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Research Article

Unipotent and Unit-Regular Elements in Certain Subrings of $M_2(\mathbb{Z})$

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Abstract

We presented a simple and direct way to construct a unipotent unit and clean but not nil-clean element in a ring. New examples of unipotent/unit-regular elements that are not nil-clean are given. We also study the product of two idempotents/unit-regulars which are unit-regular. The studies are exemplified in two subrings of $M_2(\mathbb{Z})$.

Keywords: Idempotent Element, Nil-Clean Element, Nilpotent Element, Unit Element, Unit-Regular Element

$M_2(\mathbb{Z})$ Halkasının Belirli Alt Halkalarındaki Tek Kuvvetli ve Terslenir Düzenli Elemanlar

Öz

Bir halkada, bir sıfır güçlü terslenir ve temiz ama nil-temiz olmayan bir eleman oluşturmanın basit ve doğrudan bir yolunu sunduk. Nil-temiz olmayan sıfır güçlü/terslenir-düzenli elemanların yeni örnekleri verilmiştir. Ayrıca, terslenir-düzenli olan iki eşkare/birim-düzenli elemanların çarpımları da incelenmiştir. $M_2(\mathbb{Z})$ halkasının iki alt halkasında çalışmalar örneklendirilmiştir.

Anahtar Kelimeler: Eşkare Eleman, Nil-Temiz Eleman, Sıfır Güçlü Eleman, Terslenir Eleman, Terslenir-Düzenli Eleman.

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1. Introduction

Vidinli Hüseyin Tevfik Pasha (1832-1901) was a famous Ottoman mathematician. He taught advanced algebra, high algebra, analytical geometry, differential, integral calculus, mechanics and astronomy at the Military Academy (*Harbiye Mektebi*) of the Ottoman Empire. What makes special his work Linear Algebra, written in English in 1882, is that he produced a completely original work at a time when it was tried to make progress in the sciences through translations and compilations in general.

Although his work seems to be dealing with real and complex numbers, one of the newest subjects of his time, he actually focused on three-dimensional algebras -not two dimensional- within the hypercomplex number system. In the background of this focus, pure quaternions which are a three-dimensional vector subspace of quaternions and a four-dimensional algebra have the purpose of repeating the mutation application in three-dimensional Euclidean geometry in two dimensions.

In short, Tevfik Pasa's Linear Algebra tries to spread the complex or virtual value system to three-dimensional space by making use of Argand's concept of a vector calculus.

Here we reconsider this problem by working on unipotent elements, unit-regular elements, nil-clean elements and clean elements based on the Tevfik Pasha's adaptation of linear algebra, which is one of the most important fundamental theories of modern mathematics. We present a simple and direct way to construct a unipotent unit and clean but not nil-clean element in the ring $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ for every positive integer $s \geq 3$.

We show that the product of two unit-regulars in R_i is unit-regular if and only if the product of two idempotents in R_i is unit-regular where $R_1 := \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ and $R_2 := \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ with $s \geq 3$. Because of this observation, we also obtain that the rings R_i $(i = 1,2)$ are SSP if and only if product of two idempotents in R_i $(i = 1,2)$ is unit-regular.

2. Preliminaries

Throughout, R is an associative ring with unity.

We write $\mathbb{Z}$ is the ring of integers, $M_2(\mathbb{Z})$ is the 2×2 matrix ring over $\mathbb{Z}$ whose identity is denoted by I_2 over R.

A ring R is called *clean* if each element of its can be written as the sum of a unit and an idempotent. Clean rings were introduced by W. K. Nicholson [7].

In [1], Andrica and Calugareanu found a counter example and gave a structure theorem which is nil-clean but not clean element in the matrix ring $M_2(\mathbb{Z})$. In [8] the authors considered this problem on the subring $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ of $R \coloneqq M_2(\mathbb{Z})$ instead of R since the subring $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ contains much less clean elements than $M_2(\mathbb{Z})$, a huge advantage. The authors of [8] gave also many counter-examples of unit-regular elements (an element in a ring is unit-regular if it is a product of an idempotent and a unit, and a ring is unit-regular if its every element is unit-regular) and nil-clean elements that are not clean in the ring $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$.

An element $a \in R$ in a ring is called *unipotent*, if $a - 1$ is nilpotent.

An element a in any ring R is said to *have (right) stable range 1* ($sr(a) = 1$) if $aR + bR = R$ (for any $b \in R$) implies that $a + br$ is a unit for some $r \in R$. We recall that if a is a unit-regular element in a ring R, then $sr(a) = 1$.

A ring R is said to being the *summand sum property (briefly SSP)* if the sum of two direct summands of R_R is also a direct summand of R ([5]). It is well known that $M_2(\mathbb{Z})$ is not SSP while $\mathbb{Z}$ is an SSP ring.

3. Main Theorem and Proof

We begin recalling the following basic facts over the matrix ring $M_2(\mathbb{Z})$.

- The units in $M_2(\mathbb{Z})$ are the 2×2 matrices of $det = \mp 1$.
- A non-trivial idempotent matrix in $M_2(\mathbb{Z})$ has $rank$ 1.
- A nilpotent matrix in $M_2(\mathbb{Z})$ has the characteristic polynomial t^2 and so it has the trace which is equal to 1.

Lemma 3.1. ([1]) Let $s \in \mathbb{Z}$. Nontrivial idempotents and nilpotents in the ring $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ are matrices $\begin{pmatrix} \alpha + 1 & u \\ sv & -\alpha \end{pmatrix}$ with $\alpha^2 + \alpha + suv = 0$ and $\begin{pmatrix} \beta & x \\ sy & -\beta \end{pmatrix}$ with $\beta^2 + sxy = 0$ respectively.

Proposition 3.2. For rings $R_1 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ and $R_2 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ with $s \geq 3$ an even number, there exist no any invertible matrices U_i in R_i $(i = 1,2)$ such that $I_2 + U_i$ are invertible in R_i $(i = 1,2)$.

Proof: We only give proof for the ring R_1. The other is similar.

Assume the contrary that there exists an invertible element $U_1 = \begin{pmatrix} a & b \\ 4c & d \end{pmatrix}$ in R_1 such that $det(U_1) = ad - 4bc = \mp 1$. By the assumption, $I_2 + U_1$ must be also invertible in R_1, i.e.,

$$I_2 + U_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a & b \\ 4c & d \end{pmatrix} = \begin{pmatrix} a + 1 & b \\ 4c & d + 1 \end{pmatrix}$$

and $det(I_2 + U_1) = (ad - 4bc) + (a + d + 1) = \mp 1$. Now we can proceed with the following cases.

Case 1. If $ad - 4bc = 1$, then $a + d = -1$ and $a + d = -3$.

Firstly, assume $a + d = -1$. Then $(-1 - d)d - 4bc = d + d^2 + 4bc = 1$. Since $4bc$ is an even number, the number $d + d^2 = d(d + 1)$ must be odd, a contradiction. If $a + d = -3$, we get $(-3 - d)d - 4bc = (3 + d)d + 4bc = -1$. Since $4bc$ is an even number, we get $d(d + 3)$ must be odd, a contradiction.

Case 2. If $ad - 4bc = -1$, then $a + d = -1$ and $a + d = 1$.

If we repeat the procedure of Case 1, we can obtain similar contradictions.

By [3, Corollary 3.3], $M_2(\mathbb{Z})$ is not UU (UR=1+NR) (i.e. U (R) =1+N (R)) since $I_2 + U_1$ in $M_2(\mathbb{Z})$ are not unipotent.■

Theorem 3.3. There exist unipotent unit, clean matrices which are not nil-clean in $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$

Proof: This is clear from Proposition 3.2. ■

Example 3.4. The matrix

$$\mathrm{A} = \begin{pmatrix} -3 & -2 \\ 8 & 5 \end{pmatrix}$$

is a unipotent unit in $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ since $\mathrm{A} - \mathrm{I}_2 = \begin{pmatrix} -4 & -2 \\ 8 & 4 \end{pmatrix}$ is a nilpotent. As units are clean, the matrix A is clean but is not nil-clean in $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ by [8, Theorem 3.3].

Example 3.5. The matrix

$$A = \begin{pmatrix} s+1 & 1 \\ -s^2 & -s+1 \end{pmatrix}$$

is a unipotent unit in the ring $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$, where $s \geq 3$ is an even number since $I_2 - U = \begin{pmatrix} -s & -1 \\ s^2 & s \end{pmatrix}$ is a nilpotent matrix in $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$. As units are clean, the matrix A is clean but is not nil-clean in $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ by [8, Theorem 3.4].

Lemma 3.6. ([6]) Let $s \in \mathbb{Z}$. Unit-regular elements in the ring $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ are matrices

$$\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & u \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ sz & t \end{pmatrix},$$

where $E = \begin{pmatrix} 1 & u \\ 0 & 0 \end{pmatrix}$ is an idempotent and $U = \begin{pmatrix} x & y \\ sz & t \end{pmatrix}$ is a unit.

The following examples show that the product of two idempotents (or unit-regulars) in $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ need not be unit-regular, in general.

Example 3.7. Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$. Consider the idempotents

$$E_1 = \begin{pmatrix} 1 & 0 \\ 4 & 0 \end{pmatrix} \text{ and } E_2 = \begin{pmatrix} 9 & 3 \\ -24 & -8 \end{pmatrix}$$

Then

$$E_1E_2 = \begin{pmatrix} 9 & 3 \\ 36 & 0 \end{pmatrix}$$

is not unit-regular.

Example 3.8. Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$. Consider the unit-regulars

$$A = \begin{pmatrix} 11 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 11 & 1 \\ 32 & 3 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 13 & 5 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 13 & 5 \\ 8 & 3 \end{pmatrix}$$

in $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$. Then

$$AB = \begin{pmatrix} 143 & 55 \\ 0 & 0 \end{pmatrix}$$

is not unit-regular. In fact, if

$$AB = \begin{pmatrix} 143 & 55 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 143 & 55 \\ 4a & b \end{pmatrix}$$

then $220a - 143b = 11(20a - 13b)$ can not be -1 or 1 for any integers a and b.

The following examples show that the product of two idempotents (or unit-regulars) in $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ need not be unit-regular, in general.

Example 3.9. Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ with $s \geq 3$. Consider the idempotents

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } E_2 = \begin{pmatrix} 1 & 0 \\ s^2 & 0 \end{pmatrix}$$

in R. Then

$$E_1E_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

is not unit-regular.

Example 3.10. Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ with $s \geq 3$. Consider the unit-regulars

$$A = \begin{pmatrix} 6 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 6 & 1 \\ -25 & -4 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 4 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 4 & 1 \\ -9 & -2 \end{pmatrix}$$

in $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$. Then

$$AB = \begin{pmatrix} 24 & 6 \\ 0 & 0 \end{pmatrix}$$

is not unit-regular. In fact, if

$$AB = \begin{pmatrix} 24 & 6 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 24 & 6 \\ s^2z & t \end{pmatrix}$$

then $24t - 6zs^2$ can not be -1 or 1 for any integers a and b.

Proposition 3.11. The following conditions are equivalent for the rings $R_1 := \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ and $R_2 := \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ with $s \geq 3$:

(1) The product of two unit-regulars in R_i $(i = 1,2)$ is unit-regular,

(2) The product of two idempotents in R_i $(i = 1,2)$ is unit-regular.

Proof. We only give proof for the ring R_1. The other is similar.

$(1) \Rightarrow (2)$: Suppose that the product of two idempotents in R_1 is unit-regular. Let $A = E_1U_1$ and $B = E_2U_2$ be two unit-regular in R_1, where $E_1, E_2 \in Id(R_1)$ and $U_1, U_2 \in U(R_1)$. It is easy to see that $U_1E_2U_1^{-1}$ is an idempotent and $AB = E_1(U_1E_2U_1^{-1})U_1U_2$. Put $E_3 := U_1E_2U_1^{-1}$. Then we conclude that

$$AB = E_1E_3U_1U_2$$

By the assumption, E_1E_3 is unit-regular and hence AB is unit-regular.

$(2) \Rightarrow (1)$: It is clear. ■

Example 3.12. Let $R = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$. Consider the idempotents

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } E_2 = \begin{pmatrix} 9 & -18 \\ 4 & -8 \end{pmatrix}$$

in R. Then

$$E_1E_2 = \begin{pmatrix} 9 & -18 \\ 0 & 0 \end{pmatrix}$$

is unit-regular. Let

$$A = \begin{pmatrix} 9 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 9 & 2 \\ 4 & 1 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 7 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 7 & 1 \\ 8 & 1 \end{pmatrix}$$

Then AB is unit-regular. Since

$$AB = \begin{pmatrix} 63 & 9 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 9 & -18 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 79 & 11 \\ 36 & 5 \end{pmatrix}$$

Example 3.13. In $R_2 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$, sonsider the idempotents

$$E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \text{ and } E_2 = \begin{pmatrix} -24 & -6 \\ 100 & 25 \end{pmatrix}.$$

Then

$$E_1E_2 = \begin{pmatrix} -24 & -6 \\ 0 & 0 \end{pmatrix}$$

is unit-regular. Let

$$A = \begin{pmatrix} 6 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 6 & 1 \\ -25 & -4 \end{pmatrix}$$

and

$$B = \begin{pmatrix} 11 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 11 & 2 \\ 16 & 3 \end{pmatrix}$$

Then

$$AB = \begin{pmatrix} 66 & 12 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -24 & -6 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 82 & 15 \\ -339 & -62 \end{pmatrix}$$

is unit-regular.

Corollary 3.14. The rings R_1 and R_2 are SSP if and only if the product of two unit-regulars in R_i $(i = 1,2)$ is unit-regular in R_i $(i = 1,2)$.

Proof. Assume the contrary that R_1 is SSP. Let E_1, E_2 be two idempotents in R_1. Since R_1 is SSP, we get $(I_2 - E_1)R_1 + E_2R_1$ is a direct summand of R_1, and so $E_1E_2R_1$ is a direct summand of R_1. It follows that E_1E_2 is regular. Take $A = E_1E_2$ and $B \in R$ with $A = ABA$. Since all idempotents of R_1 have right stable range 1, we obtain that $sr(A) = 1$ by [4, Proposition 2]. Now, $AR_1 + (I_2 - AB)R_1 = R_1$. There exists C in R_1 such that $A + (I_2 - AB)C$ is a unit. Let U be a unit of R with $[A + (I_2 - AB)C]U = I_2$. Then, we have

$$E_1E_2 = A = ABA = AB[A + (I_2 - AB)C] = ABAUA = AUA$$

which implies that E_1E_2 is unit-regular.

For the converse, let E_1, E_2 be two idempotents of in R_1. By the assumption (and hence from Proposition 3.11), we obtain that $(I_2 - E_1)E_2$ is unit-regular. Hence $(I_2 - E_1)E_2R_1$ is a direct summand of R_1. Let I be a right ideal of R_1 such that $(I_2 - E_1)E_2R_1 \oplus I = R_1$. Then,

$$(I_2 - E_1)R_1 = (I_2 - E_1)E_2R_1 \oplus [(I_2 - E_1)R_1 \cap I]$$

In as much as $E_1R_1 + E_2R_1 = E_1R_1 \oplus (I_2 - E_1)E_2R_1$, we have

$$R_1 = E_1R_1 \oplus (I_2 - E_1)E_2R_1 \oplus [(I_2 - E_1)R_1 \cap I]$$

$$= (E_1R_1 + E_2R_1) \oplus [(I_2 - E_1)R_1 \cap I].$$

This shows that R_1 has SSP. ■

One can easily see that unit-regular elements can not be unipotents because of the structure in the rings $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ and $\begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ s^2\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ with $s \geq 3$. The following gives us that there exists unit-regular elements which may be unipotents in these rings, but we don't know them, unfortunately.

Theorem 3.15. For rings R_1 and R_2, there exist no any unit-regular matrices A_i in R_i $(i = 1,2)$ such that $I_2 + A_i$ are invertible in R_i $(i = 1,2)$.

Proof. We only give proof for the ring R_1. The other is similar. Assume on contrary that there exists a unit-regular matrix A_1 in $R_1 = \begin{pmatrix} \mathbb{Z} & \mathbb{Z} \\ 4\mathbb{Z} & \mathbb{Z} \end{pmatrix}$ such that $I_2 + A_1$ are invertible in R_1. In the general case, we consider the unit-regular element

$$A_1 = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} x & y \\ 4z & t \end{pmatrix}$$

where $E = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ is an idempotent and $U = \begin{pmatrix} x & y \\ 4z & t \end{pmatrix}$ is a unit. By the assumption, $I_2 + A_1$ must be also invertible in R_1, i.e.,

$$I_2 + A_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a+1 & b \\ 0 & 1 \end{pmatrix}$$

and $det(I_2 + A_1) = a + 1 = \mp 1$. Now we can proceed with the following cases.

Case 1. If $a + 1 = 1$, then $a = 0$.

Hence $A_1 = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ which is not a unit-regular element in R_1.

Case 2. If $a + 1 = -1$, then $a = -2$.

Hence $A_1 = \begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} -2 & b \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -2 & b \\ 4z & t \end{pmatrix}$, which gives us that $2t + 4bz$ should be ∓ 1. Clearly, this equation has no integer solutions. ■

4. Conclusion

In this paper, we focus two subring of $M_2(\mathbb{Z})$. We give basic way to find not nil-clean elements which are unipotent and clean. We give examples of the product of two idempotent (unit-regulars) not be unit-regular in two subring of $M_2(\mathbb{Z})$.

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

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