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ABSORBING MULTIPLICATION MODULES OVER PULLBACK RINGS

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Abstract. Following some ideas and a technique introduced in [Comm. Algebra 41 (2013), pp. 776-791] we give a complete classification, up to isomorphism, of all indecomposable 2-absorbing multiplication modules with finite-dimensional top over pullback of two discrete valuation domains with the same residue field.

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1. Introduction

In this paper all rings are commutative with identity and all modules unitary. Let $v_1 : R_1 \to \bar{R}$ and $v_2 : R_2 \to \bar{R}$ be homomorphisms of two discrete valuation domains R_i onto a common field $\bar{R}$. Denote the pullback $R = \{(r_1, r_2) \in R_1 \oplus R_2 : v_1(r_1) = v_2(r_2)\}$ by $(R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2)$, where $\bar{R} = R_1/J(R_1) = R_2/J(R_2)$. Then R is a ring under coordinate-wise multiplication. Denote the kernel of v_i, $i = 1, 2$, by P_i. Then $\mathrm{Ker}(R \to \bar{R}) = P = P_1 \times P_2$, $R/P \cong \bar{R} \cong R_1/P_1 \cong R_2/P_2$, and $P_1P_2 = P_2P_1 = 0$ (so R is not a domain). Furthermore, for $i \neq j$, $0 \to P_i \to R \to R_j \to 0$ is an exact sequence of R-modules (see [20]). Modules over pullback rings has been studied by several authors (see for example, [1,5,9,15,19,25,32]). Notably, there is the important work of Levy [22], resulting in the classification of all finitely generated indecomposable modules over Dedekind-like rings. Klingler [19] extended this classification to lattices over certain non-commutative Dedekind-like rings, and Haefner and Klingler classified lattices over certain non-commutative pullback rings, which they called special quasi triads, see [16,17]. Common to all these classification is the reduction to a "matrix problem" over a division ring, see [6] and [29, Section 17.9] for a background of matrix problems and their applications. Here we should point out that the classification of all indecomposable modules over an arbitrary unitary ring (including finite-dimensional algebras over an algebraically closed field) is an impossible task. In particular, an infinite-dimensional version of tame representation type is in fact wild representation type. For a discussion of this kind of problems the reader is referred to the papers by Ringel [28] and Simson [30].

The concept of 2-absorbing ideal, which is a generalization of prime ideal, was introduced and studied by Badawi in [2]. Various generalizations of prime ideals are also studied in [3] and [4]. Recall that a proper ideal I of a ring R is called a 2-absorbing ideal of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. Recently (see [26,33]), the concept of 2-absorbing ideal is extended to the context of 2-absorbing submodule which is a generalization of prime submodule. Recall from [26] that a proper R-submodule N of a module M is said to be a 2-absorbing submodule of M if whenever $a, b \in R$, $m \in M$ and $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$.

In the present paper we introduce a new class of R-modules, called 2-absorbing multiplication modules, and we study it in details from the classification problem point of view. We are mainly interested in case either R is a discrete valuation domain or R is a pullback of two discrete valuation domains. First, we give a complete description of the 2-absorbing multiplication modules over a discrete valuation domain. Let R be a pullback of two discrete valuation domains over a common factor field. Next, the main purpose of this paper is to give a complete description of the indecomposable 2-absorbing multiplication R-modules with finite-dimensional top over $R/\mathrm{rad}(R)$ (for any module M we define its top as $M/\mathrm{Rad}(R)M$). The classification is divided into two stages: the description of all indecomposable separated 2-absorbing multiplication R-modules and then, using this list of separated 2-absorbing multiplication modules we show that non-separated indecomposable 2-absorbing multiplication R-modules with finite-dimensional top are factor modules of finite direct sums of separated indecomposable 2-absorbing multiplication R-modules. Then we use the classification of separated indecomposable 2-absorbing multiplication modules from Section 3, together with results of Levy [21,22] on the possibilities for amalgamating finitely generated separated modules, to classify the non-separated indecomposable 2-absorbing multiplication modules M with finite-dimensional top (see Theorem 4.5). We will see that the non-separated modules may be represented by certain amalgamation chains of separated indecomposable 2-absorbing multiplication modules (where infinite length 2-absorbing multiplication modules can occur only at the ends) and where adjacency corresponds to amalgamation in the socles of these separated 2-absorbing multiplication modules.

For the sake of completeness, we state some definitions and notations used throughout. Let R be the pullback ring as mentioned in the beginning of introduction. An R-module S is defined to be separated if there exist R_i-modules S_i, $i = 1, 2$, such that S is a submodule of $S_1 \oplus S_2$ (the latter is made into an R-module by setting $(r_1, r_2)(s_1, s_2) = (r_1 s_1, r_2 s_2)$). Equivalently, S is separated if it is a pullback of an R_1-module and an R_2-module and then, using the same notation for pullbacks of modules as for rings, $S = (S/P_2S \to S/PS \leftarrow S/P_1S)$ [20, Corollary 3.3] and $S \subseteq (S/P_2S) \oplus (S/P_1S)$. Also S is separated if and only if $P_1S \cap P_2S = 0$ [20, Lemma 2.9].

If R is a pullback ring, then every R-module is an epimorphic image of a separated R-module, indeed every R-module has a "minimal" such representation: a separated representation of an R-module M is an epimorphism $\varphi = (S \xrightarrow{f} S' \to M)$ of R-modules where S is separated and, if φ admits a factorization $\varphi : S \xrightarrow{f} S' \to M$ with S' separated, then f is one-to-one. The module $K = \text{Ker}(\varphi)$ is then an $\bar{R}$-module, since $\bar{R} = R/P$ and $PK = 0$ [20, Proposition 2.3]. An exact sequence $0 \to K \to S \to M \to 0$ of R-modules with S separated and K an $\bar{R}$-module is a separated representation of M if and only if $P_iS \cap K = 0$ for each i and $K \subseteq PS$ [20, Proposition 2.3]. Every module M has a separated representation, which is unique up to isomorphism [20, Theorem 2.8]. Moreover, R-homomorphisms lift to a separated representation, preserving epimorphisms and monomorphisms [20, Theorem 2.6].

Definition 1.1. (a) If R is a ring and N is a submodule of an R-module M, the ideal $\{r \in R : rM \subseteq N\}$ is denoted by $(N : M)$. Then $(0 : M)$ is the annihilator of M. A proper submodule N of a module M over a ring R is said to be a *prime submodule* if whenever $rm \in N$, for some $r \in R$, $m \in M$, then $m \in N$ or $r \in (N : M)$, so $(N : M) = P$ is a prime ideal of R, and N is said to be a *P-prime submodule*. The set of all prime submodules in an R-module M is denoted by $\text{Spec}(M)$ [23,24].

(b) An R-module M is defined to be a *multiplication module* if for each submodule N of M, $N = IM$, for some ideal I of R. In this case we can take $I = (N :_R M)$ [14].

(c) A proper submodule N of a module M is said to be *semiprime* if whenever $r^k m \in N$ for some $m \in M$, $r \in R$, and positive integer k, then $rm \in N$. The set of all semiprime submodules in an R-module M is denoted by $\text{seSpec}(M)$. An R-module M is defined to be a *semiprime multiplication module* if $\text{seSpec}(M) = \emptyset$ or for every semiprime submodule N of M, $N = IM$, for some ideal I of R [12].

(d) A proper submodule N of a module M is said to be a 2-*absorbing submodule* if whenever $a, b \in R$, $m \in M$ and $abm \in N$, then $am \in N$ or $bm \in N$ or $ab \in (N :_R M)$ [26,33]. The set of all 2-absorbing submodules in an R-module M is denoted by abSpec(M).

(e) A submodule N of an R-module M is called a *pure submodule* if any finite system of equations over N which is solvable in M is also solvable in N. A submodule N of an R-module M is called *relatively divisible* (or an *RD*-submodule) in M if $rN = N \cap rM$ for all $r \in R$ [27,31].

(f) A module M is *pure-injective* if it has the injective property relative to all pure exact sequences [27,31].

Remark 1.2. *(i) Let R be a Dedekind domain, M an R-module and N a submodule of M. Then N is pure in M if and only if $IN = N \cap IM$ for each ideal I of R. Moreover, N is pure in M if and only if N is an RD-submodule of M* [27,31].

(ii) Let N be an R-submodule of M. It is clear that N is an RD-submodule of M if and only if for all $m \in M$ and $r \in R$, $rm \in N$ implies that $rm = rn$ for some $n \in N$. Furthermore, if M is torsion-free, then N is an RD-submodule if and only if for all $m \in M$ and for all non-zero $r \in R$, $rm \in N$ implies that $m \in N$. In this case, N is an RD-submodule if and only if N is a prime submodule.

2. Basic properties of 2-absorbing multiplication modules

In this section, we give a complete description of the 2-absorbing multiplication modules over a discrete valuation domain. Our starting point is the following definition.

Definition 2.1. Let R be a commutative ring. An R-module M is defined to be a 2-*absorbing multiplication module* if abSpec$(M) = \emptyset$ or for every 2-absorbing submodule N of M, $N = IM$, for some ideal I of R.

One can easily show that if M is a 2-absorbing multiplication module, then $N = (N :_R M)M$ for every 2-absorbing submodule N of M. We need the following lemma proved in [33, Lemma 2.4] and [26, Lemmas 2.1, 2.2, and Theorem 2.3], respectively.

Lemma 2.2. (i) *Let $K \subseteq N$ be submodules of an R-module M. Then N is a 2-absorbing submodule of M if and only if N/K is a 2-absorbing submodule of M/K.*

(ii) *Let I be an ideal of R and N be a 2-absorbing submodule of M. If $a \in R$, $m \in M$ and $Iam \subseteq N$, then $am \in N$ or $Im \subseteq N$ or $Ia \subseteq (N : M)$.*

(iii) *Let I, J be ideals of R and N be a 2-absorbing submodule of M. If $m \in M$ and $IJm \subseteq N$, then $Im \in N$ or $Jm \subseteq N$ or $IJ \subseteq (N : M)$.*

(iv) *Let N be a proper submodule of M. Then N is a 2-absorbing submodule of M if and only if $IJK \subseteq N$ for some ideals I, J of R and a submodule K of M implies that $IK \subseteq N$ or $JK \subseteq N$ or $IJ \subseteq (N : M)$.*

Proposition 2.3. *Let M be a 2-absorbing multiplication module over a commutative ring R. Then the following hold:*

(i) *If I is an ideal of R and N a non-zero R-submodule of M with $I \subseteq (N : M)$, then M/N is a 2-absorbing multiplication R/I-module.*

(ii) *If N is a submodule of M, then M/N is a 2-absorbing multiplication R-module.*

(iii) *Every direct summand of M is a 2-absorbing multiplication submodule.*

(iv) *If I is an ideal of R with $I \subseteq (0 : M)$, then M is a 2-absorbing multiplication R-module if and only if M is 2-absorbing multiplication as an R/I-module.*

Proof. (i) Let K/N be a 2-absorbing submodule of M/N. Then by Lemma 2.1 (i), K is a 2-absorbing submodule of M, so $K = (K : M)M$, where $I \subseteq (N : M) \subseteq (K : M) = J$. An inspection will show that $K/N = (J/I)(M/N)$.

(ii) Take $I = 0$ in (i). (iii) Follows from (ii).

(iv) It is easy to see that N is a 2-absorbing R-submodule of M if and only if N is a 2-absorbing R/I-submodule of M. Now the assertion follows the fact that $(N :_R M) = (N :_{R/I} M)$. □

Remark 2.4. *(i) Let R and R' be any commutative rings, $g : R \to R'$ a surjective homomorphism and M an R'-module. It is clear that if N is a 2-absorbing R-submodule of M, then N is a 2-absorbing R'-submodule of M. Suppose that M is a 2-absorbing multiplication R'-module and let N be a 2-absorbing R-submodule of M. Then $N = JM$ for some ideal J of R'. It follows that $I = g^{-1}(J)$ is an ideal of R with $g(I) = J$. Then $IM = g(I)M = JM = N$. Thus M is a 2-absorbing multiplication R-module.*

(ii) Let M be a 2-absorbing multiplication module over an integral domain R (which is not a field), and let $T(M)$ be the torsion submodule of M with $T(M) \neq M$. Then $T(M)$ is a prime (so 2-absorbing) submodule M such that $(T(M) : M) = 0$ (see [24, Lemma 3.8]*); hence $T(M) = 0$. Thus M is either torsion or torsion-free.*

(iii) Let $R = M = \mathbb{Z}$ be the ring of integers. If $N = 4\mathbb{Z}$, then N is a 2-absorbing submodule of M, but it is not semiprime. So a 2-absorbing does not need to be semiprime. If $K = 30\mathbb{Z}$, then an inspection will show that K is a semiprime

submodule of M that it is not 2-absorbing. Hence a semiprime does not need to be 2-absorbing. So the class of semiprime multiplication and 2-absorbing multiplication modules are different concepts.

Proposition 2.5. *Let R be a discrete valuation domain with unique maximal ideal $P = Rp$. Then R, $E = E(R/P)$, the injective hull of R/P, $Q(R)$, the field of fractions of R, and R/P^n $(n \geq 1)$ are 2-absorbing multiplication modules.*

Proof. By [8, Lemma 2.6], every non-zero proper submodule L of E is of the form $L = A_n = (0 :_E P^n)$ $(n \geq 1)$, $L = A_n = Ra_n$ and $PA_{n+1} = A_n$. However no A_n is a 2-absorbing submodule of E, for if n is a positive integer then $P^3 A_{n+3} = A_n$, but $PA_{n+3} = A_{n+2} \nsubseteq A_n$, $P^2 A_{n+3} = A_{n+1} \nsubseteq A_n$ and $P^3 E = E \nsubseteq A_n$ (see Lemma 2.1). Now we conclude that $\text{abSpec}(E) = \emptyset$. Thus E is a 2-absorbing multiplication module.

Clearly, 0 is a 2-absorbing submodule of $Q(R)$. To show that 0 is the only 2-absorbing submodule of $Q(R)$, we assume the contrary and let N be a non-zero 2-absorbing submodule of $Q(R)$. Since N is a non-zero submodule, there exists a/b, where $a, b \in R$, so that $a/b \in N$. Clearly, $1/b \notin N$ (otherwise, $b/b = 1/1 \in N$ which is a contradiction). Now we have $a^2(1/ab) = a/b \in N$, but $a(1/ab) = 1/b \notin N$ and $a^2 Q(R) \nsubseteq N$. This contradicts the fact that N is a 2-absorbing submodule. Thus $\text{seSpec}(Q(R)) = \{(0)\}$ and hence $Q(R)$ is 2-absorbing multiplication. Finally, in the cases of R and R/P^n these follows because they are multiplication modules. □

Theorem 2.6. *Let R be a discrete valuation domain with a unique maximal ideal $P = Rp$. Then the class of indecomposable 2-absorbing multiplication modules over R, up to isomorphism, consists of the following:*

(i) *R;*
(ii) *R/P^n, $n \geq 1$, the indecomposable torsion modules;*
(iii) *$E(R/P)$, the injective hull of R/P;*
(iv) *$Q(R)$, the field of fractions of R.*

Proof. By [7, Proposition 1.3], these modules are indecomposable. Being 2-absorbing multiplication follows from Proposition 2.5. Now let M be an indecomposable 2-absorbing multiplication and choose any non-zero element $a \in M$. Let $h(a) = \sup\{n : a \in P^n M\}$ (so $h(a)$ is a nonnegative integer or ∞). Also let $(0 : a) = \{r \in R : ra = 0\}$: thus $(0 : a)$ is an ideal of the form P^m or 0. Because $(0 : a) = P^{m+1}$ implies that $p^m a \neq 0$ and $p.p^m a = 0$, we can choose a so that $(0 : a) = P$ or 0. Let $\text{abSpec}(M) = \emptyset$. Since $\text{Spec}(M) \subseteq \text{abSpec}(M)$, it follows from [23, Lemma 1.3, Proposition 1.4] that M is a torsion divisible R-module with $PM = M$ and

M is not finitely generated. We may assume that $(0 : a) = P$. By an argument like that in [8, Proposition 2.7 Case 2], $M \cong E(R/P)$. So we may assume that $\mathrm{abpSpec}(M) \neq \emptyset$.

If $h(a) = n$ and $(0 : a) = 0$, (resp. $h(a) = n$ and $(0 : a) = P$), then by a similar argument like that in [12, Theorem 3.8 Case 2] (resp. ([12, Theorem 3.8 Case 3] and [18, Theorem 5]), we get $M \cong R$ (resp. $M \cong R/P^{n+1}$). So we may assume that $h(a) = \infty$.

If $(0 : a) = P$, then by an argument like that in [8, Proposition 2.7 Case 2], we get $M \cong E(R/P)$; so $\mathrm{abSpec}(M) = \emptyset$ by Proposition 2.5, contrary to assumption. So we may assume that $h(a) = \infty$ and $(0 : a) = 0$. By an argument like that in [10, Theorem 2.12 Case 3], we get $M \cong Q(R)$. □

Theorem 2.7. *Let M be a 2-absorbing multiplication module over a discrete valuation domain with a maximal ideal $P = Rp$. Then M is of the form $M = N \oplus K$, where N is a direct sum of copies of R/P^n $(n \geq 1)$ and K is a direct sum of copies of $E(R/P)$ and $Q(R)$. In particular, every 2-absorbing multiplication R-module not isomorphic with R is pure-injective.*

Proof. Let T denote the indecomposable summand of M. Then by Proposition 2.2 (iii), T is an indecomposable 2-absorbing multiplication module. Now the assertion follows from Theorem 2.6 and [7, Proposition 1.3]. □

3. The separated case

Throughout this section we shall assume unless otherwise stated, that

$$R = (R_1 \xrightarrow{v_1} \bar{R} \xleftarrow{v_2} R_2) \tag{1}$$

is the pullback of two discrete valuation domains R_1, R_2 with maximal ideals P_1, P_2 generated respectively by p_1, p_2, P denotes $P_1 \oplus P_2$ and $R_1/P_1 \cong R_2/P_2 \cong R/P \cong \bar{R}$ is a field. In particular, R is a commutative Noetherian local ring with unique maximal ideal P. The other prime ideals of R are easily seen to be P_1 (that is $P_1 \oplus 0$) and P_2 (that is $0 \oplus P_2$). Let T be an R-submodule of a separated module $S = (S_1 \xrightarrow{f_1} \bar{S} \xleftarrow{f_2} S_2)$, with projection maps $\pi_i : S \twoheadrightarrow S_i$. Set $T_1 = \{t_1 \in S_1 : (t_1, t_2) \in T \text{ for some } t_2 \in S_2\}$ and $T_2 = \{t_2 \in S_2 : (t_1, t_2) \in T \text{ for some } t_1 \in S_1\}$. Then for each i, $i = 1, 2$, T_i is an R_i-submodule of S_i and $T \leq T_1 \oplus T_2$. Moreover, we can define a mapping $\pi'_1 = \pi_1|T : T \twoheadrightarrow T_1$ by sending (t_1, t_2) to t_1; hence $T_1 \cong T/(0 \oplus Ker(f_2) \cap T) \cong T/(T \cap P_2S) \cong (T + P_2S)/P_2S \subseteq S/P_2S$. So we may assume that T_1 is a submodule of S_1. Similarly, we may assume that T_2 is a submodule of S_2 (note that $\mathrm{Ker}(f_1) = P_1S_1$ and $\mathrm{Ker}(f_2) = P_2S_2$).

Proposition 3.1. *Let $S = (S/P_2S = S_1 \xrightarrow{f_1} \bar{S} = S/PS \xleftarrow{f_2} S_2 = S/P_1S)$ be any separated module over the pullback ring as in (1).*

(i) *If T is a 2-absorbing submodule of S, then T_1 is a 2-absorbing submodule S_1 and T_2 is a 2-absorbing submodule S_2.*

(ii) $\mathrm{abSpec}(S) = \emptyset$ *if and only if* $\mathrm{abSpec}(S_i) = \emptyset$ *for $i = 1, 2$.*

Proof. (i) Let $abs_1 \in T_1$ for some $a, b \in R_1$ and $s_1 \in S_1$. If $a \notin P_1$, then $bs_1 \in T_1$ since a is invertible, and so we are done. Similarly, if $b \notin P_1$, then $as_1 \in T_1$. So we may assume that $a, b \in P_1$. Then $v_1(ab) = v_2(0) = 0$; hence $(ab, 0) \in R$. By assumption, $(s_1, s_2) \in S$ for some $s_2 \in S_2$. Since $abs_1 \in T_1 \cap P_1S$, $0 \in T_2 \cap P_2S$ and $f_1(abs_1) = f_2(0)$, we get $(abs_1, 0) = (a, 0)(b, 0)(s_1, s_2) \in T$. Now T is a 2-absorbing submodule gives $(as_1, 0) \in T$ or $(bs_1, 0) \in T$ or $(ab, 0) \in (T :_R S) = (T_1 :_{R_1} S_1) \times (T_2 :_{R_2} S_2)$ which implies that $as_1 \in T_1$ or $bs_1 \in T_1$ or $ab \in (T_1 :_{R_1} S_1)$. Thus T_1 is a 2-absorbing submodule S_1. Similarly, T_2 is a 2-absorbing submodule S_2.

(ii) Assume that $\mathrm{abSpec}(S) = \emptyset$ and let π be the projection map of R onto R_1. Suppose that $\mathrm{abSpec}(S_1) \neq \emptyset$ and let T_1 be a 2-absorbing submodule of S_1, so T_1 is a 2-absorbing R-submodule of $S_1 = S/(0 \oplus P_2)S$; hence $\mathrm{abSpec}(S) \neq \emptyset$ by Lemma 2.2 (i), which is a contradiction. Similarly, $\mathrm{abSpec}(S_2) = \emptyset$. The other implication is clear by (i). □

Theorem 3.2. *Let $S = (S/P_2S = S_1 \xrightarrow{f_1} \bar{S} = S/PS \xleftarrow{f_2} S_2 = S/P_1S)$ be any separated module over the pullback ring as (1). Then S is a 2-absorbing multiplication R-module if and only if each S_i is a 2-absorbing multiplication R_i-module, $i = 1, 2$.*

Proof. By Proposition 3.1 (ii), $\mathrm{abSpec}(S) = \emptyset$ if and only if $\mathrm{abSpec}(S_i) = \emptyset$ for $i = 1, 2$. So we may assume that $\mathrm{abSpec}(S) \neq \emptyset$. Assume that S is a separated 2-absorbing multiplication R-module. If $\bar{S} = 0$, then by [7, Lemma 2.7], $S = S_1 \oplus S_2$; hence for each i, S_i is 2-absorbing multiplication by Proposition 2.3 (iii). So we may assume that $\bar{S} \neq 0$. Since $(0 \oplus P_2) \subseteq ((0 \oplus P_2)S : S)$, Proposition 2.3 (i) gives $S_1 \cong S/(0 \oplus P_2)S$ is a 2-absorbing multiplication $R/(0 \oplus P_2) \cong R_1$-module. Similarly, S_2 is a 2-absorbing multiplication R_2-module.

Conversely, assume that each S_i is a 2-absorbing multiplication R_i-module and let $T = (T_1 \to \bar{T} \leftarrow T_2)$ be a 2-absorbing submodule of S. We may assume that $(T : S) \neq 0$. If $(T : S) = P_1^n \oplus P_2^m$ for some positive integers m, n, then $S_i \neq 0$ for $i = 1, 2$, $(T_1 :_{R_1} S_1) = P_1^n$, and $(T_2 :_{R_2} S_2) = P_2^m$ by [12, Proposition 4.2 (i)]. Now by Proposition 3.1 (i), $T_1 = P_1^n S_1 \subseteq P_1 S_1$ since S_1 is 2-absorbing multiplication. Similarly, $T_2 = P_2^m S_2 \subseteq P_2 S_2$. If $k = \min\{m, n\}$, then by an argument like that in [12, Proposition 4.5 Case 1], we get $T = P^k S$, and so S is 2-absorbing multiplication. If $(T : S) = P_1^n \oplus 0$ for some positive integer n, then T_2 is a 2-absorbing R_2-submodule of S_2 with $(T_2 :_{R_2} S_2) = 0$; so $T_2 = 0$. Similarly, $T_1 = P_1^n S_1$. It follows that $T \subseteq T_1 \oplus T_2 = (P_1^n \oplus 0)S$. For the other inclusion, assume that $t = (p_1^n, 0)(s_1, s_2) = (p_1^n s_1, 0) \in (P_1^n \oplus 0)S$. Then $t \in T$ since $p_1^n s_1 \in T_1$ and $f_1(p_1^n s_1) = 0 = f_2(0)$ (note that $\mathrm{Ker}(f_1) = P_1 S_1$ and $\mathrm{Ker}(f_2) = P_2 S_2$); hence $T = (P_1^n \oplus 0)S$. Similarly, if $(T : S) = 0 \oplus P_2^m$ for some positive integer m, then we get $T = (0 \oplus P_2^m)S$. Thus S is a 2-absorbing multiplication R-module. □

Lemma 3.3. *Let R be the pullback ring as in (1). Then, up to isomorphism, the following separated R-modules are indecomposable and 2-absorbing multiplication:*

(i) *R;*

(ii) *$S = (E(R_1/P_1) \to 0 \leftarrow 0)$, $(0 \to 0 \leftarrow E(R_2/P_2))$, where $E(R_i/P_i)$ is the R_i-injective hull of R_i/P_i for $i = 1, 2$;*

(iii) *$S = (Q(R_1) \to 0 \leftarrow 0)$, $(0 \to 0 \leftarrow Q(R_2))$, where $Q(R_i)$ is the field of fractions of R_i for $i = 1, 2$;*

(iv) *$S = (R_1/P_1^n \to \bar{R} \leftarrow R_2/P_2^m)$ for all positive integers n, m.*

Proof. By [7, Lemma 2.8], these modules are indecomposable. Being 2-absorbing multiplication follows from Theorem 2.6 and Theorem 3.2. □

For each i, let E_i be the R_i-injective hull of R_i/P_i, regarded as an R-module, so E_1, E_2 are the modules listed under (ii) in Lemma 3.3. We refer to modules of type (ii) in Lemma 3.3 as P_1-Prüfer and P_2-Prüfer, respectively.

Proposition 3.4. *Let R be the pullback ring as in (1), and let $S \neq R$ be a separated 2-absorbing multiplication R module. Then the following hold:*

(i) *S is of the form $S = M \oplus N$, where M is a direct sum of copies of the modules as in (iv), N is a direct sum of copies of the modules as in (ii)-(iii) of Lemma 3.3.*

(ii) *Every separated 2-absorbing multiplication R-module not isomorphic with R is pure-injective.*

Proof. (i) Let T denote an indecomposable summand of S. Then we can write $T = (T_1 \to \bar{T} \leftarrow T_2)$, and T is a 2-absorbing multiplication R-module by Proposition 2.2 (iii). First suppose that $\bar{T} = 0$. Then by [7, Lemma 2.7 (i)], $T = T_1$ or T_2 and so T is an indecomposable 2-absorbing multiplication R_i-module for some i and, since $T = PT$, is type (ii) or (iii) in the list Lemma 3.3. So we may assume that $\bar{T} \neq 0$.

By Theorem 2.6 and Theorem 3.2, T_i is an indecomposable 2-absorbing multiplication R_i-module, for each $i = 1, 2$. Hence, by the structure of 2-absorbing multiplication modules over a discrete valuation domain (see Theorem 2.7), we must have $T_i = E(R_i/P_i)$ or $Q(R_i)$ or R_i/P_i^n ($n \geq 1$). Since $T \neq PT$ it follows that for each $i = 1, 2$, T_i is torsion and it is not divisible R_i-module. Then there are positive integers m, n and k such that $P_1^m T_1 = 0$, $P_2^k T_2 = 0$ and $P^n T = 0$. For $t \in T$, let $o(t)$ denote the least positive integer m such that $P^m t = 0$. Now choose $t \in T_1 \cup T_2$ with $\bar{t} \neq 0$ and such that $o(t)$ is maximal (given that $\bar{t} \neq 0$). There exists a $t = (t_1, t_2)$ such that $o(t) = n$, $o(t_1) = m$ and $o(t_2) = k$. Then for each $i = 1, 2$, $R_i t_i$ is pure in T_i (see [7, Theorem 2.9]). Thus, $R_1 t_1 \cong R_1/(0 : t_1) \cong R_1/P_1^m$ is a direct summand of T_1 since $R_1 t_1$ is pure-injective; hence $T_1 = R_1 t_1$ since T_1 is indecomposable. Similarly, $T_2 = R_2 t_2 \cong R_2/P_2^k$. Let $\bar{M}$ be the $\bar{R}$-subspace of $\bar{T}$ generated by $\bar{t}$. Then $\bar{M} \cong \bar{R}$. Let $M = (R_1 t_1 \to \bar{M} \leftarrow R_2 t_2)$. Then $T = M$, and T satisfies the case (iv) (see [7, Theorem 2.9]).

(ii) Apply (i) and [7, Theorem 2.9]. □

Theorem 3.5. *Let $S \neq R$ be an indecomposable separated 2-absorbing multiplication module over the pullback ring as in (1). Then S is isomorphic to one of the modules listed in Lemma 3.3.*

Proof. Apply Proposition 3.4 and Lemma 3.3. □

4. The nonseparated case

We continue to use the notation already established, so R is the pullback ring as in (1). In this section we find the indecomposable non-separated 2-absorbing multiplication modules with finite-dimensional top. It turns out that each can be obtained by amalgamating finitely many separated indecomposable 2-absorbing multiplication modules.

Proposition 4.1. *Let R be a pullback ring as in (1).*

(i) *$E(R/P)$ is a non-separated 2-absorbing multiplication R-module.*

(ii) *If $0 \to K \to S \to M \to 0$ is a separated representation of an R-module M, then $\mathrm{abSpec}_R(S) = \emptyset$ if and only if $\mathrm{abSpec}_R(M) = \emptyset$.*

Proof. (i) It is enough to show that $\mathrm{abpSpec}(E(R/P)) = \emptyset$. Assume that L is any submodule of $E(R/P)$ described in [13, Proposition 3.1 (iii)]. However no L, say $E_1 + A_n$, is a 2-absorbing submodule of $E(R/P)$, for if n is any positive integer, then $P^3(E_1 + A_{n+3}) = E_1 + A_n$, but $P(E_1 + A_{n+3}) = E_1 + A_{n+2} \nsubseteq E_1 + A_n$

and $P^3E(R/P) = E(R/P) \nsubseteq E_1 + A_n$ (see Lemma 2.2). Therefore, $E(R/P)$ is a non-separated 2-absorbing multiplication R-module (see [7, p. 4053]).

(ii) Assume that $\text{abSpec}_R(S) = \emptyset$ and let $\text{abSpec}_R(M) \neq \emptyset$. Then there exists a submodule T/K of $M \cong S/K$ such that $T/K \in \text{abSpec}_R(M)$; so $T \in \text{abSpec}_R(S)$ by Lemma 2.2 (i) which is a contradiction. Therefore $\text{abSpec}_R(M) = \emptyset$. For the other implication, suppose that $\text{abSpec}_R(M) = \emptyset$, and let $\text{abSpec}_R(S) \neq \emptyset$. So S has a 2-absorbing submodule T with $K \subseteq T$ by [11, Proposition 4.3 (ii)]; hence T/K is a 2-absorbing submodule of M by Lemma 2.2 (i) which is a contradiction. Thus $\text{abSpec}_R(S) = \emptyset$. □

Theorem 4.2. *Let R be a pullback ring as in (1) and let M be any non-separated R-module. Let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then S is 2-absorbing multiplication if and only if M is 2-absorbing multiplication.*

Proof. By Proposition 4.1 (ii), we may assume that $\text{abSpec}(S) \neq \emptyset$. Suppose that M is a 2-absorbing multiplication R-module and let U be a non-zero 2-absorbing submodule of S. Then by [11, Proposition 4.3], $K \subseteq U$, and so U/K is a 2-absorbing submodule of $S \cong M/\text{K}$ by Lemma 2.2 (i). By an argument like that in [12, Theorem 5.3], we get S is 2-absorbing multiplication. Conversely, assume that S is a 2-absorbing multiplication R-module. Then $S \cong M/\text{K}$ is 2-absorbing multiplication by Proposition 2.3 (ii), as required. □

Proposition 4.3. *Let R be a pullback ring as in (1) and let M be an indecomposable 2-absorbing multiplication non-separated R-module with finite-dimensional top over $\bar{R}$. Let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then the following hold:*

(i) *S is pure-injective.*

(ii) *R do not occur among the direct summands of S.*

Proof. (i) Since $S/PS \cong M/PM$ by [7, Proposition 2.6 (i)], we get S has finite-dimensional top. Now the assertion follows from Theorem 4.2 and Proposition 3.4. (ii) follows from [12, Lemma 5.5]. □

Let R be a pullback ring as in (1) and let M be an indecomposable 2-absorbing multiplication non-separated R-module with finite-dimensional top over $\bar{R}$. Consider the separated representation $0 \to K \to S \to M \to 0$. By Proposition 4.3, S is pure-injective. So in the proofs of [7, Lemma 3.1, Propositions 3.2 and 3.4] (here the pure-injectivity of M implies the pure-injectivity of S by [7, Proposition 2.6 (ii)] we can replace the statement "M is an indecomposable pure-injective non-separated

R-module" by "M is an indecomposable 2-absorbing multiplication non-separated R-module": because the main key in those results are the pure-injectivity of S, the indecomposability and the non-separability of M. So we have the following result:

Corollary 4.4. *Let R be a pullback ring as in (1) and let M be an indecomposable 2-absorbing multiplication non-separated R-module with M/PM finite-dimensional over $\bar{R}$, and let $0 \to K \to S \to M \to 0$ be a separated representation of M. Then the following hold:*

- (i) *the quotient fields $Q(R_1)$ and $Q(R_2)$ of R_1 and R_2 do not occur among the direct summands of S.*
- (ii) *S is a direct sum of finitely many indecomposable 2-absorbing multiplication modules.*
- (iii) *At most two copies of modules of infinite length can occur among the indecomposable summands of S.*

Recall that every indecomposable R-module of finite length is 2-absorbing multiplication since it is a quotient of a 2-absorbing multiplication R-module (see Proposition 2.2 (ii)). So by Corollary 4.4 (iii), the infinite length non-separated indecomposable 2-absorbing multiplication modules are obtained in just the same way as the deleted cycle type indecomposable ones are, except that at least one of the two "end" modules must be a separated indecomposable 2-absorbing multiplication of infinite length (that is, P_1-Prüfer and P_2-Prüfer). Note that one can not have, for instance, a P_1-Prüfer module at each end (consider the alternation of primes P_1, P_2 along the amalgamation chain). So, apart form any finite length modules: we have amalgamations involving two Prüfer modules as well as modules of finite length (the injective hull $E(R/P)$ is the simplest module of this type), a P_1-Prüfer module and a P_2-Prüfer module. If the P_1-Prüfer and the P_2-Prüfer are direct summands of S then we will describe these modules as **doubly infinite**. Those where S has just one infinite length summand we will call **singly infinite** (the reader is referred to [7] for more details). It remains to show that the modules obtained by these amalgamations are, indeed, indecomposable 2-absorbing multiplication modules.

Theorem 4.5. *Let $R = (R_1 \to \bar{R} \leftarrow R_2)$ be the pullback of two discrete valuation domains R_1, R_2 with common factor field $\bar{R}$. Then the class of indecomposable non-separated 2-absorbing multiplication modules with finite-dimensional top consists of the following:*

- (i) *The indecomposable modules of finite length (apart from R/P which is separated);*

(ii) *The doubly infinite* 2*-absorbing multiplication modules;*

(iii) *The singly infinite* 2*-absorbing multiplication modules (except the two prüfer modules (ii) in Lemma 3.3).*

Proof. We know already that every indecomposable 2-absorbing multiplication non-separated module has one of these forms so it remains to show that the modules obtained by these amalgamation are, indeed, indecomposable 2-absorbing multiplication modules. Let M be an indecomposable non-separated 2-absorbing multiplication R-module with finite-dimensional top and let $0 \to K \xrightarrow{i} S \xrightarrow{\varphi} M \to 0$ be a separated representation of M.

(i) Every indecomposable R-module of finite length is 2-absorbing multiplication since it is a quotient of a 2-absorbing multiplication R-module (see Proposition 2.3 (ii)). The indecomposability follows from [21, 1.9].

(ii) and (iii) (involving one or two Prüfer modules) M is 2-absorbing multiplication since they are a quotient of a 2-absorbing multiplication R-module (also see Proposition 4.1 (i)). Finally, the indecomposability follows from [7, Theorem 3.5]. □

Remark 4.6. *(i) Let R be the pullback ring as described in Theorem 4.5. Then by* [7, Theorem 3.5] *and Theorem 4.5, every indecomposable* 2*-absorbing multiplication R-module with finite-dimensional top is pure-injective.*

(ii) For a given field k, the infinite-dimensional k-algebra $k[x, y : xy = 0]_{(x,y)}$ *is the pullback* $(k[x]_{(x)} \to k \leftarrow k[y]_{(y)})$ *of two discrete valuation domains* $k[x]_{(x)}, k[y]_{(y)}$ *(see* [1, Section 6]*). This paper includes the classification of those indecomposable* 2*-absorbing multiplication modules over k-algebra* $k[x, y : xy = 0]_{(x,y)}$ *which have finite-dimensional top.*

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