiliary problem by the contraction mappings principle. Based on the equivalency of these problems, the existence and uniqueness of the classical solution of the original problem are shown.

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1. Introduction and formulation of inverse problem

It is well known that mathematical modeling of many real processes that arise during experiments in the field of some natural sciences leads to the study of inverse problems for partial differential equations. In general, inverse problems are understood as the problem of determining the parameters of the model when the structure of the mathematical model of the studied process is known. In other words, the problems of simultaneous determination of unknown coefficients and right-hand side of partial differential equations from some additional data are called inverse problems in the theory of equations of mathematical physics. The theoretical foundations of the study of inverse problems were established and developed in the works by Tikhonov [1], Lavrentiev [2], Ivanov [3], and their followers (see, e.g., [4–10], and the references therein).

The importance of the application of inverse problems (for instance, seismology, mineral exploration, biology, medicine, desalination of seawater, movement of liquid in a porous

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medium, acoustics, electromagnetics, fluid dynamics, remote sensing, nondestructive evaluation, and many other areas, etc.) makes them one of the most relevant branch of modern mathematics.

Inverse boundary-value problems for second-order partial differential equations have been actually well studied by many authors using different methods and boundary conditions. But it should be noted that inverse problems for pseudohyperbolic equations are less developed than for second-order equations. This is explained by the fact that the study of inverse problems for pseudohyperbolic differential equations is closely related to the solution of the corresponding direct problem. As is known that pseudohyperbolic equations arise in the theory of unsteady flow of a viscous gas during the propagation of initial densifications in a viscous gas [11], in the theory of solitons [12,13] when describing the process of electron motion in the system superconductor dielectric with tunneling conductivity superconductor. In fundamental science, pseudohyperbolic equations are considered as a Sobolev type equations and some works have been devoted to investigate them (see for example, [14–19], and references therein).

Let us now browse the content of some works devoted to inverse coefficient problems for pseudohyperbolic equations: In the paper by Kozhanov and Safiullova [20] the existence theorems are proved for the regular solutions of the pseudohyperbolic inverse problems with integral overdetermination conditions. In [21], the classical solvability of an inverse boundary-value problem for a fourth-order pseudohyperbolic equation with non-self-adjoint boundary conditions is studied. The existence of regular solutions to the inverse problem for a pseudohyperbolic equation with an additional integral condition is studied by Namsaraeva [22]. In the work [23] published by Kurmanbaeva the local existence and uniqueness theorems were proved for inverse coefficient problems for a pseudohyperbolic equation. The unique existence theorem for a time-nonlocal inverse boundary-value problem of recovering unknown external sources for the longitudinal wave propagation equation were proved in the paper [24]. In the article [25] the inverse coefficient problem for the pseudohyperbolic equation with non-self-adjoint boundary conditions is investigated. The authors proved the existence and uniqueness of the classical solutions for the considered inverse coefficient problem. In [26], the existence of a solution to a fractional integral equation involving (k, z)-RiemannLiouville fractional integral was studied. To establish the existence result, the authors used shifting distance functions and introduced a new generalization of the Dorbo-type fixed point theorem.

Motivated by these works, we study in this paper the existence and uniqueness of a classical solution for the inverse problem for a pseudohyperbolic equation with nonclassical boundary conditions: Let $T > 0$ be a fixed time moment and let D_T denotes the rectangular region in the xt-plane defined by the inequalities $0 \le x \le 1,\ 0 \le t \le T$. We further assume that $f(x,t), \varphi(x), \psi(x), \omega_i(x)$, and $h_i(t)$ $(i = 1,2,3)$ are given functions of $x \in [0,1]$ and $t \in [0,T]$. Consider the one-dimensional inverse boundary-value problem of defining an unknown quadruple of functions $u(x,t), a(t), b(t)$, and $c(t)$ for the equation [27,28]

$$u_{tt}(x,t) - \alpha u_{ttxx}(x,t) - u_{xx}(x,t)$$
$$= a(t)u(x,t) + b(t)u_t(x,t) + c(t)g(x,t) + f(x,t) \quad (x,t) \in D_T, \tag{1.1}$$

with the initial conditions

$$u(x,0) = \varphi(x),\ u_t(x,0) = \psi(x),\ 0 \le x \le 1, \tag{1.2}$$

the boundary conditions

$$u(0,t) = 0,\ 0 \le t \le T, \tag{1.3}$$

$$u_x(1,t) + du_{xx}(1,t) = 0,\ 0 \le t \le T, \tag{1.4}$$

and the overdetermination conditions

$$U_i(u) := u(x_i,t) + \int_0^1 \omega_i(x)u(x,t)dt = h_i(t),\ i=1,2,3;\ x_1 \neq x_2 \neq x_3,\ 0 \le t \le T, \quad (1.5)$$

where $x_i \in (0,1)$ $(i=1,2,3)$ are known fixed points, and $\alpha > 0$ and $d > 0$ are given numbers.

Definition 1.1. The quadruple $\{u(x,t), a(t), b(t), c(t)\}$ is said to be a classical solution of problem (1.1)–(1.5), if the functions $u(x,t), a(t), b(t)$, and $c(t)$ satisfy the following conditions:

i) The function $u(x,t)$ and its derivatives $u_t(x,t), u_{tt}(x,t), u_{xx}(x,t), u_{ttx}(x,t)$, and $u_{ttxx}(x,t)$ are continuous in the domain D_T;
ii) the functions $a(t)$, $b(t)$, and $c(t)$ are continuous on the interval $[0,T]$;
iii) equation (1.1) and conditions (1.2)–(1.5) are satisfied in the classical (usual) sense.

To study problem (1.1)–(1.5), we first consider the following spectral problem:

$$\begin{gathered} y''(x) + \lambda y(x) = 0,\ 0 \le x \le 1, \\ y(0) = 0,\ y'(1) = d\lambda y(1),\ d > 0. \end{gathered} \quad (1.6)$$

It is clear that (cf.[29], p.1071) the problem (1.6) has only eigenfunctions $y_k(x) = \sqrt{2}\sin(\sqrt{\lambda_k}x)$, $k = 0,1,2,...$, with positive eigenvalues λ_k from equation $ctg\sqrt{\lambda} = d\sqrt{\lambda}$. The zero index is assigned to any eigenfunction, and all others are numbered in ascending order of eigenvalues.

The following theorem is valid.

Theorem 1.2. *Suppose that* $f(x,t) \in C(D_T),\ \varphi(x), \psi(x) \in C^2[0,1],\ h_i(t) \in C^2[0,T]$ $(i=1,2,3),\ h(t) \neq 0\ (0 \le t \le T)$,

$$\varphi(1) + \frac{1}{d\sin\sqrt{\lambda_0}} \int_0^1 \varphi(x)\sin(\sqrt{\lambda_0}x)dx = 0, \quad (1.7)$$

$$\psi(1) + \frac{1}{d\sin\sqrt{\lambda_0}} \int_0^1 \psi(x)\sin(\sqrt{\lambda_0}x)dx = 0, \quad (1.8)$$

$$\begin{gathered} f(1,t) + \frac{1}{d\sin\sqrt{\lambda_0}} \int_0^1 f(x,t)\sin(\sqrt{\lambda_0}x)dx = 0,\ 0 \le t \le T, \\ g(1,t) + \frac{1}{d\sin\sqrt{\lambda_0}} \int_0^1 g(x,t)\sin(\sqrt{\lambda_0}x)dx = 0,\ 0 \le t \le T, \end{gathered} \quad (1.9)$$

and the compatibility conditions

$$\varphi'(1) + d\varphi''(1) = 0,\ \psi'(1) + d\psi''(1) = 0,\ U_i(\varphi) = h_i(0),\ U_i(\psi) = h_i'(0)\ (i=1,2,3) \quad (1.10)$$

hold. Then the problem of finding a classical solution of (1.1)–(1.5) *is equivalent to the problem of determining the functions* $u(x,t) \in C(D_T)$, $a(t), b(t) \in C[0,T]$, *and* $c(t) \in C[0,T]$ *satisfying* (1.1)–(1.3), *and the conditions*

$$u(1,t) + \frac{1}{d\sin\sqrt{\lambda_0}} \int_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)dx = 0,\ 0 \le t \le T, \quad (1.11)$$

$$\begin{gathered} h_i''(t) - \alpha U_i(u_{ttxx}) - U_i(u_{xx}) \\ = a(t)h_i(t) + b(t)h_i'(t) + c(t)U_i(g) + U_i(f),\ i=1,2,3;\ 0 \le t \le T, \end{gathered} \quad (1.12)$$

where

$$h(t) \equiv \begin{vmatrix} h_1(t) & h_1'(t) & U_1(g) \\ h_2(t) & h_2'(t) & U_2(g) \\ h_3(t) & h_3'(t) & U_3(g) \end{vmatrix} \neq 0,\ 0 \le t \le T.$$

Proof. Let $\{u(x,t), a(t), b(t), c(t)\}$ be the classical solution to problem (1.1)–(1.5). Then from equation (1.1), taking into account (1.9), we have

$$u_{tt}(1,t) + \frac{1}{d\sin\sqrt{\lambda_0}} \int\limits_0^1 u_{tt}(x,t)\sin(\sqrt{\lambda_0}x)dx$$

$$-\alpha\left(u_{ttxx}(1,t) + \frac{1}{d\sin\sqrt{\lambda_0}} \int\limits_0^1 u_{ttxx}(x,t)\sin(\sqrt{\lambda_0}x)dx\right)$$

$$-\left(u_{xx}(1,t) + \frac{1}{d\sin\sqrt{\lambda_0}} \int\limits_0^1 u_{xx}(x,t)\sin(\sqrt{\lambda_0}x)dx\right)$$

$$= a(t)\left[u(1,t) + \frac{1}{d\sin\sqrt{\lambda_0}} \int\limits_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)dx\right]$$

$$+b(t)\left[u_t(1,t) + \frac{1}{d\sin\sqrt{\lambda_0}} \int\limits_0^1 u_t(x,t)\sin(\sqrt{\lambda_0}x)dx\right],\ 0 \le t \le T. \tag{1.13}$$

Taking into consideration (1.3), it is easy to see that

$$\int\limits_0^1 u_{xx}(x,t)\sin(\sqrt{\lambda_0}x)dx$$

$$= u_x(1,t)\sin\sqrt{\lambda_0} - \sqrt{\lambda_0}u(1,t)\cos\sqrt{\lambda_0} - \lambda_0 \int\limits_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)dx,\ 0 \le t \le T.$$

Exploiting the latter relation we have

$$u_{xx}(1,t) + \frac{1}{d\sin\sqrt{\lambda_0}} \int\limits_0^1 u_{xx}(x,t)\sin(\sqrt{\lambda_0}x)dx$$

$$= \frac{1}{d}(u_x(1,t) + du_{xx}(1,t)) - \lambda_0\left[u(1,t) + \frac{1}{d\sin\sqrt{\lambda_0}} \int\limits_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)dx\right]. \tag{1.14}$$

Substituting (1.14) into (1.3) gives

$$\frac{d^2}{dt^2}\left[u(1,t) + \frac{1}{d\sin\sqrt{\lambda_0}} \int\limits_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)dx\right]$$

$$-\alpha\frac{d^2}{dt^2}\left\{\frac{1}{d}(u_x(1,t) + du_{xx}(1,t))\right.$$

$$\left.-\lambda_0\left[u(1,t) + \frac{1}{d\sin\sqrt{\lambda_0}} \int\limits_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)dx\right]\right\}$$

$$-\left\{\frac{1}{d}(u_x(1,t) + du_{xx}(1,t)) - \lambda_0\left[u(1,t) + \frac{1}{d\sin\sqrt{\lambda_0}} \int\limits_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)dx\right]\right\}$$

$$= a(t)\left[u(1,t) + \frac{1}{d\sin\sqrt{\lambda_0}} \int\limits_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)dx\right]$$

$$+b(t)\left[u_t(1,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u_t(x,t)\sin(\sqrt{\lambda_0}x)dx\right],\ 0\le t\le T. \tag{1.15}$$

From (1.15) by virtue of (1.4), we find that

$$(1+\alpha\lambda_0)\omega''(t)-b(t)\omega'(t)-(a(t)-\lambda_0)\omega(t)=0,\ 0\le t\le T, \tag{1.16}$$

where

$$\omega(t)=u(1,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 u(x,t)\sin(\sqrt{\lambda_0}x)dx,\ 0\le t\le T. \tag{1.17}$$

Furthermore, using (1.2) and conditions (1.7), (1.8), we obtain

$$\begin{aligned}\omega(0)&=\varphi(1)+\tfrac{1}{d\sin\sqrt{\lambda_0}}\int_0^1\varphi(x)\sin(\sqrt{\lambda_0}x)dx=0,\\ \omega'(0)&=\psi(1)+\tfrac{1}{d\sin\sqrt{\lambda_0}}\int_0^1\psi(x)\sin(\sqrt{\lambda_0}x)dx=0.\end{aligned} \tag{1.18}$$

From (1.16) and (1.18) it is evident that $\omega(t)=0,\ 0\le t\le T$. Hence, taking into account (1.17), we conclude that condition (1.11) is satisfied.

In turn, from (1.5) it can be seen that

$$U_i(u_t)=h_i'(t),\ U_i(u_{tt})=h_i''(t),\ i=1,2,3;\ 0\le t\le T. \tag{1.19}$$

Then from equation (1.1), we have

$$U_i(u_{tt})-\alpha U_i(u_{ttxx})-U_i(u_{xx})$$

$$=a(t)U_i(u)+b(t)U_i(u_t)+c(t)U_i(g)+U_i(f),\ i=1,2,3;\ 0\le t\le T. \tag{1.20}$$

Henceforth, taking into account (1.5) and (1.19), we arrive at the fulfillment of (1.12).

Now assume that the quadruple $\{u(x,t),a(t),b(t),c(t)\}$ is a solution to the problem (1.1)–(1.3), (1.11), (1.12). Then from (1.11) and (1.15), we get

$$\alpha\frac{d^2}{dt^2}(u_x(1,t)+du_{xx}(1,t))+(u_x(1,t)+du_{xx}(1,t))=0. \tag{1.21}$$

Using (1.2) and the two equalities $\varphi'(1)+d\varphi''(1)=0,\ \psi'(1)+d\psi''(1)=0$, we obtain the following relations:

$$\begin{aligned}u_x(1,0)+du_{xx}(1,0)&=\varphi'(1)+d\varphi''(1)=0,\\ u_{tx}(1,0)+du_{txx}(1,0)&=\psi'(1)+d\psi''(1)=0.\end{aligned} \tag{1.22}$$

Hence relations (1.12) and (1.22) enable us to conclude that (1.4) is satisfied.

Further, from (1.12) and (1.20), we obtain

$$\frac{d^2}{dt^2}(U_i(u)-h_i(t))-b(t)\frac{d}{dt}(U_i(u)-h_i(t))$$

$$-a(t)(U_i(u)-h_i(t))=0,\ i=1,2,3;\ 0\le t\le T. \tag{1.23}$$

From $U_i(\varphi)=h_i(0),\ U_i(\psi)=h_i'(0)(i=1,2,3)$ by virtue of (1.2), we find

$$\begin{cases}U_i(u)(0)-h_i(0)=U_i(\varphi)-h_i(0)=0,\\ U_i(u_t)(0)-h_i'(0)=U_i(\psi)-h_i'(0)=0\ (i=1,2,3).\end{cases} \tag{1.24}$$

Thus, from (1.23) and (1.24) we conclude that condition (1.5) is satisfied. □

2. Auxiliary facts and notations

Solving the homogeneous problem corresponding to problem (1.1)–(1.3), (1.11) by the method of separation of variables, we arrive at the spectral problem:

$$\begin{gathered} y''(x)+\lambda y(x)=0,\ 0\le x\le 1, \\ y(0)=0,\ y(1)+\tfrac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 y(x)\sin(\sqrt{\lambda_0}x)dx=0. \end{gathered} \tag{2.1}$$

It is obvious that the spectral problem (2.1) is equivalent to the spectral problem (1.6) without an eigenfunction corresponding to the eigenvalue λ_0.

Consequently, the spectral problem (2.1) has only eigenfunctions $y_k(x)=\sqrt{2}\sin(\sqrt{\lambda_k}x)$, $k=1,2,...$, with positive eigenvalues λ_k, determined from the equation $ctg\sqrt{\lambda}=d\sqrt{\lambda}$, numbered in ascending order.

We will use as auxiliary facts the following assertions formulated and substantiated in the work of N.Yu.Kapustin and E.I.Moiseev (see, p.1071, [29]).

Lemma 2.1. *Starting from some integer N, the estimate*

$$0<\sqrt{\lambda_k}-\pi k<(d\pi k)^{-1}$$

holds true.

Corollary 2.2. *Let $v_k(x)=\sqrt{2}\sin(\sqrt{\mu_k}x)$, for $\sqrt{\mu_k}=\pi k,\ k=1,2,....$ Then the inequalities*

$$\max_{x\in[0,1]}|y_k(x)-v_k(x)|\le\sqrt{2}(d\pi k)^{-1},\ k\ge N,$$

$$\sum_{k=N}^{\infty}\|y_k(x)-v_k(x)\|^2_{L_2(0,1)}\le\frac{1}{9d^2}$$

are fulfilled.

Lemma 2.3. *The biorthogonal conjugate system $\{z_k(x)\}_{k=1}^{\infty}$ to the system $\{y_k(x)\}_{k=1}^{\infty}$ is determined by the formula*

$$z_k(x)=\sqrt{2}\left(\sin\sqrt{\lambda_k}x-\frac{\sin\sqrt{\lambda_k}\sin\sqrt{\lambda_0}x}{\sin\sqrt{\lambda_0}}\right)/(1+d\sin^2\sqrt{\lambda_k}).$$

Theorem 2.4. *The system of eigenfunctions $\{y_k(x)\}_{k=1}^{\infty}$ forms a basis in the space $L_2(0,1)$.*

Using Lemma 2.1 and Corollary 2.2, we obtain the validity of the following estimates: if $g(x)\in C[0,1],\ g'(x)\in L_2(0,1),\ g(0)=0$, then

$$\left(\sum_{k=1}^{\infty}(\sqrt{\lambda_k}\,|g_k|)^2\right)^{\frac{1}{2}}\le M\,\|g'(x)\|_{L_2(0,1)}, \tag{2.2}$$

where

$$g_k=\int_0^1 g(x)z_k(x)dx,\ M=\left[\sum_{k=1}^{N}\int_0^1 y_k^2(x)dx+\frac{2}{9d^2}+2\right];$$

if $g(x)\in C^1[0,1],\ g''(x)\in L_2(0,1),\ g''(x)\in L_2(0,1)$, and

$$J(g)\equiv g(1)+\frac{1}{d\sin\sqrt{\lambda_0}}\int_0^1 g(x)\sin(\sqrt{\lambda_0}x)dx=0,$$

then

$$\left(\sum_{k=1}^{\infty}(\lambda_k|g_k|)^2\right)^{\frac{1}{2}}\le m|g'(0)|+\sqrt{2}M\,\|g''(x)\|_{L_2(0,1)}, \tag{2.3}$$

where

$$m=\frac{\sqrt{2}}{d}\left(\sum_{k=1}^{\infty}\frac{1}{\lambda_k}\right)^{\frac{1}{2}};$$

and if $g(x)\in C^2[0,1],\ g'''(x)\in L_2(0,1),\ g(0)=0,\ J(g)=0,\ g''(0)=0$, and $dg''(1)+g'(1)=0$, then

$$\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\,|g_k|)^2\right)^{\frac{1}{2}}\le M||g'''(x)||_{L_2(0,1)}. \tag{2.4}$$

Let us consider the functional space $B_{2,T}^{\frac{3}{2},1}$ that is introduced in the study of [30], where $B_{2,T}^{\frac{3}{2},1}$ denotes a set of all functions of the form

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)y_k(x),$$

considered in D_T. Moreover, the functions $u_k(t)\in C^1[0,T],\ k=1,2,...$, contained in last sum are continuously differentiable on $[0,T]$ and

$$I(u)\equiv\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}||u_k(t)||_{C[0,T]})^2\right)^{\frac{1}{2}}+\left(\sum_{k=1}^{\infty}(\lambda_k|u_k'(t)||_{C[0,T]})^2\right)^{\frac{1}{2}}<+\infty.$$

The norm on the set $I(u)$ is established as follows:

$$||u||_{B_{2,T}^{\frac{3}{2},1}}=I(u).$$

Let $E_T^{\frac{3}{2},1}$ denote the space consisting of the topological product $B_{2,T}^{\frac{3}{2},1}\times C[0,T]\times C[0,T]\times C[0,T]$, which is the norm of the element $z=\{u,a,b,c\}$ defined by the formula

$$||z||_{E_T^{\frac{3}{2},1}}=||u(x,t)||_{B_{2,T}^{\frac{3}{2},1}}+||a(t)||_{C[0,T]}+||b(t)||_{C[0,T]}+||c(t)||_{C[0,T]}.$$

It is known [30] that the spaces $B_{2,T}^{\frac{3}{2},1}$ and $E_T^{\frac{3}{2},1}$ are Banach spaces.

3. Classical solvability of inverse boundary-value problem

Taking into consideration Lemma 2.3 and Theorem 2.4, we will seek the first component of classical solution $\{u(x,t),a(t),b(t),c(t)\}$ of the problem (1.1)–(1.3), (1.11), (1.12) in the form:

$$u(x,t)=\sum_{k=1}^{\infty}u_k(t)y_k(x), \tag{3.1}$$

where

$$u_k(t)=\int_0^1 u(x,t)z_k(x)dx,\ k=1,2,....$$

Applying the method of separation of variables to determine the desired coefficients $u_k(t),\ k=1,2,...$ of the function $u(x,y,t)$ from (1.1) and (1.2), we obtain:

$$u_k''(t)+\frac{\lambda_k}{1+\alpha\lambda_k}u_k(t)=\frac{1}{1+\alpha\lambda_k}F_k(t;u,a,b,c),\ k=1,2,\ \ldots,\ 0\le t\le T, \tag{3.2}$$

$$u_k(0)=\varphi_k,\ u_k'(0)=\psi_k,\ k=1,2,..., \tag{3.3}$$

where

$$F_k(t;u,a,b,c)=a(t)u_k(t)+b(t)u_k'(t)+c(t)g_k(t)+f_k(t),$$

$$f_k(t)=\int_0^1 f(x,t)z_k(x)dx,\ g_k(t)=\int_0^1 g(x,t)z_k(x)dx,\ k=1,2,...,$$

$$\varphi_k=\int_0^1 \varphi(x)z_k(x)dx,\ \psi_k=\int_0^1 \psi(x)z_k(x)dx,\ k=1,2,....$$

Solving the problem (3.2), (3.3) gives

$$u_k(t)=\varphi_k\cos\beta_k t+\frac{1}{\beta_k}\psi_k\sin\beta_k t$$

$$+\frac{1}{\beta_k(1+\alpha\lambda_k)}\int_0^t F_k(\tau;u,a,b,c)\sin\beta_k(t-\tau)d\tau,\ k=1,2,..., \tag{3.4}$$

where

$$\beta_k=\sqrt{\frac{\lambda_k}{1+\alpha\lambda_k}}.$$

From (3.4) it is easy to find

$$u_k'(t)=-\varphi_k\beta_k\sin\beta_k t+\psi_k\cos\beta_k t$$

$$+\frac{1}{(1+\alpha\lambda_k)}\int_0^t F_k(\tau;u,a,b,c)\cos\beta_k(t-\tau)d\tau,\ k=1,2,.... \tag{3.5}$$

Substituting the expressions $u_k(t)$ $(k=1,2,\dots)$ described by (3.4) into (3.1), to determine the first component of the solution (1.1)–(1.3), (1.11), (1.12) we obtain

$$u(x,t)=\sum_{k=1}^{\infty}\left\{\varphi_k\cos\beta_k t+\frac{1}{\beta_k}\psi_k\sin\beta_k t\right.$$

$$\left.+\frac{1}{\beta_k(1+\alpha\lambda_k)}\int_0^t F_k(\tau;u,a,b,c)\sin\beta_k(t-\tau)d\tau\right\}y_k(x). \tag{3.6}$$

Now, in order to obtain an equation for the components $a(t)$, $b(t)$, and $c(t)$ of the solution $\{u(x,t),a(t),b(t),c(t)\}$ of problem (1.1)–(1.3), (1.11), (1.12), using the (3.1), we have

$$a(t)h_i(t)+b(t)h_i'(t)+c(t)U_i(g)+U_i(f)$$

$$=h_i''(t)+\sum_{k=1}^{\infty}\lambda_k(\alpha u_k''(t)+u_k(t))U_i(y_k),\ i=1,2,3;\ 0\le t\le T, \tag{3.7}$$

Under the assumption

$$h(t)\equiv\begin{vmatrix} h_1(t) & h_1'(t) & U_1(g)\\ h_2(t) & h_2'(t) & U_2(g)\\ h_3(t) & h_3'(t) & U_3(g)\end{vmatrix}\ne 0,\ 0\le t\le T,$$

from (3.7), we will find

$$a(t)=[h(t)]^{-1}\left\{q_1(t)+\sum_{k=1}^{\infty}\lambda_k(\alpha u_k''(t)+u_k(t))q_{1k}(t)\right\}, \tag{3.8}$$

$$b(t)=[h(t)]^{-1}\left\{q_2(t)+\sum_{k=1}^{\infty}\lambda_k(\alpha u_k''(t)+u_k(t))q_{2k}(t)\right\}, \tag{3.9}$$

$$c(t)=[h(t)]^{-1}\left\{q_3(t)+\sum_{k=1}^{\infty}\lambda_k(\alpha u_k''(t)+u_k(t))q_{3k}(t)\right\}, \tag{3.10}$$

where

$$q_1(t) \equiv (h_1''(t) - U_1(f)) \begin{vmatrix} h_2'(t) & U_2(g) \\ h_3'(t) & U_3(g) \end{vmatrix}$$

$$-(h_2''(t) - U_2(f)) \begin{vmatrix} h_1'(t) & U_1(g) \\ h_3'(t) & U_3(g) \end{vmatrix} + (h_3''(t) - U_3(f)) \begin{vmatrix} h_1'(t) & U_1(g) \\ h_3'(t) & U_2(g) \end{vmatrix},$$

$$q_{1k}(t) \equiv U_1(y_k) \begin{vmatrix} h_2'(t) & U_2(g) \\ h_3'(t) & U_3(g) \end{vmatrix}$$

$$-U_2(y_k) \begin{vmatrix} h_1'(t) & U_1(g) \\ h_3'(t) & U_3(g) \end{vmatrix} + U_3(y_k) \begin{vmatrix} h_1'(t) & U_1(g) \\ h_3'(t) & U_2(g) \end{vmatrix},$$

$$q_2(t) \equiv -(h_1''(t) - U_1(f)) \begin{vmatrix} h_2(t) & U_2(g) \\ h_3(t) & U_3(g) \end{vmatrix}$$

$$+(h_2''(t) - U_2(f)) \begin{vmatrix} h_1(t) & U_1(g) \\ h_3(t) & U_3(g) \end{vmatrix} - (h_3''(t) - U_3(f)) \begin{vmatrix} h_1(t) & U_1(g) \\ h_2(t) & U_2(g) \end{vmatrix},$$

$$q_{2k}(t) \equiv -U_1(y_k) \begin{vmatrix} h_2(t) & U_2(g) \\ h_3(t) & U_3(g) \end{vmatrix}$$

$$+U_2(y_k) \begin{vmatrix} h_1(t) & U_1(g) \\ h_3(t) & U_3(g) \end{vmatrix} - U_3(y_k) \begin{vmatrix} h_1'(t) & U_1(g) \\ h_2'(t) & U_2(g) \end{vmatrix},$$

$$q_3(t) \equiv (h_1''(t) - U_1(f)) \begin{vmatrix} h_2(t) & h_2'(t) \\ h_3(t) & h_3'(t) \end{vmatrix}$$

$$-(h_2''(t) - U_2(f)) \begin{vmatrix} h_1(t) & h_1'(t) \\ h_3(t) & h_3'(t) \end{vmatrix} + (h_3''(t) - U_3(f)) \begin{vmatrix} h_1(t) & h_1'(t) \\ h_2(t) & h_2'(t) \end{vmatrix},$$

$$q_{3k}(t) \equiv U_1(y_k) \begin{vmatrix} h_2(t) & h_2'(t) \\ h_3'(t) & h_3'(t) \end{vmatrix}$$

$$-U_2(y_k) \begin{vmatrix} h_1(t) & h_1'(t) \\ h_3'(t) & h_3'(t) \end{vmatrix} + U_3(y_k) \begin{vmatrix} h_1(t) & h_1'(t) \\ h_2(t) & h_2'(t) \end{vmatrix}.$$

The formulas (3.2) and (3.4) enables us to write

$$\lambda_k(\alpha u_k''(t) + u_k(t)) = \frac{\lambda_k}{1+\alpha\lambda_k} u_k(t) + \frac{\alpha\lambda_k}{1+\alpha\lambda_k} F_k(t; u, a, b, c)$$

$$= \frac{\alpha\lambda_k}{1+\alpha\lambda_k} F_k(t; u, a, b, c) + \frac{\lambda_k}{1+\alpha\lambda_k} \left(\varphi_k \cos\beta_k t + \frac{1}{\beta_k} \psi_k \sin\beta_k t \right.$$

$$\left. + \frac{1}{\beta_k(1+\alpha\lambda_k)} \int_0^t F_k(\tau; u, a, b, c) \sin\beta_k(t-\tau) d\tau \right). \tag{3.11}$$

Then from (3.8), (3.9),and (3.10), taking into account (3.11), we respectively find

$$a(t) = [h(t)]^{-1} \left\{ q_1(t) + \sum_{k=1}^{\infty} \left[\frac{\alpha\lambda_k}{1+\alpha\lambda_k} F_k(t; u, a, b, c) \right.\right.$$

$$+ \frac{\lambda_k}{1+\alpha\lambda_k} \left(\varphi_k \cos\beta_k t + \frac{1}{\beta_k} \psi_k \sin\beta_k t \right.$$

$$\left.\left.\left. + \frac{1}{\beta_k(1+\alpha\lambda_k)} \int_0^t F_k(\tau; u, a, b, c) \sin\beta_k(t-\tau) d\tau \right) \right] q_{1k}(t) \right\}, \tag{3.12}$$

$$b(t) = [h(t)]^{-1} \left\{ q_2(t) + \sum_{k=1}^{\infty} \left[\frac{\alpha\lambda_k}{1+\alpha\lambda_k} F_k(t; u, a, b, c) \right.\right.$$

$$+ \frac{\lambda_k}{1+\alpha\lambda_k} \left(\varphi_k \cos\beta_k t + \frac{1}{\beta_k} \psi_k \sin\beta_k t \right.$$

$$+\frac{1}{\beta_k(1+\alpha\lambda_k)}\int\limits_0^t F_k(\tau;u,a,b,c)\sin\beta_k(t-\tau)d\tau\Bigg)\Bigg]q_{2k}(t)\Bigg\}, \tag{3.13}$$

$$c(t)=[h(t)]^{-1}\Bigg\{q_3(t)+\sum_{k=1}^{\infty}\Bigg[\frac{\alpha\lambda_k}{1+\alpha\lambda_k}F_k(t;u,a,b,c)$$

$$+\frac{\lambda_k}{1+\alpha\lambda_k}\Bigg(\varphi_k\cos\beta_k t+\frac{1}{\beta_k}\psi_k\sin\beta_k t$$

$$+\frac{1}{\beta_k(1+\alpha\lambda_k)}\int\limits_0^t F_k(\tau;u,a,b,c)\sin\beta_k(t-\tau)d\tau\Bigg)\Bigg]q_{3k}(t)\Bigg\}. \tag{3.14}$$

Thus, the solution of problem (1.1)–(1.3), (1.11), (1.12) was reduced to the solution of systems (3.6), (3.12), (3.13), and (3.14), with respect to unknown functions $u(x,t), a(t), b(t)$, and $c(t)$.

Lemma 3.1. *If* $\{u(x,t), a(t), b(t), c(t)\}$ *is any solution of* (1.1)–(1.3), (1.11), (1.12)*, then the functions*

$$u_k(t)=\int\limits_0^1 u(x,t)z_k(x)dx,\ \ k=1,2,...$$

satisfy the relation (3.4) *on the interval* $[0,T]$.

Corollary 3.2. *Assume that the system* (3.6), (3.12), (3.13), *and* (3.14) *has a unique solution. Then the problem* (1.1)–(1.3), (1.11), (1.12) *has at most one solution; in other words, if the problem* (1.1)–(1.3), (1.11), (1.12) *has a solution, then it is unique.*

Let us now consider the operator

$$\Phi(u,a,b)=\{\Phi_1(u,a,b,c),\Phi_2(u,a,b,c),\Phi_3(u,a,b,c),\Phi_4(u,a,b,c)\},$$

in the space $E_T^{\frac{3}{2},1}$, where

$$\Phi_1(u,a,b,c)=\tilde{u}(x,t)\equiv\sum_{k=1}^{\infty}\tilde{u}_k(t)y_k(x),$$

$$\Phi_2(u,a,b,c)=\tilde{a}(t),\ \ \Phi_3(u,a,b,c)=\tilde{b}(t),\ \ \Phi_3(u,a,b,c)=\tilde{c}(t),$$

and the functions $\tilde{u}_k(t), k=1,2,..., \tilde{a}(t), \tilde{b}(t)$, and $\tilde{c}(t)$ are defined by the right-hand sides of (3.6), (3.12), (3.13), and (3.14), respectively.

It is obvious that

$$|U_i(y_k)|\le\sqrt{2}(1+\|\omega(x)\|_{C[0,1]})\equiv p,$$

$$1+\alpha\lambda_k>\alpha\lambda_k,\quad \frac{1}{\sqrt{\alpha+1}}<\beta_k<\frac{1}{\sqrt{\alpha}},\quad \sqrt{\alpha}<\frac{1}{\beta_k}<\sqrt{\alpha+1},$$

$$\frac{1}{(1+\alpha\lambda_k)\beta_k}=\frac{1}{\sqrt{(1+\alpha\lambda_k)\lambda_k}}<\frac{1}{\sqrt{\alpha}\lambda_k}.$$

With the help of easy transformations, we find that the following inequalities are valid

$$\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\,\|\tilde{u}_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\le\sqrt{6}\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\,|\varphi_k|)^2\right)^{\frac{1}{2}}$$

$$+\sqrt{6(\alpha+1)}\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\,|\psi_k|)^2\right)^{\frac{1}{2}}+\sqrt{\frac{6T}{\alpha}}\left(\int\limits_0^T\sum_{k=1}^{\infty}(\sqrt{\lambda_k}\,|f_k(\tau)|)^2\right)^{\frac{1}{2}}$$

$$+\sqrt{\frac{6}{\alpha}}T\,\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\sqrt{\frac{6}{\alpha}}T\left\|b(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k\left\|u'_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\sqrt{\frac{6T}{\alpha}}\left\|c(t)\right\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}(\sqrt{\lambda_k}\left|g_k(\tau)\right|)^2\right)^{\frac{1}{2}}, \tag{3.15}$$

$$\left(\sum_{k=1}^{\infty}(\lambda_k\left\|\tilde{u}'_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\leq\sqrt{\frac{6}{\alpha}}\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\left|\varphi_k\right|)^2\right)^{\frac{1}{2}}+\sqrt{6}\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\left|\psi_k\right|)^2\right)^{\frac{1}{2}}$$

$$+\frac{\sqrt{6T}}{\alpha}\left(\int_0^T\sum_{k=1}^{\infty}(\sqrt{\lambda_k}\left|f_k(\tau)\right|)^2\right)^{\frac{1}{2}}+\frac{\sqrt{6}}{\alpha}T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\left\|u_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\frac{\sqrt{6}}{\alpha}T\left\|b(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k\left\|u'_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\sqrt{\frac{6T}{\alpha}}\left\|c(t)\right\|_{C[0,T]}\left(\int_0^T\sum_{k=1}^{\infty}(\sqrt{\lambda_k}\left|g_k(\tau)\right|)^2\right)^{\frac{1}{2}}, \tag{3.16}$$

$$\left\|\tilde{a}(t)\right\|_{C[0,T]}\leq\left\|[h(t)]^{-1}\right\|_{C[0,T]}\Bigg\{\left\|q_1(t)\right\|_{C[0,T]}+p_1(T)\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}$$

$$\times\Bigg[\left(\sum_{k=1}^{\infty}(\sqrt{\lambda_k}\left\|f_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\left\|u_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\left\|b(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k\left\|u'_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\left\|c(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\sqrt{\lambda_k}\left\|g_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\frac{1}{\alpha}\left(\sum_{k=1}^{\infty}(\sqrt{\lambda_k}\left|\varphi_k\right|)^2\right)^{\frac{1}{2}}+\frac{1}{\sqrt{\alpha}}\left(\sum_{k=1}^{\infty}(\sqrt{\lambda_k}\left|\psi_k\right|)^2\right)^{\frac{1}{2}}$$

$$+\frac{1}{\alpha}\sqrt{\frac{T}{\alpha}}\left(\int_0^T\sum_{k=1}^{\infty}(\sqrt{\lambda_k}\left|f_k(\tau)\right|)^2d\tau\right)^{\frac{1}{2}}$$

$$+\frac{1}{\alpha\sqrt{\alpha}}T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\left\|u_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\frac{1}{\alpha\sqrt{\alpha}}T\left\|b(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k\left\|u'_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\frac{1}{\alpha}\sqrt{\frac{T}{\alpha}}\left\|c(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\sqrt{\lambda_k}\left\|g_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}\Bigg]\Bigg\}, \tag{3.17}$$

$$\left\|\tilde{b}(t)\right\|_{C[0,T]}\leq\left\|[h(t)]^{-1}\right\|_{C[0,T]}\Bigg\{\left\|q_2(t)\right\|_{C[0,T]}+p_2(T)\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}$$

$$\times\Bigg[\left(\sum_{k=1}^{\infty}(\sqrt{\lambda_k}\left\|f_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\left\|a(t)\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\left\|u_k(t)\right\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\|b(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k\|u_k'(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\|c(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\sqrt{\lambda_k}\|g_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\frac{1}{\alpha}\left(\sum_{k=1}^{\infty}(\sqrt{\lambda_k}|\varphi_k|)^2\right)^{\frac{1}{2}}+\frac{1}{\sqrt{\alpha}}\left(\sum_{k=1}^{\infty}(\sqrt{\lambda_k}|\psi_k|)^2\right)^{\frac{1}{2}}$$

$$+\frac{1}{\alpha}\sqrt{\frac{T}{\alpha}}\left(\int_0^T\sum_{k=1}^{\infty}(\sqrt{\lambda_k}|f_k(\tau)|)^2d\tau\right)^{\frac{1}{2}}$$

$$+\frac{1}{\alpha\sqrt{\alpha}}T\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\frac{1}{\alpha\sqrt{\alpha}}T\|b(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k\|u_k'(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$\left.\left.+\frac{1}{\alpha}\sqrt{\frac{T}{\alpha}}\|c(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\sqrt{\lambda_k}\|g_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\right\}, \tag{3.18}$$

$$\|\tilde{c}(t)\|_{C[0,T]}\le\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\|q_3(t)\|_{C[0,T]}+p_3(T)\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}\right.$$

$$\times\left[\left(\sum_{k=1}^{\infty}(\sqrt{\lambda_k}\|f_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right.$$

$$+\|b(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k\|u_k'(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}+\|c(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\sqrt{\lambda_k}\|g_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\frac{1}{\alpha}\left(\sum_{k=1}^{\infty}(\sqrt{\lambda_k}|\varphi_k|)^2\right)^{\frac{1}{2}}+\frac{1}{\sqrt{\alpha}}\left(\sum_{k=1}^{\infty}(\sqrt{\lambda_k}|\psi_k|)^2\right)^{\frac{1}{2}}+\frac{1}{\alpha}\sqrt{\frac{T}{\alpha}}\left(\int_0^T\sum_{k=1}^{\infty}(\sqrt{\lambda_k}|f_k(\tau)|)^2d\tau\right)^{\frac{1}{2}}$$

$$+\frac{1}{\alpha\sqrt{\alpha}}T\|a(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k\sqrt{\lambda_k}\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$+\frac{1}{\alpha\sqrt{\alpha}}T\|b(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\lambda_k\|u_k'(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}$$

$$\left.\left.+\frac{1}{\alpha}\sqrt{\frac{T}{\alpha}}\|c(t)\|_{C[0,T]}\left(\sum_{k=1}^{\infty}(\sqrt{\lambda_k}\|g_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}}\right]\right\}, \tag{3.19}$$

whose $\tilde{u}_k'(t),\ k=1,2,...$, equal to the right sides of (3.5), and

$$p_1(T)\equiv p\left\|\begin{matrix}h_2'(t) & U_2(g)\\ h_3'(t) & U_3(g)\end{matrix}\right\|_{C[0,T]}$$

$$+p\left\|\begin{matrix}h_1'(t) & U_1(g)\\ h_3'(t) & U_3(g)\end{matrix}\right\|_{C[0,T]}+p\left\|\begin{matrix}h_1'(t) & U_1(g)\\ h_2'(t) & U_2(g)\end{matrix}\right\|_{C[0,T]},$$

$$p_2(T)\equiv p\left\|\begin{matrix}h_2(t) & U_2(g)\\ h_3(t) & U_3(g)\end{matrix}\right\|_{C[0,T]}$$

$$+p\left\|\begin{matrix}h_1(t) & U_1(g)\\ h_3(t) & U_3(g)\end{matrix}\right\|_{C[0,T]}+p\left\|\begin{matrix}h_1'(t) & U_1(g)\\ h_2'(t) & U_2(g)\end{matrix}\right\|_{C[0,T]},$$

$$p_3(T)\equiv p\left\| \begin{matrix} h_2(t) & h_2'(t) \\ h_3(t) & h_3'(t) \end{matrix} \right\|_{C[0,T]}$$

$$+p\left\| \begin{matrix} h_1(t) & h_1'(t) \\ h_3'(t) & h_3(t) \end{matrix} \right\|_{C[0,T]}+p\left\| \begin{matrix} h_1(t) & h_1'(t) \\ h_2(t) & h_2'(t) \end{matrix} \right\|_{C[0,T]}.$$

Suppose that the data of the problem (1.1)–(1.3), (1.11), (1.12), satisfy the following statements:

$S_1)$ $\varphi(x)\in C^2[0,1],\ \varphi'''(x)\in L_2(0,1),\ \varphi(0)=\varphi''(0)=0$, and
$\varphi(1)+\frac{1}{d\sin\sqrt{\lambda_0}}\int\limits_0^1\varphi(x)\sin(\sqrt{\lambda_0}x)dx=0,\ d\varphi''(1)+\varphi'(1)=0;$

$S_2)$ $\psi(x)\in C^2[0,1],\ \psi'''(x)\in L_2(0,1),\ \psi(0)=\psi''(0)=0$, and
$\psi(1)+\frac{1}{d\sin\sqrt{\lambda_0}}\int\limits_0^1\psi(x)\sin\sqrt{\lambda_0}x dx=0,\ d\psi''(1)+\psi'(1)=0;$

$S_3)$ $f(x,t)\in C(D_T),\ f_x(x,t)\in L_2(D_T),\ f(0,t)=0$, and
$f(1,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int\limits_0^1 f(x,t)\sin(\sqrt{\lambda_0}x)dx=0,\ 0\le t\le T;$

$S_4)$ $g(x,t)\in C(D_T),\ g_x(x,t)\in L_2(D_T),\ g(0,t)=0$, and
$g(1,t)+\frac{1}{d\sin\sqrt{\lambda_0}}\int\limits_0^1 g(x,t)\sin(\sqrt{\lambda_0}x)dx=0,\ 0\le t\le T;$

$S_5)$ $h_i(t)\in C^2[0,T]\ (i=1,2,3),\ h(t)\equiv\begin{vmatrix} h_1(t) & h_1'(t) & U_1(g) \\ h_2(t) & h_2'(t) & U_2(g) \\ h_3(t) & h_3'(t) & U_3(g)\end{vmatrix}\neq 0,\ 0\le t\le T.$

Then, taking into account (2.2), (2.4), from (3.15)–(3.19), we obtain

$$\|\tilde u(x,t)\|_{B^{\frac{3}{2},1}_{2,T}}+\|\tilde a(t)\|_{C[0,T]}+\left\|\tilde b(t)\right\|_{C[0,T]}+\|\tilde c(t)\|_{C[0,T]}$$

$$\le A(T)+B(T)(\|a(t)\|_{C[0,T]}+\|b(t)\|_{C[0,T]})\,\|u(x,t)\|_{B^{\frac{3}{2},1}_{2,T}}+D(T)\,\|c(t)\|_{C[0,T]}, \qquad (3.20)$$

where

$$A(T)=A_1(T)+A_2(T)+A_3(T)+A_4(T),$$
$$B(T)=B_1(T)+B_2(T)+B_3(T)+B_4(T),$$
$$D(T)=D_1(T)+D_2(T)+D_3(T)+D_4(T),$$

in which

$$A_1(T)=\left(\sqrt{6}+\sqrt{\frac{6}{\alpha}}\right)M\,\|\varphi'''(x)\|_{L_2(0,1)}+M(\sqrt{6(\alpha+1)}+\sqrt{6})\,\|\psi'''(x)\|_{L_2(0,1)}$$

$$+M\left(\sqrt{\frac{6T}{\alpha}}+\frac{\sqrt{6T}}{\alpha}\right)\|f_x(x,t)\|_{L_2(D_T)},$$

$$B_1(T)=\left(\sqrt{\frac{6}{\alpha}}+\frac{\sqrt{6}}{\alpha}\right)T,$$

$$D_1(T)=M\left(\sqrt{\frac{6T}{\alpha}}+\frac{\sqrt{6T}}{\alpha}\right)\|g_x(x,t)\|_{L_2(D_T)},$$

$$A_2(T)=\left\|[h(t)]^{-1}\right\|\left\{\|q_1(t)\|_{C[0,T]}+p_1(T)\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}\left[M\left\|\|f_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}\right.\right.$$

$$\left.\left.+\frac{1}{\alpha}M\,\|\varphi'''(x)\|_{L_2(0,1)}+\frac{1}{\sqrt{\alpha}}M\,\|\psi'''(x)\|_{L_2(0,1)}+\frac{1}{\alpha}\sqrt{\frac{T}{\alpha}}M\,\|f_x(x,t)\|_{L_2(D_T)}\right]\right\},$$

$$B_2(T)=p_1(T)\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}\left(1+\frac{T}{\alpha\sqrt{\alpha}}\right),$$

$$D_2(T)=p_1(T)\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}M$$

$$\times\left(\left\|\|g_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}+\frac{1}{\alpha}\sqrt{\frac{T}{\alpha}}\|g_x(x,t)\|_{L_2(D_T)}\right),$$

$$A_3(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\|\left\{\|q_2(t)\|_{C[0,T]}+p_2(T)\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}\left[M\left\|\|f_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}\right.\right.$$

$$\left.\left.+\frac{1}{\alpha}M\|\varphi'''(x)\|_{L_2(0,1)}+\frac{1}{\sqrt{\alpha}}M\|\psi'''(x)\|_{L_2(0,1)}+\frac{1}{\alpha}\sqrt{\frac{T}{\alpha}}M\|f_x(x,t)\|_{L_2(D_T)}\right]\right\},$$

$$B_3(T)=p_2(T)\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}\left(1+\frac{T}{\alpha\sqrt{\alpha}}\right),$$

$$D_3(T)=p_2(T)\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}M$$

$$\times\left(\left\|\|g_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}+\frac{1}{\alpha}\sqrt{\frac{T}{\alpha}}\|g_x(x,t)\|_{L_2(D_T)}\right),$$

$$A_4(T)=\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\|q_3(t)\|_{C[0,T]}+p_3(T)\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}\left[M\left\|\|f_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}\right.\right.$$

$$\left.\left.+\frac{1}{\alpha}M\|\varphi'''(x)\|_{L_2(0,1)}+\frac{1}{\sqrt{\alpha}}M\|\psi'''(x)\|_{L_2(0,1)}+\frac{1}{\alpha}\sqrt{\frac{T}{\alpha}}M\|f_x(x,t)\|_{L_2(D_T)}\right]\right\},$$

$$B_4(T)=p_3(T)\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}\left(1+\frac{T}{\alpha\sqrt{\alpha}}\right),$$

$$D_4(T)=p_3(T)\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{k=1}^{\infty}\lambda_k^{-1}\right)^{\frac{1}{2}}M$$

$$\times\left(\left\|\|g_x(x,t)\|_{C[0,T]}\right\|_{L_2(0,1)}+\frac{1}{\alpha}\sqrt{\frac{T}{\alpha}}\|g_x(x,t)\|_{L_2(D_T)}\right).$$

Let us denote by K_R a closed ball in the space $E_T^{\frac{3}{2},1}$ centered at zero, of radius R.

Theorem 3.3. *Suppose that the conditions S_1) - S_5) and*

$$(A(T)+2)((A(T)+2)B(T)+D(T))<1 \tag{3.21}$$

are satisfied. Then, problem (1.1)–(1.3), (1.11), (1.12) *has a unique solution in the ball* $K_R\subset E_T^{\frac{3}{2},1}$ $(R=A(T)+2)$.

Remark 3.4. Inequality (3.21) is satisfied for sufficiently small values of $T+\left\|[h(t)]^{-1}\right\|_{C[0,T]}$.

Proof. Let us consider the following operator equation

$$z = \Phi z, \tag{3.22}$$

in the space $E_T^{\frac{3}{2},1}$ whose $z = \{u, a, b, c\}$ and the components Φ_i $(i = 1, 2, 3, 4)$ of operator $\Phi(u, a, b, c)$ defined by the right side of equations (3.6), (3.12), (3.13), and (3.14), respectively.

Similarly to (3.20), we obtain that for any $z, z_1, z_2 \in K_R$ the following two estimates hold:

$$\|\Phi z\|_{E_T^{3/2;1}} \le A(T) + B(T)(\|a(t)\|_{C[0,T]} + \|b(t)\|_{C[0,T]}) \|u(x,t)\|_{B_{2,T}^{3/2;1}} + D(T)\|c(t)\|_{C[0,T]}, \tag{3.23}$$

$$\|\Phi z_1 - \Phi z_2\|_{E_T^{3/2;1}} \le B(T)R(\|a_1(t) - a_2(t)\|_{C[0,T]} + \|b_1(t) - b_2(t)\|_{C[0,T]} + \|u_1(x,t) - u_2(x,t)\|_{B_{2,T}^{3/2;1}}) + D(T)\|c_1(t) - c_2(t)\|_{C[0,T]}. \tag{3.24}$$

Then (3.21), (3.23), and (3.24) implies that the operator Φ acts in the ball K_R and is contractive. Therefore, the operator Φ has a unique fixed point $\{u, a, b, c\}$ in the ball K_R. Consequently, $z = \{u, a, b, c\}$ is the unique solution of system (3.6), (3.12), (3.23), (3.14) in the ball K_R.

Thus, we obtain that the function $u(x,t)$ as an element of the space $B_{2,T}^{\frac{3}{2},1}$ is continuous and has continuous derivatives $u_x(x,t), u_{xx}(x,t), u_t(x,t)$, and $u_{tx}(x,t)$ in D_T.

Exploiting the inequality (2.2), from (3.2) we get

$$\left(\sum_{k=1}^{\infty} (\lambda_k\sqrt{\lambda_k}\,\|u_k''(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} \le \sqrt{2}\left[\frac{1}{\alpha}\left(\sum_{k=1}^{\infty} (\lambda_k\sqrt{\lambda_k}\,\|u_k(t)\|_{C[0,T]})^2\right)^{\frac{1}{2}} + \frac{M}{\alpha}\|f_x(x,t) + a(t)u_x(x,t) + b(t)u_{tx}(x,t) + c(t)g_x(x,t)\|_{L_2(D_T)}\right],$$

whence it follows that $u_{tt}(x,t)$ and $u_{ttxx}(x,t)$ are continuous in the region D_T. Furthermore, it is not hard to verify that equation (1.1), and conditions (1.2), (1.3), (1.11), and (1.12) are satisfied in the usual sense.

Consequently, $\{u(x,t), a(t), b(t), c(t)\}$ is a solution of problem (1.1)–(1.3), (1.11), (1.12), and by Corollary 3.2 this solution is unique in the ball K_R. □

Finally, from Theorem 1.2 and Theorem 3.3 we arrive at the following desired result

Theorem 3.5. *Assume that the assumptions of Theorem 3.3 and compatibility conditions*

$$U_i(\varphi) = h_i(0),\ U_i(\psi) = h_i'(0)\ (i = 1, 2, 3)$$

hold. Then problem (1.1)–(1.5) *has a unique classical solution in the ball* K_R *of space* $E_T^{\frac{3}{2},1}$.

4. Conclusions

In the work, the classical solvability of a nonlinear inverse boundary-value problem for a one-dimensional pseudohyperbolic equation with nonclassical conditions was studied. First, the considered problem was reduced to an auxiliary inverse boundary-value problem in a certain sense, then using the Fourier method and contraction mappings principle, the existence and uniqueness theorem for auxiliary problem is proved. Further, on the basis of the equivalency of these problems, the existence and uniqueness theorem for the classical solution of the original inverse coefficient problem is established.

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