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# GENERALIZED TOPOLOGICAL OPERATOR THEORY IN GENERALIZED TOPOLOGICAL SPACES

## *PART I. GENERALIZED INTERIOR AND GENERALIZED CLOSURE*

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Abstract. In a generalized topological space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ ($\mathscr{T}_{\mathfrak{g}}$-*space*), various ordinary topological operators ($\mathfrak{T}_{\mathfrak{g}}$-*operators*), namely, $\mathrm{int}_{\mathfrak{g}}$, $\mathrm{cl}_{\mathfrak{g}}$, $\mathrm{ext}_{\mathfrak{g}}$, $\mathrm{fr}_{\mathfrak{g}}$, $\mathrm{der}_{\mathfrak{g}}$, $\mathrm{cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ ($\mathfrak{T}_{\mathfrak{g}}$-*interior*, $\mathfrak{T}_{\mathfrak{g}}$-*closure*, $\mathfrak{T}_{\mathfrak{g}}$-*exterior*, $\mathfrak{T}_{\mathfrak{g}}$-*frontier*, $\mathfrak{T}_{\mathfrak{g}}$-*derived*, $\mathfrak{T}_{\mathfrak{g}}$-*coderived operators*), are defined in terms of ordinary sets ($\mathfrak{T}_{\mathfrak{g}}$-*sets*). Accordingly, generalized $\mathfrak{T}_{\mathfrak{g}}$-operators ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*operators*), namely, $\mathfrak{g}$-$\mathrm{Int}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathrm{Cl}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathrm{Ext}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathrm{Fr}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathrm{Der}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathrm{Cod}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*interior*, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*closure*, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*exterior*, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*frontier*, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*derived*, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*coderived operators*) may be defined in terms of generalized $\mathfrak{T}_{\mathfrak{g}}$-sets ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-*sets*), thereby making $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators theory in $\mathscr{T}_{\mathfrak{g}}$-spaces an interesting subject of inquiry. In this paper, we introduce the definitions and study the essential properties of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $\mathfrak{g}$-$\mathrm{Int}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathrm{Cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$, respectively, in terms of a new class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets which we studied earlier. The major findings to which the study has led to are: Firstly, $(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}},\ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}):\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$ is $(\Omega,\emptyset)$-*grounded*, (*expansive, non-expansive*), (*idempotent, idempotent*) and $(\cap,\cup)$-*additive*. Secondly, $\mathfrak{g}$-$\mathrm{Int}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathrm{int}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ and $\mathfrak{g}$-$\mathrm{Cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathrm{cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$. The elements supporting these facts are reported therein as sources of inspiration for more generalized operations.

## 1. Introduction

Just as the concepts of $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-interior operators in $\mathscr{T}$-spaces (ordinary and generalized interior operators in ordinary topological spaces) and $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-closure operators in $\mathscr{T}$-spaces (ordinary and generalized closure operators in ordinary topological spaces) are essential operators in the study of $\mathfrak{T}$-sets in $\mathscr{T}$-spaces (arbitrary sets in ordinary topological spaces) [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12], so are the concepts

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of $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operators in $\mathscr{T}_{\mathfrak{g}}$-spaces (ordinary and generalized interior operators in generalized topological spaces) and $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}_{\mathfrak{g}}$-spaces (ordinary and generalized closure operators in generalized topological spaces) essential operators in the study of $\mathfrak{T}_{\mathfrak{g}}$-sets in $\mathscr{T}_{\mathfrak{g}}$-spaces (arbitrary sets in generalized topological spaces) [13, 14, 15, 16, 17, 18, 19].

Intuitively, $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-interior operators, respectively, in a $\mathscr{T}$-space can be characterized as one-valued maps int, $\mathfrak{g}\text{-Int} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ from the power set $\mathscr{P}(\Omega)$ of $\Omega$ into itself, assigning to each $\mathfrak{T}$-set in the $\mathscr{T}$-space the $\cup$-operation (union operation) of all $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-open subsets of the $\mathfrak{T}$-set [20, 21, 22, 23]. When the role of $\cup$-operation and $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-open subsets, respectively, are given to $\cap$-operation (intersection operation) and $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-closed supersets of the $\mathfrak{T}$-set, the dual notions, called $\mathfrak{T}$, $\mathfrak{g}$-$\mathfrak{T}$-closure operators in the $\mathscr{T}$-space follow [21, 23, 24, 25, 26], which can likewise be characterized as one-valued maps cl, $\mathfrak{g}\text{-Cl} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$. Finally, when $(\mathscr{T}, \mathfrak{T}, \mathfrak{g}\text{-}\mathfrak{T}) \longmapsto (\mathscr{T}_{\mathfrak{g}}, \mathfrak{T}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}})$, the notions of $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in a $\mathscr{T}_{\mathfrak{g}}$-space follow [15, 16, 27, 28, 29, 30, 31], which can in a similar manner be characterized as one-valued maps of the types $\text{int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and $\text{cl}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively.

Thus, in a $\mathscr{T}$-space, int, $\mathfrak{g}\text{-Int} : \mathscr{S} \longmapsto \text{int}(\mathscr{S})$, $\mathfrak{g}\text{-Int}(\mathscr{S})$ describe two types of collections of points interior in $\mathscr{S}$ and, cl, $\mathfrak{g}\text{-Cl} : \mathscr{S} \longmapsto \text{cl}(\mathscr{S})$, $\mathfrak{g}\text{-Cl}(\mathscr{S})$ describe another two types of collections of points but close to $\mathscr{S}$. Similarly, in a $\mathscr{T}_{\mathfrak{g}}$-space, $\text{int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \text{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ describe two types of collections of points interior in $\mathscr{S}_{\mathfrak{g}}$ and, $\text{cl}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \text{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ describe another two types of collections of points but close to $\mathscr{S}_{\mathfrak{g}}$. Of all such operators int, cl, $\mathfrak{g}$-Int, $\mathfrak{g}\text{-Cl} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in $\mathscr{T}$-spaces and $\text{int}_{\mathfrak{g}}$, $\text{cl}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ in $\mathscr{T}_{\mathfrak{g}}$-spaces, int, $\text{cl} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are the oldest and $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ are the newest. Hence, the studies of operators of these kinds have evolved from the studies of ordinary operators in ordinary topological spaces to the studies of generalized operators in generalized topological spaces.

In the literature of $\mathscr{T}_{\mathfrak{g}}$-spaces on $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators, some new types of one-valued maps $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ have been defined and investigated by Mathematicians.

Based on $\theta$-sets in $\mathscr{T}_{\mathfrak{g}}$-spaces, Min, W. K. [32, 33, 28] has introduced the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $i_\theta$, $c_\theta : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, and used them to study some properties of $\theta(g, g')$-continuity in $\mathscr{T}_{\mathfrak{g}}$-spaces. Cao, Yan, Wang and Wang [34] have introduced and then used the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $i_\lambda$, $c_\lambda : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ ($\lambda$-*interior* and $\lambda$-*closure operators*), respectively, where $\lambda \in \{\alpha, \beta, \sigma, \pi\}$ in $\mathscr{T}$-spaces. Saravanakumar, Kalaivani and Krishnan [30] have studied the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $i_{\tilde{\mu}}$, $c_{\tilde{\mu}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ ($\tilde{\mu}$-*interior* and $\tilde{\mu}$-*closure operators*), respectively, in terms of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-sets ($\tilde{\mu}$-*open sets*) in $\mathscr{T}_{\mathfrak{g}}$-spaces. Srija and Jayanthi [35] have introduced the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $\text{si}_{\text{g}}$, $\text{sc}_{\text{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ (g-*semi interior* and g-*semi closure operators*), respectively. Boonpok, C. [36] has introduced the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $i_{\delta(\mu)}$, $c_{\delta(\mu)} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ ($\delta(\mu)$-*interior* and $\delta(\mu)$-*closure operators*), respectively, and utilized them to study the properties of $\zeta_{\delta(\mu)}$, $(\zeta, \delta(\mu))$-closed sets in strong $\mathscr{T}_{\mathfrak{g}}$-spaces. Later on, in extending the notion of $\mu$-$\hat{\beta}g$-closed set introduced by Kannan and Nagaveni [37] in $\mathscr{T}$-spaces to $\mathscr{T}_{\mathfrak{g}}$-spaces and then studying their properties, Camargo, J. F. Z. [27] has also investigated the related $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $\hat{\beta}gi_\mu$,

$\hat{\beta}gc_{\mu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ ($\mu$-$\hat{\beta}g$-*interior* and $\mu$-$\hat{\beta}g$-*closure operators*), respectively. Relative to the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators introduced by Császár, A. [3, 38], the author found that the image of a $\mathfrak{T}_{\mathfrak{g}}$-set under $\hat{\beta}gi_{\mu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is a superset of that under $i_{\mu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ and, the image of the $\mathfrak{T}_{\mathfrak{g}}$-set under $\hat{\beta}gc_{\mu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is a subset of its image under $c_{\mu} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$.

In this paper, the essential properties of a new class of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}_{\mathfrak{g}}$-spaces are presented.

The rest of the paper is structured as: In Section 2, necessary and sufficient preliminary notions are described and the main results are reported in Section 3. In Section 4, various relationships between these $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators are discussed and an application of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in a $\mathscr{T}_{\mathfrak{g}}$-space is presented. Finally, the work is concluded in Section 5.

## 2. Theory

2.1. **Necessary Preliminaries.** The standard reference for notations and concepts is the Ph.D. Thesis of Khodabocus M. I. [16].

Throughout, $\mathfrak{U}$ is the *universe* of discourse, fixed within the framework of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operator theory in $\mathscr{T}_{\mathfrak{g}}$-spaces; $I_n^0,\ I_n^* \subset \mathbb{Z}_+^0$ and $I_\infty^0,\ I_\infty^* \subset \mathbb{Z}_+^0$ are index sets including and excluding 0 [15, 16]. To abstract definitions of concepts, let $\mathfrak{a} \in \{\mathfrak{o}, \mathfrak{g}\}$.

**Definition 2.1** ($\mathscr{T}_{\mathfrak{a}}$-Space [15, 16])**.** *A topological structure* $\mathfrak{T}_{\mathfrak{a}} \overset{\text{def}}{=} (\Omega, \mathscr{T}_{\mathfrak{a}})$*, consisting of an underlying set* $\Omega \subset \mathfrak{U}$ *and an* $\mathfrak{a}$*-topology* $\begin{array}{rccc} \mathscr{T}_{\mathfrak{a}}: & \mathscr{P}(\Omega) & \longrightarrow & \mathscr{P}(\Omega) \\ & \mathscr{O}_{\mathfrak{a}} & \longmapsto & \mathscr{T}_{\mathfrak{a}}(\mathscr{O}_{\mathfrak{a}}) \end{array}$ *satisfying the compound* $\mathscr{T}_{\mathfrak{a}}$*-axiom:*

$$\mathrm{Ax}(\mathscr{T}_{\mathfrak{a}}) \overset{\text{def}}{\Longleftrightarrow} \begin{cases} \big(\mathscr{T}_{\mathfrak{o}}(\emptyset) = \emptyset\big) \wedge \big(\mathscr{T}_{\mathfrak{o}}(\mathscr{O}_{\mathfrak{o},\nu}) \subseteq \mathscr{O}_{\mathfrak{o},\nu}\big) \\ \qquad \wedge\big(\mathscr{T}_{\mathfrak{o}}\big(\bigcap_{\nu \in I_n^*} \mathscr{O}_{\mathfrak{o},\nu}\big) = \bigcap_{\nu \in I_n^*} \mathscr{T}_{\mathfrak{o}}(\mathscr{O}_{\mathfrak{o},\nu})\big) \\ \qquad \wedge\big(\mathscr{T}_{\mathfrak{o}}\big(\bigcup_{\nu \in I_\infty^*} \mathscr{O}_{\mathfrak{o},\nu}\big) = \bigcup_{\nu \in I_\infty^*} \mathscr{T}_{\mathfrak{o}}(\mathscr{O}_{\mathfrak{o},\nu})\big) \quad (\mathfrak{a} = \mathfrak{o}), \\ \\ \big(\mathscr{T}_{\mathfrak{g}}(\emptyset) = \emptyset\big) \wedge \big(\mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu}) \subseteq \mathscr{O}_{\mathfrak{g},\nu}\big) \\ \qquad \wedge\big(\mathscr{T}_{\mathfrak{g}}\big(\bigcup_{\nu \in I_\infty^*} \mathscr{O}_{\mathfrak{g},\nu}\big) = \bigcup_{\nu \in I_\infty^*} \mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})\big) \quad (\mathfrak{a} = \mathfrak{g}), \end{cases}$$

*is called a* $\mathscr{T}_{\mathfrak{a}}$*-space.*

On $\mathscr{T}_{\mathfrak{a}}$-spaces, neither ordinary nor generalized separation axioms are assumed unless otherwise stated. If $\mathfrak{a} = \mathfrak{o}$ (*ordinary*), then $\mathrm{Ax}(\mathscr{T}_{\mathfrak{o}})$ stands for an ordinary topology and if $\mathfrak{a} = \mathfrak{g}$ (*generalized*), then $\mathrm{Ax}(\mathscr{T}_{\mathfrak{g}})$ stands for a generalized topology. Accordingly, $\mathfrak{T} = (\Omega, \mathscr{T}) = (\Omega, \mathscr{T}_{\mathfrak{o}}) = \mathfrak{T}_{\mathfrak{o}} \neq \mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. If $\Omega \in \mathscr{T}_{\mathfrak{g}}$, then $\mathfrak{T}_{\mathfrak{g}}$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space [3, 39] and if $\mathscr{T}_{\mathfrak{g}}\big(\bigcap_{\nu \in I_n^*} \mathscr{O}_{\mathfrak{g},\nu}\big) = \bigcap_{\nu \in I_n^*} \mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})$ for any $I_n^* \subset I_\infty^*$, then $\mathfrak{T}_{\mathfrak{g}}$ is a quasi $\mathscr{T}_{\mathfrak{g}}$-space [40].

Typically, $\big(\Gamma, \{\mathscr{O}_{\mathfrak{a}}\}, \mathscr{S}_{\mathfrak{a}}\big) \subset \Omega \times \mathscr{T}_{\mathfrak{a}} \times \mathfrak{T}_{\mathfrak{a}}$ denotes a triple of a $\Omega$-subset, a unit set containing a $\mathscr{T}_{\mathfrak{a}}$-open set and a $\mathfrak{T}_{\mathfrak{a}}$-set. By $\complement_{\Omega}(\mathscr{O}_{\mathfrak{a}}) = \mathscr{K}_{\mathfrak{a}} \in \neg\mathscr{T}_{\mathfrak{a}} \overset{\text{def}}{=} \big\{\mathscr{K}_{\mathfrak{a}} : \complement(\mathscr{K}_{\mathfrak{a}}) \in \mathscr{T}_{\mathfrak{a}}\big\}$ is meant a $\mathscr{T}_{\mathfrak{a}}$-closed set. On the other hand, the operators $\begin{array}{rcl} \mathrm{int}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}}: & \mathscr{P}(\Omega) & \longrightarrow \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto \mathrm{int}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}),\ \mathrm{cl}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}) \end{array}$ are called $\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{T}_{\mathfrak{a}}$-closure operators, respectively. Accordingly,

$$\mathrm{int}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}) \overset{\text{def}}{=} \bigcup_{\mathscr{O}_{\mathfrak{a}} \in \mathrm{C}^{\mathrm{sub}}_{\mathscr{T}_{\mathfrak{a}}}[\mathscr{S}_{\mathfrak{a}}]} \mathscr{O}_{\mathfrak{a}}, \quad \mathrm{cl}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}) \overset{\text{def}}{=} \bigcap_{\mathscr{K}_{\mathfrak{a}} \in \mathrm{C}^{\mathrm{sup}}_{\neg\mathscr{T}_{\mathfrak{a}}}[\mathscr{S}_{\mathfrak{a}}]} \mathscr{K}_{\mathfrak{a}}, \tag{2.1}$$

where $\mathrm{C}^{\mathrm{sub}}_{\mathscr{T}_{\mathfrak{a}}}[\mathscr{S}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathscr{O}_{\mathfrak{a}} \in \mathscr{T}_{\mathfrak{a}} : \ \mathscr{O}_{\mathfrak{a}} \subseteq \mathscr{S}_{\mathfrak{a}}\}$ and $\mathrm{C}^{\mathrm{sup}}_{\neg\mathscr{T}_{\mathfrak{a}}}[\mathscr{S}_{\mathfrak{a}}] \stackrel{\mathrm{def}}{=} \{\mathscr{K}_{\mathfrak{a}} \in \neg\mathscr{T}_{\mathfrak{a}} : \mathscr{K}_{\mathfrak{a}} \supseteq \mathscr{S}_{\mathfrak{a}}\}$. In general, $(\mathrm{int}_{\mathfrak{g}}, \mathrm{cl}_{\mathfrak{g}}) \neq (\mathrm{int}_{\mathfrak{o}}, \mathrm{cl}_{\mathfrak{o}})$ [41]. Set $\mathscr{P}^{*}(\Omega) = \mathscr{P}(\Omega) \setminus \{\emptyset\}$, $\mathscr{T}^{*}_{\mathfrak{a}} = \mathscr{T}_{\mathfrak{a}} \setminus \{\emptyset\}$, and $\neg\mathscr{T}^{*}_{\mathfrak{a}} = \neg\mathscr{T}_{\mathfrak{a}} \setminus \{\emptyset\}$.

**Definition 2.2** ($\mathfrak{g}$-Operation [15, 16])**.** *A mapping* $\begin{array}{rlll} \mathrm{op}_{\mathfrak{a}} : & \mathscr{P}(\Omega) & \longrightarrow & \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto & \mathrm{op}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}) \end{array}$ *is called a generalized operation ($\mathfrak{g}$-operation) if and only if the following statements hold:*

$$\begin{aligned} \big(\forall \mathscr{S}_{\mathfrak{a}} \in \mathscr{P}^{*}(\Omega)\big)\big(\exists (\mathscr{O}_{\mathfrak{a}}, \mathscr{K}_{\mathfrak{a}}) \in \mathscr{T}^{*}_{\mathfrak{a}} \times \neg\mathscr{T}^{*}_{\mathfrak{a}}\big)\big[\big(\mathrm{op}_{\mathfrak{a}}(\emptyset) = \emptyset\big) \vee \big(\neg\,\mathrm{op}_{\mathfrak{a}}(\emptyset) = \emptyset\big) \\ \vee\big(\mathscr{S}_{\mathfrak{a}} \subseteq \mathrm{op}_{\mathfrak{a}}(\mathscr{O}_{\mathfrak{a}})\big) \vee \big(\mathscr{S}_{\mathfrak{a}} \supseteq \neg\,\mathrm{op}_{\mathfrak{a}}(\mathscr{K}_{\mathfrak{a}})\big)\big], \end{aligned} \tag{2.2}$$

*where* $\begin{array}{rlll} \neg\,\mathrm{op}_{\mathfrak{a}} : & \mathscr{P}(\Omega) & \longrightarrow & \mathscr{P}(\Omega) \\ & \mathscr{S}_{\mathfrak{a}} & \longmapsto & \neg\,\mathrm{op}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}) \end{array}$ *is called its complementary $\mathfrak{g}$-operation, and for all $\mathfrak{T}_{\mathfrak{a}}$-sets $\mathscr{S}_{\mathfrak{a}}$, $\mathscr{S}_{\mathfrak{a},\nu}$, $\mathscr{S}_{\mathfrak{a},\mu} \in \mathscr{P}^{*}(\Omega)$, the following axioms are satisfied:*

- AX. I. $\big(\mathscr{S}_{\mathfrak{a}} \subseteq \mathrm{op}_{\mathfrak{a}}(\mathscr{O}_{\mathfrak{a}})\big) \vee \big(\mathscr{S}_{\mathfrak{a}} \supseteq \neg\,\mathrm{op}_{\mathfrak{a}}(\mathscr{K}_{\mathfrak{a}})\big)$,
- AX. II. $\big(\mathrm{op}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}) \subseteq \mathrm{op}_{\mathfrak{a}} \circ \mathrm{op}_{\mathfrak{a}}(\mathscr{O}_{\mathfrak{a}})\big) \vee \big(\neg\,\mathrm{op}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{a}} \circ \neg\,\mathrm{op}_{\mathfrak{a}}(\mathscr{K}_{\mathfrak{a}})\big)$,
- AX. III. $\big(\mathscr{S}_{\mathfrak{a},\nu} \subseteq \mathscr{S}_{\mathfrak{a},\mu} \ \longrightarrow \ \mathrm{op}_{\mathfrak{a}}(\mathscr{O}_{\mathfrak{a},\nu}) \subseteq \mathrm{op}_{\mathfrak{a}}(\mathscr{O}_{\mathfrak{a},\mu})\big)$
  $\vee\big(\mathscr{S}_{\mathfrak{a},\mu} \subseteq \mathscr{S}_{\mathfrak{a},\nu} \ \longleftarrow \ \neg\,\mathrm{op}_{\mathfrak{a}}(\mathscr{K}_{\mathfrak{a},\mu}) \supseteq \neg\,\mathrm{op}_{\mathfrak{a}}(\mathscr{K}_{\mathfrak{a},\nu})\big)$,
- AX. IV. $\big(\mathrm{op}_{\mathfrak{a}}\big(\bigcup_{\sigma=\nu,\mu} \mathscr{S}_{\mathfrak{a},\sigma}\big) \subseteq \bigcup_{\sigma=\nu,\mu} \mathrm{op}_{\mathfrak{a}}(\mathscr{O}_{\mathfrak{a},\sigma})\big)$
  $\vee\big(\neg\,\mathrm{op}_{\mathfrak{a}}\big(\bigcup_{\sigma=\nu,\mu} \mathscr{S}_{\mathfrak{a},\sigma}\big) \supseteq \bigcup_{\sigma=\nu,\mu} \neg\,\mathrm{op}_{\mathfrak{a}}(\mathscr{K}_{\mathfrak{a},\sigma})\big)$,

*for some $\mathscr{T}_{\mathfrak{a}}$-sets $\mathscr{O}_{\mathfrak{a}}$, $\mathscr{O}_{\mathfrak{a},\nu}$, $\mathscr{O}_{\mathfrak{a},\mu} \in \mathscr{T}^{*}_{\mathfrak{a}}$ and $\mathscr{K}_{\mathfrak{a}}$, $\mathscr{K}_{\mathfrak{a},\nu}$, $\mathscr{K}_{\mathfrak{a},\mu} \in \neg\mathscr{T}^{*}_{\mathfrak{a}}$.*

The formulation of DEF. 2.2 is based on the Čech closure operator axioms [42] and the axioms used by other mathematicians to define closure operators [43]. The class $\mathscr{L}_{\mathfrak{a}}[\Omega] \stackrel{\mathrm{def}}{=} \{\mathbf{op}_{\mathfrak{a},\nu} = (\mathrm{op}_{\mathfrak{a},\nu}, \neg\,\mathrm{op}_{\mathfrak{a},\nu}) : \ \nu \in I^{0}_{3}\} \subseteq \mathscr{L}^{\omega}_{\mathfrak{a}}[\Omega] \times \mathscr{L}^{\kappa}_{\mathfrak{a}}[\Omega]$, where

$$\begin{aligned} \mathrm{op}_{\mathfrak{a}} \in \mathscr{L}^{\omega}_{\mathfrak{a}}[\Omega] & \stackrel{\mathrm{def}}{=} \{\mathrm{op}_{\mathfrak{a},0},\ \mathrm{op}_{\mathfrak{a},1},\ \mathrm{op}_{\mathfrak{a},2},\ \mathrm{op}_{\mathfrak{a},3}\} \\ & = \{\mathrm{int}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}} \circ \mathrm{int}_{\mathfrak{a}},\ \mathrm{int}_{\mathfrak{a}} \circ \mathrm{cl}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}} \circ \mathrm{int}_{\mathfrak{a}} \circ \mathrm{cl}_{\mathfrak{a}}\}, \end{aligned} \tag{2.3}$$

$$\begin{aligned} \neg\,\mathrm{op}_{\mathfrak{a}} \in \mathscr{L}^{\kappa}_{\mathfrak{a}}[\Omega] & \stackrel{\mathrm{def}}{=} \{\neg\,\mathrm{op}_{\mathfrak{a},0},\ \neg\,\mathrm{op}_{\mathfrak{a},1},\ \neg\,\mathrm{op}_{\mathfrak{a},2},\ \neg\,\mathrm{op}_{\mathfrak{a},3}\} \\ & = \{\mathrm{cl}_{\mathfrak{a}},\ \mathrm{int}_{\mathfrak{a}} \circ \mathrm{cl}_{\mathfrak{a}},\ \mathrm{cl}_{\mathfrak{a}} \circ \mathrm{int}_{\mathfrak{a}},\ \mathrm{int}_{\mathfrak{a}} \circ \mathrm{cl}_{\mathfrak{a}} \circ \mathrm{int}_{\mathfrak{a}}\}, \end{aligned} \tag{2.4}$$

stands for the class of all possible pairs of $\mathfrak{g}$-operators and its complementary $\mathfrak{g}$-operators in the $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}}$. In general, $\mathscr{L}_{\mathfrak{g}}[\Omega] \ni \mathbf{op}_{\mathfrak{g}} = (\mathrm{op}_{\mathfrak{g}}, \neg\,\mathrm{op}_{\mathfrak{g}}) \neq (\mathrm{op}_{\mathfrak{o}}, \neg\,\mathrm{op}_{\mathfrak{o}}) = \mathbf{op}_{\mathfrak{o}} \in \mathscr{L}_{\mathfrak{o}}[\Omega]$.

**Definition 2.3** ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-Sets [15, 16])**.** *Let $(\mathscr{S}_{\mathfrak{a}}, \{\mathscr{O}_{\mathfrak{a}}\}, \{\mathscr{K}_{\mathfrak{a}}\}) \subset \mathfrak{T}_{\mathfrak{a}} \times \mathscr{T}_{\mathfrak{a}} \times \neg\mathscr{T}_{\mathfrak{a}}$ and let $\mathbf{op}_{\mathfrak{a},\nu} \in \mathscr{L}_{\mathfrak{a}}[\Omega]$ be a $\mathfrak{g}$-operator in a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$. Suppose the predicates*

$$\begin{aligned} \mathrm{P}_{\mathfrak{a}}\big(\mathscr{S}_{\mathfrak{a}}, \mathscr{O}_{\mathfrak{a}}, \mathscr{K}_{\mathfrak{a}}; \mathbf{op}_{\mathfrak{a},\nu}; \subseteq, \supseteq\big) & \stackrel{\mathrm{def}}{=} \mathrm{P}_{\mathfrak{a}}\big(\mathscr{S}_{\mathfrak{a}}, \mathscr{O}_{\mathfrak{a}}; \mathbf{op}_{\mathfrak{a},\nu}; \subseteq\big) \vee \mathrm{P}_{\mathfrak{a}}\big(\mathscr{S}_{\mathfrak{a}}, \mathscr{K}_{\mathfrak{a}}; \mathbf{op}_{\mathfrak{a},\nu}; \supseteq\big), \\ \mathrm{P}_{\mathfrak{a}}\big(\mathscr{S}_{\mathfrak{a}}, \mathscr{O}_{\mathfrak{a}}; \mathbf{op}_{\mathfrak{a},\nu}; \subseteq\big) & \stackrel{\mathrm{def}}{=} \big(\exists (\mathscr{O}_{\mathfrak{a}}, \mathrm{op}_{\mathfrak{a},\nu}) \in \mathscr{T}_{\mathfrak{a}} \times \mathscr{L}^{\omega}_{\mathfrak{a}}[\Omega]\big) \\ & \qquad \big[\mathscr{S}_{\mathfrak{a}} \subseteq \mathrm{op}_{\mathfrak{a},\nu}(\mathscr{O}_{\mathfrak{a}})\big], \\ \mathrm{P}_{\mathfrak{a}}\big(\mathscr{S}_{\mathfrak{a}}, \mathscr{K}_{\mathfrak{a}}; \mathbf{op}_{\mathfrak{a},\nu}; \supseteq\big) & \stackrel{\mathrm{def}}{=} \big(\exists (\mathscr{K}_{\mathfrak{a}}, \neg\,\mathrm{op}_{\mathfrak{a},\nu}) \in \neg\mathscr{T}_{\mathfrak{a}} \times \mathscr{L}^{\kappa}_{\mathfrak{a}}[\Omega]\big) \\ & \qquad \big[\mathscr{S}_{\mathfrak{a}} \supseteq \neg\,\mathrm{op}_{\mathfrak{a},\nu}(\mathscr{K}_{\mathfrak{a}})\big] \end{aligned} \tag{2.5}$$

*be Boolean-valued on* $\mathfrak{T}_{\mathfrak{a}} \times (\mathscr{T}_{\mathfrak{a}} \cup \neg \mathscr{T}_{\mathfrak{a}}) \times \mathscr{L}_{\mathfrak{a}}[\Omega] \times \{\subseteq, \supseteq\}$, *then*

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{a}}] &\stackrel{\text{def}}{=} \big\{\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}} : \mathrm{P}_{\mathfrak{a}}\big(\mathscr{S}_{\mathfrak{a}}, \mathscr{O}_{\mathfrak{a}}, \mathscr{K}_{\mathfrak{a}}; \mathbf{op}_{\mathfrak{a},\nu}; \subseteq, \supseteq\big)\big\}, \\ \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}] &\stackrel{\text{def}}{=} \big\{\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}} : \mathrm{P}_{\mathfrak{a}}\big(\mathscr{S}_{\mathfrak{a}}, \mathscr{O}_{\mathfrak{a}}; \mathbf{op}_{\mathfrak{a},\nu}; \subseteq\big)\big\}, \\ \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}] &\stackrel{\text{def}}{=} \big\{\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}} : \mathrm{P}_{\mathfrak{a}}\big(\mathscr{S}_{\mathfrak{a}}, \mathscr{K}_{\mathfrak{a}}; \mathbf{op}_{\mathfrak{a},\nu}; \supseteq\big)\big\}, \end{aligned} \tag{2.6}$$

*respectively, are called the classes of all* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$*-sets,* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$*-open sets and* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$*-closed sets of category* $\nu$ *in* $\mathfrak{T}_{\mathfrak{a}}$.

In particular, $\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\text{def}}{=} \big\{\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}} : \mathrm{P}_{\mathfrak{a}}\big(\mathscr{S}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}; \mathbf{op}_{\mathfrak{a},0}; \subseteq\big)\big\}$ and $\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\text{def}}{=} \big\{\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}} : \mathrm{P}_{\mathfrak{a}}\big(\mathscr{S}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}; \mathbf{op}_{\mathfrak{a},0}; \supseteq\big)\big\}$ denote the classes of all $\mathfrak{T}_{\mathfrak{a}}$-open and $\mathfrak{T}_{\mathfrak{a}}$-closed sets, respectively, in $\mathfrak{T}_{\mathfrak{a}}$, with $\mathrm{S}[\mathfrak{T}_{\mathfrak{a}}] = \bigcup_{\mathrm{E}\in\{\mathrm{O},\mathrm{K}\}} \mathrm{E}[\mathfrak{T}_{\mathfrak{a}}]$ [15, 16]. Clearly,

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{a}}] &\stackrel{\text{def}}{=} \bigcup_{\nu\in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{S}[\mathfrak{T}_{\mathfrak{a}}] \\ &= \bigcup_{(\nu,\mathrm{E})\in I_3^0\times\{\mathrm{O},\mathrm{K}\}} \mathfrak{g}\text{-}\nu\text{-}\mathrm{E}[\mathfrak{T}_{\mathfrak{a}}] = \bigcup_{\mathrm{E}\in\{\mathrm{O},\mathrm{K}\}} \mathfrak{g}\text{-}\mathrm{E}[\mathfrak{T}_{\mathfrak{a}}]. \end{aligned}$$

By virtue of the foregoing descriptions, $\mathscr{S}_{\mathfrak{g}}$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-open or $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-closed of category $\nu$ ($\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}$-*open* or $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}$-*closed*) if and only if there exist $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathscr{T}_{\mathfrak{g}} \times \neg\mathscr{T}_{\mathfrak{g}}$ such that

$$\big(\mathscr{S}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{g}})\big) \vee \big(\mathscr{S}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{g}})\big), \tag{2.7}$$

where

$$\mathbf{op}_{\mathfrak{g},\nu} = \big(\mathrm{op}_{\mathfrak{g},\nu}, \neg\,\mathrm{op}_{\mathfrak{g},\nu}\big) \stackrel{\text{def}}{=} \begin{cases} \big(\mathrm{int}_{\mathfrak{g}},\ \mathrm{cl}_{\mathfrak{g}}\big) & (\nu = 0), \\ \big(\mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}},\ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}\big) & (\nu = 1), \\ \big(\mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}},\ \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}\big) & (\nu = 2), \\ \big(\mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}},\ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}\big) & (\nu = 3). \end{cases}$$

Thus, $\mathscr{R}_{\mathfrak{g}}$, $\mathscr{S}_{\mathfrak{g}}$, $\mathscr{U}_{\mathfrak{g}}$, $\mathscr{V}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ are of categories 0, 1, 2, 3, respectively, if and only if

$$\begin{aligned} &\big(\mathscr{R}_{\mathfrak{g}} \subseteq \mathrm{int}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\big) \vee \big(\mathscr{R}_{\mathfrak{g}} \supseteq \mathrm{cl}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\big), \\ &\big(\mathscr{S}_{\mathfrak{g}} \subseteq \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\big) \vee \big(\mathscr{S}_{\mathfrak{g}} \supseteq \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\big), \\ &\big(\mathscr{U}_{\mathfrak{g}} \subseteq \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\big) \vee \big(\mathscr{U}_{\mathfrak{g}} \supseteq \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\big), \\ &\big(\mathscr{V}_{\mathfrak{g}} \subseteq \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\big) \vee \big(\mathscr{V}_{\mathfrak{g}} \supseteq \mathrm{int}_{\mathfrak{g}} \circ \mathrm{cl}_{\mathfrak{g}} \circ \mathrm{int}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\big), \end{aligned} \tag{2.8}$$

for some $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathscr{T}_{\mathfrak{g}} \times \neg\mathscr{T}_{\mathfrak{g}}$. The notions of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-separateness and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-connectedness of category $\nu \in I_3^0$ are based on $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-sets of the same category $\nu$.

**Definition 2.4** ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-Separation, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-Connected [16])**.** *A* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$*-separation of category* $\nu$ *of two nonempty* $\mathfrak{T}_{\mathfrak{a}}$*-sets* $\mathscr{R}_{\mathfrak{a}}$, $\mathscr{S}_{\mathfrak{a}} \subseteq \mathfrak{T}_{\mathfrak{a}}$ *of a* $\mathscr{T}_{\mathfrak{a}}$*-space* $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ *is realised if and only if there exists either* $(\mathscr{O}_{\mathfrak{a},\xi}, \mathscr{O}_{\mathfrak{a},\zeta}) \in \times_{\alpha\in I_2^*} \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\big[\mathfrak{T}_{\mathfrak{a}}\big]$ *or* $(\mathscr{K}_{\mathfrak{a},\xi}, \mathscr{K}_{\mathfrak{a},\zeta}) \in \times_{\alpha\in I_2^*} \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\big[\mathfrak{T}_{\mathfrak{a}}\big]$ *such that:*

$$\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{O}_{\mathfrak{a},\lambda} = \mathscr{R}_{\mathfrak{a}} \sqcup \mathscr{S}_{\mathfrak{a}}\Big) \bigvee \Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{K}_{\mathfrak{a},\lambda} = \mathscr{R}_{\mathfrak{a}} \sqcup \mathscr{S}_{\mathfrak{a}}\Big). \tag{2.9}$$

*Otherwise, they are said to be* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$*-connected of category* $\nu$.

Thus, $\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}}$ is $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-connected if and only if $\mathscr{S}_{\mathfrak{a}} \in \mathfrak{g}\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{a}}\right] = \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{a}}\right]$ and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-separated if and only if $\mathscr{S}_{\mathfrak{a}} \in \mathfrak{g}\text{-}\mathrm{D}\left[\mathfrak{T}_{\mathfrak{a}}\right] = \bigcup_{\nu \in I_3^0} \mathfrak{g}\text{-}\nu\text{-}\mathrm{D}\left[\mathfrak{T}_{\mathfrak{a}}\right]$ where,

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{a}}\right] \stackrel{\text{def}}{=} \Big\{ & \mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}} : \left(\forall \left(\mathscr{O}_{\mathfrak{a},\lambda}, \mathscr{K}_{\mathfrak{a},\lambda}\right)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{a}}\right] \times \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{a}}\right]\right) \\ & \left[\neg\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{O}_{\mathfrak{a},\lambda} = \mathscr{S}_{\mathfrak{a}}\Big) \bigwedge \neg\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{O}_{\mathfrak{a},\lambda} = \mathscr{S}_{\mathfrak{a}}\Big)\right]\Big\}; \end{aligned} \tag{2.10}$$

$$\begin{aligned} \mathfrak{g}\text{-}\nu\text{-}\mathrm{D}\left[\mathfrak{T}_{\mathfrak{a}}\right] \stackrel{\text{def}}{=} \Big\{ & \mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}} : \left(\exists \left(\mathscr{O}_{\mathfrak{a},\lambda}, \mathscr{K}_{\mathfrak{a},\lambda}\right)_{\lambda=\xi,\zeta} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{a}}\right] \times \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{a}}\right]\right) \\ & \left[\Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{O}_{\mathfrak{a},\lambda} = \mathscr{S}_{\mathfrak{a}}\Big) \bigvee \Big(\bigsqcup_{\lambda=\xi,\zeta} \mathscr{K}_{\mathfrak{a},\lambda} = \mathscr{S}_{\mathfrak{a}}\Big)\right]\Big\}. \end{aligned} \tag{2.11}$$

Evidently, by $\Omega \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{Q}\left[\mathfrak{T}_{\mathfrak{a}}\right]$ or $\Omega \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{D}\left[\mathfrak{T}_{\mathfrak{a}}\right]$ is meant a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-connection of category $\nu$ or a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-separation of category $\nu$ of the $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ is realised.

2.2. **Sufficient Preliminaries.** The dual concepts called $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-closure operators of category $\nu$ in $\mathscr{T}_{\mathfrak{a}}$-spaces are presented from set-theoretic and vectorial viewpoints herein.

**Definition 2.5** ($\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{a}}$-Interior, $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{a}}$-Closure Operators)**.** *Let $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ be a $\mathscr{T}_{\mathfrak{a}}$-space, let $\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}]}\left[\mathscr{S}_{\mathfrak{a}}\right] \stackrel{\text{def}}{=} \left\{\mathscr{O}_{\mathfrak{a}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{a}}\right] :\ \mathscr{O}_{\mathfrak{a}} \subseteq \mathscr{S}_{\mathfrak{a}}\right\}$ be the family of all $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{a}}$-open subsets of $\mathscr{S}_{\mathfrak{a}} \in \mathscr{P}(\Omega)$ relative to the class $\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{a}}\right]$ of $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{a}}$-open sets, and let $\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}]}\left[\mathscr{S}_{\mathfrak{a}}\right] \stackrel{\text{def}}{=} \left\{\mathscr{K}_{\mathfrak{a}} \in \mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{a}}\right] :\ \mathscr{K}_{\mathfrak{a}} \supseteq \mathscr{S}_{\mathfrak{a}}\right\}$ be the family of all $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{a}}$-closed supersets of $\mathscr{S}_{\mathfrak{a}} \in \mathscr{P}(\Omega)$ relative to the class $\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{a}}\right]$ of $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{a}}$-closed sets. Then, the one-valued maps of the types*

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \\ \mathscr{S}_{\mathfrak{a}} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{a}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}]}[\mathscr{S}_{\mathfrak{a}}]} \mathscr{O}_{\mathfrak{a}}, \end{aligned} \tag{2.12}$$

$$\begin{aligned} \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu} : \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \\ \mathscr{S}_{\mathfrak{a}} &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{a}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\nu\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}]}[\mathscr{S}_{\mathfrak{a}}]} \mathscr{K}_{\mathfrak{a}} \end{aligned} \tag{2.13}$$

*on $\mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega)$ are called, respectively, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-closure operators of category $\nu$. The classes $\mathfrak{g}\text{-}\mathrm{I}\left[\mathfrak{T}_{\mathfrak{a}}\right] \stackrel{\text{def}}{=} \left\{\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu} :\ \nu \in I_3^0\right\}$ and $\mathfrak{g}\text{-}\mathrm{C}\left[\mathfrak{T}_{\mathfrak{a}}\right] \stackrel{\text{def}}{=} \left\{\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu} :\ \nu \in I_3^0\right\}$, respectively, are called the classes of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-closure operators.*

**Remark.** *Note that* $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a}},\ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a}} : \begin{array}{rl} \mathscr{P}(\Omega) & \longrightarrow \mathscr{P}(\Omega) \\ \mathscr{S}_{\mathfrak{a}} & \longmapsto \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}),\ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}}) \end{array}$ *are dual $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-operators because, the first is based on $\cup$, $\subseteq$, $\mathscr{O}_{\mathfrak{a},1}$, $\mathscr{O}_{\mathfrak{a},2}$, $\ldots$ while the second on $\cap$, $\supseteq$, $\mathscr{K}_{\mathfrak{a},1}$, $\mathscr{K}_{\mathfrak{a},2}$, $\ldots$.*

**Definition 2.6** ($\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-Vector Operator)**.** *Let $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$ be a $\mathscr{T}_{\mathfrak{a}}$-space. Then, an operator of the type*

$$\begin{aligned} \mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{a},\nu} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \\ (\mathscr{R}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}) &\longmapsto \left(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu}(\mathscr{R}_{\mathfrak{a}}), \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu}(\mathscr{S}_{\mathfrak{a}})\right) \end{aligned} \tag{2.14}$$

*on $\mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ ranging in $\mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ is called a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-vector operator of category $\nu$. Then, $\mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{a}}] \stackrel{\text{def}}{=} \{\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{a},\nu} = (\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu}) : \ \nu \in I_3^0\}$ is called the class of all $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-vector operators.*

**Remark.** *Observing that, for every $\nu \in I_3^*$, the first and second components of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-vector operator $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{a},\nu} = (\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{a},\nu}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{a},\nu})$ are based on $\mathfrak{g}\text{-}\nu\text{-O}[\mathfrak{T}_{\mathfrak{a}}]$ and $\mathfrak{g}\text{-}\nu\text{-K}[\mathfrak{T}_{\mathfrak{a}}]$, respectively, it follows that $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{a},\nu} = \mathbf{ic}_{\mathfrak{a}} \stackrel{\text{def}}{=} (\mathrm{int}_{\mathfrak{a}}, \mathrm{cl}_{\mathfrak{a}})$ if based on $\mathrm{O}[\mathfrak{T}_{\mathfrak{a}}]$ and $\mathrm{K}[\mathfrak{T}_{\mathfrak{a}}]$. In this way,* $\begin{aligned} \mathbf{ic}_{\mathfrak{a}} : \ \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) &\longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \\ (\mathscr{R}_{\mathfrak{a}}, \mathscr{S}_{\mathfrak{a}}) &\longmapsto (\mathrm{int}_{\mathfrak{a}}(\mathscr{R}_{\mathfrak{a}}), \mathrm{cl}_{\mathfrak{a}}(\mathscr{S}_{\mathfrak{a}})) \end{aligned}$ *is called a $\mathfrak{T}_{\mathfrak{g}}$-vector operator in a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}} = (\Omega, \mathscr{T}_{\mathfrak{a}})$.*

## 3. Main Results

The essential properties of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}_{\mathfrak{g}}$-spaces are presented below.

**Lemma 3.1.** *If $\{\mathscr{S}_{\mathfrak{g},\nu} \subset \mathfrak{T}_{\mathfrak{g}} : \ \nu \in I_\sigma^*\}$ be a collection of $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then:*

- I. $\mathrm{C}^{\mathrm{sub}}_{\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\bigcap_{\nu \in I_\sigma^*} \mathscr{S}_{\mathfrak{g},\nu}\right] = \bigcap_{\nu \in I_\sigma^*} \mathrm{C}^{\mathrm{sub}}_{\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\nu}]$,
- II. $\mathrm{C}^{\mathrm{sup}}_{\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\bigcup_{\nu \in I_\sigma^*} \mathscr{S}_{\mathfrak{g},\nu}\right] = \bigcup_{\nu \in I_\sigma^*} \mathrm{C}^{\mathrm{sup}}_{\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\nu}]$.

*Proof.* Let $\{\mathscr{S}_{\mathfrak{g},\nu} \subset \mathfrak{T}_{\mathfrak{g}} : \ \nu \in I_\sigma^*\}$ be a collection of $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then by virtue of $\mathfrak{T}_{\mathfrak{g}}$-set-theoretic $(\cap, \cup)$-operation, it results that

$$\begin{aligned} \mathrm{C}^{\mathrm{sub}}_{\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\bigcap_{\nu \in I_\sigma^*} \mathscr{S}_{\mathfrak{g},\nu}\right] &= \left\{\mathscr{O}_{\mathfrak{g}} \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] : \ \mathscr{O}_{\mathfrak{g}} \subseteq \bigcap_{\nu \in I_\sigma^*} \mathscr{S}_{\mathfrak{g},\nu}\right\} \\ &= \left\{\mathscr{O}_{\mathfrak{g}} \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] : \ \bigwedge_{\nu \in I_\sigma^*} (\mathscr{O}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g},\nu})\right\} \\ &= \bigcap_{\nu \in I_\sigma^*} \left\{\mathscr{O}_{\mathfrak{g}} \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] : \ \mathscr{O}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g},\nu}\right\} = \bigcap_{\nu \in I_\sigma^*} \mathrm{C}^{\mathrm{sub}}_{\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\nu}]; \\ \mathrm{C}^{\mathrm{sup}}_{\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\bigcup_{\nu \in I_\sigma^*} \mathscr{S}_{\mathfrak{g},\nu}\right] &= \left\{\mathscr{K}_{\mathfrak{g}} \in \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] : \ \mathscr{K}_{\mathfrak{g}} \supseteq \bigcup_{\nu \in I_\sigma^*} \mathscr{S}_{\mathfrak{g},\nu}\right\} \\ &= \left\{\mathscr{K}_{\mathfrak{g}} \in \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] : \ \bigvee_{\nu \in I_\sigma^*} (\mathscr{K}_{\mathfrak{g}} \supseteq \mathscr{S}_{\mathfrak{g},\nu})\right\} \\ &= \bigcup_{\nu \in I_\sigma^*} \left\{\mathscr{K}_{\mathfrak{g}} \in \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] : \ \mathscr{K}_{\mathfrak{g}} \supseteq \mathscr{S}_{\mathfrak{g},\nu}\right\} = \bigcup_{\nu \in I_\sigma^*} \mathrm{C}^{\mathrm{sup}}_{\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\nu}]. \end{aligned}$$

The proof of the lemma is complete. □

For any $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$, $\mathscr{O}_{\mathfrak{g}} \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})$ and $\mathscr{K}_{\mathfrak{g}} \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})$ hold, or alternatively, $\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$. Consequently,

$$\left(\mathscr{O}_{\mathfrak{g}} \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathscr{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]\right) \wedge \left(\mathscr{K}_{\mathfrak{g}} \in \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathscr{K}_{\mathfrak{g}} \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]\right).$$

As a consequence of the above lemma, the corollary follows.

**Corollary 3.2.** *If* $\{\mathscr{S}_{\mathfrak{g},\nu} \subset \mathfrak{T}_{\mathfrak{g}} :\ \nu \in I_{\sigma}^{*}\}$ *be a collection of* $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$*-sets of a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

- I. $\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\big[\bigcap_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu}\big] = \bigcap_{\nu\in I_{\sigma}^{*}} \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g},\nu}\right]$,
- II. $\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\big[\bigcup_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu}\big] = \bigcup_{\nu\in I_{\sigma}^{*}} \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g},\nu}\right]$.

**Remark.** *Clearly,* $\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\big[\bigcap_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu} = \emptyset\big] = \{\emptyset\}$ *and* $\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\big[\bigcup_{\nu\in I_{\sigma}^{*}} \mathscr{S}_{\mathfrak{g},\nu}\big] = \{\Omega\}$ *hold. Moreover,* $\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\big[\mathscr{S}_{\mathfrak{g}} = \Omega\big] = \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ *and* $\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\big[\mathscr{S}_{\mathfrak{g}} = \emptyset\big] = \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right]$.

**Proposition 3.3.** *Let* $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ *be a* $\mathfrak{T}_{\mathfrak{g}}$*-set and, let* $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$*, respectively, be a* $\mathfrak{g}$*-*$\mathfrak{T}_{\mathfrak{g}}$*-interior and a* $\mathfrak{g}$*-*$\mathfrak{T}_{\mathfrak{g}}$*-closure operators in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*. Then, the necessary and sufficient conditions for* $(\xi,\zeta) \in \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \times \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}}$ *to hold in* $\mathfrak{T}_{\mathfrak{g}}$ *are:*

- I. $\xi \in \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \ \longleftrightarrow\ \big(\exists \mathscr{O}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big)\big[\mathscr{O}_{\mathfrak{g},\xi} \subseteq \mathscr{S}_{\mathfrak{g}}\big]$,
- II. $\zeta \in \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \ \longleftrightarrow\ \big(\forall \mathscr{O}_{\mathfrak{g},\zeta} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big)\big[\mathscr{O}_{\mathfrak{g},\zeta} \cap \mathscr{S}_{\mathfrak{g}} \neq \emptyset\big]$.

*Proof.* Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be a $\mathfrak{T}_{\mathfrak{g}}$-set and, let $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}, \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Suppose

$$(\xi,\zeta) \in \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \times \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}\Bigg) \times \Bigg(\bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}}\Bigg).$$

Then, since the relations

$$\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} \ \longleftrightarrow\ \big\{\xi :\ \big(\exists \mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g}}\right]\big)\big[\xi \in \mathscr{O}_{\mathfrak{g}}\big]\big\},$$

$$\bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} \ \longleftrightarrow\ \big\{\zeta :\ \big(\forall \mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g}}\right]\big)\big[\zeta \in \mathscr{K}_{\mathfrak{g}}\big]\big\}$$

hold and $\mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right] \times \mathfrak{g}\text{-}\mathrm{K}\left[\mathfrak{T}_{\mathfrak{g}}\right] \supseteq \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g}}\right] \times \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g}}\right]$, and, on the other hand, the relation $\xi \in \mathscr{O}_{\mathfrak{g},\xi} \subseteq \mathscr{S}_{\mathfrak{g}} \subseteq \mathscr{K}_{\mathfrak{g},\xi}$ also holds for any $(\xi, \mathscr{O}_{\mathfrak{g},\xi}, \mathscr{K}_{\mathfrak{g},\xi}) \in \mathscr{S}_{\mathfrak{g}} \times \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g}}\right] \times \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g}}\right]$, it follows that

$$\begin{aligned}
\xi \in \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \ &\longleftrightarrow\ \big(\exists \mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g}}\right]\big)\big[\xi \in \mathscr{O}_{\mathfrak{g}}\big] \\
&\longleftrightarrow\ \big(\exists \mathscr{O}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big)\big[\mathscr{O}_{\mathfrak{g},\xi} \subseteq \mathscr{S}_{\mathfrak{g}}\big]; \\
\zeta \in \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \ &\longleftrightarrow\ \big(\forall \mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g}}\right]\big)\big[\zeta \in \mathscr{K}_{\mathfrak{g}}\big] \\
&\longleftrightarrow\ \big(\forall \mathscr{O}_{\mathfrak{g},\zeta} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big)\big[\mathscr{O}_{\mathfrak{g},\zeta} \cap \mathscr{S}_{\mathfrak{g}} \neq \emptyset\big].
\end{aligned}$$

Hence, $\xi \in \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ is equivalent to $\big(\exists \mathscr{O}_{\mathfrak{g},\xi} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big)\big[\mathscr{O}_{\mathfrak{g},\xi} \subseteq \mathscr{S}_{\mathfrak{g}}\big]$ and $\zeta \in \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ is equivalent to $\big(\forall \mathscr{O}_{\mathfrak{g},\zeta} \in \mathfrak{g}\text{-}\mathrm{O}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big)\big[\mathscr{O}_{\mathfrak{g},\zeta} \cap \mathscr{S}_{\mathfrak{g}} \neq \emptyset\big]$. The proof of the proposition is complete. ☐

**Theorem 3.4.** *If* $\{\mathscr{S}_{\mathfrak{g},\nu} \subset \mathfrak{T}_{\mathfrak{g}} :\ \nu \in I_{\sigma}^{*}\}$ *be a collection of* $\sigma \geq 1$ $\mathfrak{T}_{\mathfrak{g}}$*-sets of a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *then:*

– I. $\mathfrak{g}\text{-Int}_{\mathfrak{g}}:\bigcap_{\nu\in I_{\sigma}^{*}}\mathscr{S}_{\mathfrak{g},\nu}\longmapsto\bigcap_{\nu\in I_{\sigma}^{*}}\mathfrak{g}\text{-Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},\nu}\big)\quad\forall\,\mathfrak{g}\text{-Int}_{\mathfrak{g}}\in\mathfrak{g}\text{-I}\left[\mathfrak{T}_{\mathfrak{g}}\right]$,

– II. $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}:\bigcup_{\nu\in I_{\sigma}^{*}}\mathscr{S}_{\mathfrak{g},\nu}\longmapsto\bigcup_{\nu\in I_{\sigma}^{*}}\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},\nu}\big)\quad\forall\,\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\in\mathfrak{g}\text{-C}\left[\mathfrak{T}_{\mathfrak{g}}\right]$.

*Proof.* Let $\{\mathscr{S}_{\mathfrak{g},\nu}\subset\mathfrak{T}_{\mathfrak{g}}:\ \nu\in I_{\sigma}^{*}\}$ be a collection of $\sigma\geq 1$ $\mathfrak{T}_{\mathfrak{g}}$-sets of a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. Then for any $\big(\mathfrak{g}\text{-Int}_{\mathfrak{g}},\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big)\in\mathfrak{g}\text{-I}\left[\mathfrak{T}_{\mathfrak{g}}\right]\times\mathfrak{g}\text{-C}\left[\mathfrak{T}_{\mathfrak{g}}\right]$, it follows that

$$\begin{aligned}\mathfrak{g}\text{-Int}_{\mathfrak{g}}:\bigcap_{\nu\in I_{\sigma}^{*}}\mathscr{S}_{\mathfrak{g},\nu}\quad&\longmapsto\bigcup_{\mathscr{O}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\bigcap_{\nu\in I_{\sigma}^{*}}\mathscr{S}_{\mathfrak{g},\nu}\right]}\mathscr{O}_{\mathfrak{g}}\\&=\bigcup_{\mathscr{O}_{\mathfrak{g}}\in\bigcap_{\nu\in I_{\sigma}^{*}}\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\nu}]}\mathscr{O}_{\mathfrak{g}}\\&=\bigcap_{\nu\in I_{\sigma}^{*}}\Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\nu}]}\mathscr{O}_{\mathfrak{g}}\Bigg)=\bigcap_{\nu\in I_{\sigma}^{*}}\mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},\nu}\right);\\\mathfrak{g}\text{-Cl}_{\mathfrak{g}}:\bigcup_{\nu\in I_{\sigma}^{*}}\mathscr{S}_{\mathfrak{g},\nu}\quad&\longmapsto\bigcup_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\bigcup_{\nu\in I_{\sigma}^{*}}\mathscr{S}_{\mathfrak{g},\nu}\right]}\mathscr{K}_{\mathfrak{g}}\\&=\bigcup_{\mathscr{K}_{\mathfrak{g}}\in\bigcup_{\nu\in I_{\sigma}^{*}}\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\nu}]}\mathscr{K}_{\mathfrak{g}}\\&=\bigcup_{\nu\in I_{\sigma}^{*}}\Bigg(\bigcup_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g},\nu}]}\mathscr{K}_{\mathfrak{g}}\Bigg)=\bigcup_{\nu\in I_{\sigma}^{*}}\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g},\nu}\right).\end{aligned}$$

The proof of the theorem is complete. □

**Theorem 3.5.** *If $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ be any $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$, then:*

$$\big(\forall\,\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}\in\mathfrak{g}\text{-IC}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big)\big[\big(\mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\subseteq\mathscr{S}_{\mathfrak{g}}\big)\wedge\big(\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\supseteq\mathscr{S}_{\mathfrak{g}}\big)\big].\tag{3.1}$$

*Proof.* Let $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ be any $\mathfrak{T}_{\mathfrak{g}}$-set and $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}\in\mathfrak{g}\text{-IC}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. Then, by virtue of the definition of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operators $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}:\mathscr{P}\left(\Omega\right)\longrightarrow\mathscr{P}\left(\Omega\right)$, it results that,

$$\begin{aligned}\mathfrak{g}\text{-Int}_{\mathfrak{g}}:\mathscr{S}_{\mathfrak{g}}\quad&\longmapsto\bigcup_{\mathscr{O}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{O}_{\mathfrak{g}}\\\mathfrak{g}\text{-Cl}_{\mathfrak{g}}:\mathscr{S}_{\mathfrak{g}}\quad&\longmapsto\bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{K}_{\mathfrak{g}},\end{aligned}$$

respectively. But, for every $(\mathscr{O}_{\mathfrak{g}},\mathscr{K}_{\mathfrak{g}})\in\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g}}\right]\times\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}\left[\mathscr{S}_{\mathfrak{g}}\right]$, the relation $(\mathscr{O}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\subseteq(\mathscr{S}_{\mathfrak{g}},\mathscr{K}_{\mathfrak{g}})$ holds. Hence, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\subseteq\mathscr{S}_{\mathfrak{g}}$ and $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\supseteq\mathscr{S}_{\mathfrak{g}}$. This completes the proof of the theorem. □

A consequence of the above theorem is the following corollary.

**Corollary 3.6.** *If $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ be any $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$, then:*

$$\big(\forall\,\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}\in\mathfrak{g}\text{-IC}\left[\mathfrak{T}_{\mathfrak{g}}\right]\big)\big[\mathfrak{g}\text{-Int}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\subseteq\mathscr{S}_{\mathfrak{g}}\subseteq\mathfrak{g}\text{-Cl}_{\mathfrak{g}}\left(\mathscr{S}_{\mathfrak{g}}\right)\big].\tag{3.2}$$

**Remark.** *Employing the terminology of Levine, N. [10], any $\mathfrak{T}_{\mathfrak{g}}$-set $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ which satisfies the relation $\mathscr{O}_{\mathfrak{g}} = \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})$ for some $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open set $\mathscr{O}_{\mathfrak{g}} \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ may well be termed a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-semi-open set.*

**Proposition 3.7.** *If $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a strong $\mathscr{T}_{\mathfrak{g}}$-space, then:*

$$\big(\forall\, \mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]\big)\big[\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} : (\Omega, \emptyset) \longmapsto (\Omega, \emptyset)\big]. \tag{3.3}$$

*Proof.* Let $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ in a strong $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, since $\mathfrak{T}_{\mathfrak{g}}$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space, $(\Omega, \emptyset) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$ and, therefore, $\Omega$ is the biggest $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-open subset contained in itself and, $\emptyset$ is the smallest $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closed superset containing itself. Consequently,

$$\begin{aligned} \mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} : (\Omega, \emptyset) \quad &\longmapsto \Bigg( \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\Omega]} \mathscr{O}_{\mathfrak{g}}, \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\emptyset]} \mathscr{K}_{\mathfrak{g}} \Bigg) \\ &= \Bigg( \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \{\Omega\} \cup \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\Omega]} \mathscr{O}_{\mathfrak{g}}, \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \{\emptyset\} \cup \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\emptyset]} \mathscr{K}_{\mathfrak{g}} \Bigg) = (\Omega, \emptyset). \end{aligned}$$

Hence, $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} : (\Omega, \emptyset) \longmapsto (\Omega, \emptyset)$. The proof of the proposition is complete. □

**Proposition 3.8.** *If $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be any $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then:*

- I. $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall\, \mathfrak{g}\text{-Int}_{\mathfrak{g}} \in \mathfrak{g}\text{-I}[\mathfrak{T}_{\mathfrak{g}}]$,
- II. $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall\, \mathfrak{g}\text{-Cl}_{\mathfrak{g}} \in \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g}}]$.

*Proof.* Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be any $\mathfrak{T}_{\mathfrak{g}}$-set and let $\big(\mathfrak{g}\text{-Int}_{\mathfrak{g}}, \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big) \in \mathfrak{g}\text{-I}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-C}[\mathfrak{T}_{\mathfrak{g}}]$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then,

$$\begin{aligned} \mathfrak{g}\text{-Int}_{\mathfrak{g}} :\ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{O}_{\mathfrak{g}}; \\ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} :\ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{K}_{\mathfrak{g}}. \end{aligned}$$

But, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathscr{S}_{\mathfrak{g}} \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and consequently,

$$\begin{aligned} \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{O}_{\mathfrak{g}} \quad &= \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}; \\ \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{K}_{\mathfrak{g}} \quad &= \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}}. \end{aligned}$$

Hence, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. This completes the proof of the proposition. □

**Proposition 3.9.** *If $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be any $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, then:*

- I. $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall \big(\mathfrak{g}\text{-Int}_{\mathfrak{g}}, \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big) \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$,
- II. $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad \forall \big(\mathfrak{g}\text{-Int}_{\mathfrak{g}}, \mathfrak{g}\text{-Cl}_{\mathfrak{g}}\big) \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$.

*Proof.* Let $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ be any $\mathfrak{T}_{\mathfrak{g}}$-set and let $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ be a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-operator in a $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, the first and second components of $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} : \mathscr{P}(\Omega) \times \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$ operated on $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}}$ gives

$$\begin{aligned}
\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \ &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{O}_{\mathfrak{g}} \\
&= \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \left(\mathscr{O}_{\mathfrak{g}} \cap \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right) \\
&= \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \left(\mathscr{O}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}\right) = \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}, \\
\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \ &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{K}_{\mathfrak{g}} \\
&= \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \left(\mathscr{K}_{\mathfrak{g}} \cup \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\right) \\
&= \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \left(\mathscr{K}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}\right) = \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}},
\end{aligned}$$

respectively. Hence, $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \longmapsto \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the proposition is complete. □

**Theorem 3.10.** *If* $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ *then, for every* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}}$ *such that* $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}$*:*

$$\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}). \tag{3.4}$$

*Proof.* Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Suppose $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}\left[\mathfrak{T}_{\mathfrak{g}}\right]$ be given and $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}}$ such that $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}$ be an arbitrary pair of $\mathfrak{T}_{\mathfrak{g}}$-sets. Then, since for any $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}_{\mathfrak{g}}(\Omega)$, $(\mathscr{O}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \subseteq (\mathscr{S}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}})$ for every $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}] \times \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]$, it follows by virtue of the relation $\mathscr{R}_{\mathfrak{g}} \subseteq \mathscr{S}_{\mathfrak{g}}$ that $(\mathscr{O}_{\mathfrak{g}}, \mathscr{R}_{\mathfrak{g}}) \subseteq (\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \subseteq (\mathscr{S}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}})$ for any $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{R}_{\mathfrak{g}}] \times \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]$. Consequently, it results on the one hand that

$$\begin{aligned}
\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{R}_{\mathfrak{g}} \ &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{R}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} = \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{R}_{\mathfrak{g}}]} \left(\mathscr{O}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}\right) \\
&\subseteq \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \left(\mathscr{O}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}\right) = \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} = \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),
\end{aligned}$$

and on the other hand,

$$\begin{aligned}\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}:\mathscr{R}_{\mathfrak{g}} &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{R}_{\mathfrak{g}}]}\mathscr{K}_{\mathfrak{g}} = \bigcap_{\mathscr{K}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{R}_{\mathfrak{g}}]}(\mathscr{K}_{\mathfrak{g}}\cap\mathscr{R}_{\mathfrak{g}})\\ &\subseteq \bigcap_{\mathscr{K}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]}(\mathscr{K}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}}) = \bigcap_{\mathscr{K}_{\mathfrak{g}}\in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{K}_{\mathfrak{g}} = \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

These show that the images of $\mathscr{R}_{\mathfrak{g}}$ under $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$, respectively, are subsets of $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Hence, $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}},\mathscr{R}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})$. The proof of the theorem is complete. □

**Theorem 3.11.** *If* $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *and* $\mathbf{ic}_{\mathfrak{g}}\in\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{int}_{\mathfrak{g}}$, $\mathrm{cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$*, then:*

$$\big(\forall\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}\big)\big[\big(\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\subseteq\big(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\big]. \tag{3.5}$$

*Proof.* Let $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Suppose $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathbf{ic}_{\mathfrak{g}}\in\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ be given and $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ be an arbitrary $\mathfrak{T}_{\mathfrak{g}}$-set. Then,

$$\begin{aligned}\mathrm{int}_{\mathfrak{g}}:\mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcup_{\mathscr{O}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sub}}_{\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{O}_{\mathfrak{g}}\subseteq\bigcup_{\mathscr{O}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{O}_{\mathfrak{g}}=\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}});\\ \mathrm{cl}_{\mathfrak{g}}:\mathscr{S}_{\mathfrak{g}} &\longmapsto \bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sup}}_{\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{K}_{\mathfrak{g}}\supseteq\bigcap_{\mathscr{K}_{\mathfrak{g}}\in\mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]}\mathscr{K}_{\mathfrak{g}}=\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).\end{aligned}$$

Therefore, it follows that the images of $\mathscr{S}_{\mathfrak{g}}$ under $\mathrm{int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$, respectively, are subsets of $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. Hence, $\big(\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\subseteq\big(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)$. The proof of the theorem is complete. □

**Proposition 3.12.** *If* $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *and* $\mathbf{ic}_{\mathfrak{g}}\in\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{int}_{\mathfrak{g}}$, $\mathrm{cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ *then, for any* $\mathfrak{T}_{\mathfrak{g}}$*-set* $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$*,*

$$\big(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}\subseteq\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\longrightarrow\big(\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}\subseteq\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big). \tag{3.6}$$

*Proof.* If $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}\in\mathfrak{g}\text{-}\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathbf{ic}_{\mathfrak{g}}\in\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ be given and, let $\mathscr{S}_{\mathfrak{g}}\subset\mathfrak{T}_{\mathfrak{g}}$ be an arbitrary $\mathfrak{T}_{\mathfrak{g}}$-set in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$. Then, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}\subseteq\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. But since $\big(\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\subseteq\big(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}),\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)$ it follows that

$$\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}\subseteq\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).$$

Hence, $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}\subseteq\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ implies $\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathscr{S}_{\mathfrak{g}}\subseteq\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the proposition is complete. □

**Remark.** *If* $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}\gtrsim\mathrm{int}_{\mathfrak{g}}$ *stands for* $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\supseteq\mathrm{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ *and* $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}\precsim\mathrm{cl}_{\mathfrak{g}}$*, for* $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\subseteq\mathrm{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$*, then the outstanding facts are:* $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *is* finer *(or,* larger*,* stronger*) than* $\mathrm{int}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *or,* $\mathrm{int}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *is* coarser *(or,* smaller*,* weaker*) than* $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$*;* $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *is* coarser *(or,* smaller*,* weaker*) than* $\mathrm{cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *or,* $\mathrm{cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *is* finer *(or,* larger*,* stronger*) than* $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$.

**Proposition 3.13.** *If* $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathbf{ic}_{\mathfrak{g}} \in \mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathrm{int}_{\mathfrak{g}}$, $\mathrm{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$*, and* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}}$ *be any pair of* $\mathscr{T}_{\mathfrak{g}}$*-sets in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

$$(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \longrightarrow \mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) = \mathbf{ic}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}). \tag{3.7}$$

*Proof.* Let $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathbf{ic}_{\mathfrak{g}} \in \mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ be given and, let $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \subset \mathfrak{T}_{\mathfrak{g}} \times \mathfrak{T}_{\mathfrak{g}}$ be arbitrary in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$. Then, since $\mathrm{S}[\mathfrak{T}_{\mathfrak{g}}] = \mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cup \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$ and, $\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \subseteq \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \supseteq \mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]$, it follows that

$$\begin{aligned}
\mathfrak{g}\text{-}\mathbf{ic}_{\mathfrak{g}} : (\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \quad &\longmapsto \quad \Bigg( \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{R}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}, \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} \Bigg) \\
&= \quad \Bigg( \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{R}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}, \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}] \cap \mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} \Bigg) \\
&= \quad \Bigg( \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{R}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}, \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} \Bigg) \\
&= \quad \mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}).
\end{aligned}$$

Hence, $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) = \mathbf{ic}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}})$. The proof of the proposition is complete. □

**Proposition 3.14.** *If* $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

$$\begin{aligned}
\big(\forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)\big)\big[&\big(\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big) \\
&\wedge \big(\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big)\big].
\end{aligned} \tag{3.8}$$

*Proof.* Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Suppose $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-}\mathrm{IC}[\mathfrak{T}_{\mathfrak{g}}]$ be given and $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be an arbitrary $\mathfrak{T}_{\mathfrak{g}}$-set. Then,

$$\begin{aligned}
\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} : \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad &\longmapsto \quad \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{O}_{\mathfrak{g}} \\
&\supseteq \quad \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} = \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}); \\
\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \quad &\longmapsto \quad \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})]} \mathscr{K}_{\mathfrak{g}} \\
&\subseteq \quad \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-}\mathrm{K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} = \mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}).
\end{aligned}$$

Hence, the image of $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ under $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is a superset of $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and that of $\mathfrak{g}\text{-}\mathrm{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ under $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is a subset of $\mathfrak{g}\text{-}\mathrm{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$. The proof of the proposition is complete. □

**Theorem 3.15.** *If* $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ *be a given pair of* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then:*

$$\big(\forall \mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)\big)\big[\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]\big]. \tag{3.9}$$

*Proof.* Let $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ be a $\mathscr{T}_{\mathfrak{g}}$-space. Suppose $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}[\mathfrak{T}_{\mathfrak{g}}]$ be given and $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ be an arbitrary $\mathfrak{T}_{\mathfrak{g}}$-set. Then, by virtue of the definition of $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}$, it results that,

$$\begin{aligned}
\mathfrak{g}\text{-Int}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \quad &\longmapsto \quad \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}} \\
&\subseteq \quad \bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathscr{T}_{\mathfrak{g}}}[\mathscr{S}_{\mathfrak{g}}]} \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}}) = \mathrm{op}_{\mathfrak{g}}\Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathscr{T}_{\mathfrak{g}}}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}\Bigg); \\
\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{S}_{\mathfrak{g}} \quad &\longmapsto \quad \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}} \\
&\supseteq \quad \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\neg\mathscr{T}_{\mathfrak{g}}}[\mathscr{S}_{\mathfrak{g}}]} \mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}}) = \mathrm{op}_{\mathfrak{g}}\Bigg(\bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\neg\mathscr{T}_{\mathfrak{g}}}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}}\Bigg).
\end{aligned}$$

But since

$$\Bigg(\bigcup_{\mathscr{O}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sub}}_{\mathscr{T}_{\mathfrak{g}}}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{O}_{\mathfrak{g}}, \bigcap_{\mathscr{K}_{\mathfrak{g}} \in \mathrm{C}^{\mathrm{sup}}_{\neg\mathscr{T}_{\mathfrak{g}}}[\mathscr{S}_{\mathfrak{g}}]} \mathscr{K}_{\mathfrak{g}},\Bigg) \in \mathscr{T}_{\mathfrak{g}} \times \neg\mathscr{T}_{\mathfrak{g}},$$

it follows, consequently, that $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}]$ and $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$. Hence, $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \in \mathfrak{g}\text{-O}[\mathfrak{T}_{\mathfrak{g}}] \times \mathfrak{g}\text{-K}[\mathfrak{T}_{\mathfrak{g}}]$. This proves the theorem. □

**Corollary 3.16.** *If* $\mathfrak{g}\text{-}\mathbf{Ic}_{\mathfrak{g}} \in \mathfrak{g}\text{-IC}\big[\Omega\big]$ *be a given pair of* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-operators* $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *and* $\mathscr{S}_{\mathfrak{g}} \subset \mathfrak{T}_{\mathfrak{g}}$ *be any* $\mathfrak{T}_{\mathfrak{g}}$*-set in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*, then there exists* $(\mathscr{O}_{\mathfrak{g}}, \mathscr{K}_{\mathfrak{g}}) \in \mathscr{T}_{\mathfrak{g}} \times \neg\mathscr{T}_{\mathfrak{g}}$ *such that:*

$$\big[\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \mathrm{op}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g}})\big] \wedge \big[\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \neg\,\mathrm{op}_{\mathfrak{g}}(\mathscr{K}_{\mathfrak{g}})\big]. \tag{3.10}$$

In view of THMS 3.2, 3.4 and PROPS 3.7, 3.8, it follows immediately that the $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$-closure operators $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}$: $\mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively possess similar properties analogous to the *Kuratowski closure Axioms* which can be grouped and stated in the form of a corollary.

**Corollary 3.17.** *Let* $\mathfrak{g}\text{-Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ *be a* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-interior and a* $\mathfrak{g}\text{-}\mathfrak{T}_{\mathfrak{g}}$*-closure operators in a strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$*. Then:*

- *For every* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$,
  - I. $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\Omega) = \Omega$,
  - II. $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \subseteq \mathscr{R}_{\mathfrak{g}}$,
  - III. $\mathfrak{g}\text{-Int}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$,
  - IV. $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cap \mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cap \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$.
- *For every* $(\mathscr{R}_{\mathfrak{g}}, \mathscr{S}_{\mathfrak{g}}) \in \mathscr{P}(\Omega) \times \mathscr{P}(\Omega)$,
  - V. $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\emptyset) = \emptyset$,
  - VI. $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \supseteq \mathscr{R}_{\mathfrak{g}}$,
  - VII. $\mathfrak{g}\text{-Cl}_{\mathfrak{g}} \circ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})$,
  - VIII. $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}} \cup \mathscr{S}_{\mathfrak{g}}) = \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}) \cup \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$.

Some nice Mathematical vocabulary follow. In COR. 3.17, ITEMS I., II., III. and IV. state that the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operator $\mathfrak{g}\text{-Int}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is $\Omega$-*grounded*, *non-expansive*, *idempotent* and $\cap$-*additive*, respectively. ITEMS V., VI., VII. and VIII. state that the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operator $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ is $\emptyset$-*grounded*, *expansive*, *idempotent* and $\cup$-*additive*, respectively.

The axiomatic definitions of the concepts of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\nu$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}_{\mathfrak{g}}$-spaces follow.

**Definition 3.1** (Axiomatic Definition: $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Interior Operator)**.** *A one-valued map of the type* $\mathfrak{g}\text{-Int}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *in a* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ *is called a "*$\mathfrak{g}$*-*$\mathfrak{T}_{\mathfrak{g}}$*-interior operator" on* $\mathscr{P}(\Omega)$ *ranging in* $\mathscr{P}(\Omega)$ *if and only if, for any* $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$*, it satisfies the following axioms:*

– AX. I. $\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\subseteq\mathscr{R}_{\mathfrak{g}}$,
– AX. II. $\mathscr{R}_{\mathfrak{g}}\subseteq\mathscr{S}_{\mathfrak{g}}\ \longrightarrow\ \mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$.

Thus, a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operator $\mathfrak{g}\text{-Int}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ is a non-expansive $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set-valued set map forming a generalization of the $\mathfrak{T}_{\mathfrak{g}}$-set-valued set map $\text{int}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, provided

$$\big[\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\subseteq\mathscr{R}_{\mathfrak{g}}\big]\wedge\big[\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cap\mathscr{S}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cap\mathfrak{g}\text{-Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big] \tag{3.11}$$

holds for any $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$.

**Definition 3.2** (Axiomatic Definition: $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-Closure Operator)**.** *A one-valued map of the type* $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ *in a strong* $\mathscr{T}_{\mathfrak{g}}$*-space* $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ *is called a "*$\mathfrak{g}$*-*$\mathfrak{T}_{\mathfrak{g}}$*-closure operator" on* $\mathscr{P}(\Omega)$ *ranging in* $\mathscr{P}(\Omega)$ *if and only if, for any* $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$*, it satisfies the following axioms:*

– AX. I. $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\supseteq\mathscr{R}_{\mathfrak{g}}$,
– AX. II. $\mathscr{R}_{\mathfrak{g}}\subseteq\mathscr{S}_{\mathfrak{g}}\ \longrightarrow\ \mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\subseteq\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$.

Hence, a $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operator $\mathfrak{g}\text{-Cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ in a $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}=(\Omega,\mathscr{T}_{\mathfrak{g}})$ is an expansive $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-set-valued set map forming a generalization of the $\mathfrak{T}_{\mathfrak{g}}$-set-valued set map $\text{cl}_{\mathfrak{g}}:\mathscr{P}(\Omega)\longrightarrow\mathscr{P}(\Omega)$ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, provided

$$\big[\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\supseteq\mathscr{R}_{\mathfrak{g}}\big]\wedge\big[\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}}\cup\mathscr{S}_{\mathfrak{g}})\supseteq\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{R}_{\mathfrak{g}})\cup\mathfrak{g}\text{-Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})\big] \tag{3.12}$$

holds for any $(\mathscr{R}_{\mathfrak{g}},\mathscr{S}_{\mathfrak{g}})\in\mathscr{P}(\Omega)\times\mathscr{P}(\Omega)$.

## 4. DISCUSSION

4.1. **Categorical Classifications.** The notions of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-closure operators of category $\nu$ have been defined in terms of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-sets of the same category $\nu$. Having adopted such a categorical approach in the classifications of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-closure operators, the twofold purposes here are, firstly, to establish the various relationships amongst the classes of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-closure operators, $\mathfrak{a}\in\{\mathfrak{o},\mathfrak{g}\}$, in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$, and secondly, to illustrate them through diagrams.

In a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}}$, $\text{op}_{\mathfrak{a},0}(\mathscr{O}_{\mathfrak{a}})\subseteq\text{op}_{\mathfrak{a},1}(\mathscr{O}_{\mathfrak{a}})\subseteq\text{op}_{\mathfrak{a},3}(\mathscr{O}_{\mathfrak{a}})\supseteq\text{op}_{\mathfrak{a},2}(\mathscr{O}_{\mathfrak{a}})$ for every $\mathscr{O}_{\mathfrak{a}}\in\text{O}\big[\mathfrak{T}_{\mathfrak{a}}\big]$. Consequently, $\mathfrak{g}\text{-Int}_{\mathfrak{a},0}(\mathscr{S}_{\mathfrak{a}})\subseteq\mathfrak{g}\text{-Int}_{\mathfrak{a},1}(\mathscr{S}_{\mathfrak{a}})\subseteq\mathfrak{g}\text{-Int}_{\mathfrak{a},3}(\mathscr{S}_{\mathfrak{a}})\supseteq\mathfrak{g}\text{-Int}_{\mathfrak{a},2}(\mathscr{S}_{\mathfrak{a}})$ for any $\mathscr{S}_{\mathfrak{a}}\in\mathfrak{T}_{\mathfrak{a}}$. But, $\mathscr{O}_{\mathfrak{a}}\subseteq\text{op}_{\mathfrak{o},\nu}(\mathscr{O}_{\mathfrak{a}})\subseteq\text{op}_{\mathfrak{g},\nu}(\mathscr{O}_{\mathfrak{a}})$ for every $\nu\in I_3^0$, implying $\mathfrak{g}\text{-Int}_{\mathfrak{o},\nu}(\mathscr{S}_{\mathfrak{a}}) \subseteq \mathfrak{g}\text{-Int}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{a}})$ for any $(\nu, \mathscr{S}_{\mathfrak{a}}) \in I_3^0 \times \mathfrak{T}_{\mathfrak{g}}$. Thus, this diagram, which is to be read horizontally, from left to right and vertically, from top to bottom, follows:

$$\begin{array}{ccccccc} \mathscr{O}_{\mathfrak{a}} & = & \mathscr{O}_{\mathfrak{a}} & = & \mathscr{O}_{\mathfrak{a}} & = & \mathscr{O}_{\mathfrak{a}} \\ \cap\!| & & \cap\!| & & \cap\!| & & \cap\!| \\ \text{op}_{\mathfrak{o},0}(\mathscr{O}_{\mathfrak{a}}) & \subseteq & \text{op}_{\mathfrak{o},1}(\mathscr{O}_{\mathfrak{a}}) & \subseteq & \text{op}_{\mathfrak{o},3}(\mathscr{O}_{\mathfrak{a}}) & \supseteq & \text{op}_{\mathfrak{o},2}(\mathscr{O}_{\mathfrak{a}}) \\ \cap\!| & & \cap\!| & & \cap\!| & & \cap\!| \\ \text{op}_{\mathfrak{g},0}(\mathscr{O}_{\mathfrak{a}}) & \subseteq & \text{op}_{\mathfrak{g},1}(\mathscr{O}_{\mathfrak{a}}) & \subseteq & \text{op}_{\mathfrak{g},3}(\mathscr{O}_{\mathfrak{a}}) & \supseteq & \text{op}_{\mathfrak{g},2}(\mathscr{O}_{\mathfrak{a}}). \end{array} \tag{4.1}$$

In FIG. 1, we present the relationships between the elements of the collections $\{\mathfrak{g}\text{-Int}_{\mathfrak{o},\nu}(\mathscr{S}_{\mathfrak{a}}) : \nu \in I_3^0\}$ in the $\mathscr{T}_{\mathfrak{o}}$-space $\mathfrak{T}_{\mathfrak{o}}$ and $\{\mathfrak{g}\text{-Int}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{a}}) : \nu \in I_3^0\}$ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$; FIG. 1 may well be called a $(\mathfrak{g}\text{-Int}_{\mathfrak{o}}, \mathfrak{g}\text{-Int}_{\mathfrak{g}})$-*valued diagram.*

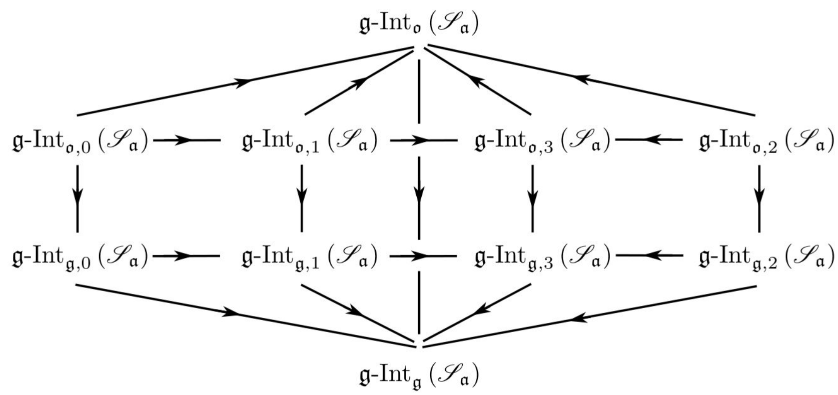


FIGURE 1. Relationships: $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{o}}$-interior operators in $\mathscr{T}_{\mathfrak{o}}$-spaces and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.

In a $\mathscr{T}_{\mathfrak{a}}$-space $\mathfrak{T}_{\mathfrak{a}}$, $\neg\,\text{op}_{\mathfrak{a},0}(\mathscr{K}_{\mathfrak{a}}) \supseteq \neg\,\text{op}_{\mathfrak{a},1}(\mathscr{K}_{\mathfrak{a}}) \supseteq \neg\,\text{op}_{\mathfrak{a},3}(\mathscr{K}_{\mathfrak{a}}) \subseteq \neg\,\text{op}_{\mathfrak{a},2}(\mathscr{K}_{\mathfrak{a}})$ for every $\mathscr{K}_{\mathfrak{a}} \in \mathrm{K}[\mathfrak{T}_{\mathfrak{a}}]$. Consequently, $\mathfrak{g}\text{-Cl}_{\mathfrak{a},0}(\mathscr{S}_{\mathfrak{a}}) \supseteq \mathfrak{g}\text{-Cl}_{\mathfrak{a},1}(\mathscr{S}_{\mathfrak{a}}) \supseteq \mathfrak{g}\text{-Cl}_{\mathfrak{a},3}(\mathscr{S}_{\mathfrak{a}}) \subseteq \mathfrak{g}\text{-Cl}_{\mathfrak{a},2}(\mathscr{S}_{\mathfrak{a}})$ for any $\mathscr{S}_{\mathfrak{a}} \in \mathfrak{T}_{\mathfrak{a}}$. But, $\mathscr{K}_{\mathfrak{a}} \supseteq \neg\,\text{op}_{\mathfrak{o},\nu}(\mathscr{K}_{\mathfrak{a}}) \supseteq \neg\,\text{op}_{\mathfrak{g},\nu}(\mathscr{K}_{\mathfrak{a}})$ for every $\nu \in I_3^0$, implying, $\mathfrak{g}\text{-Cl}_{\mathfrak{o},\nu}(\mathscr{S}_{\mathfrak{a}}) \supseteq \mathfrak{g}\text{-Cl}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{a}})$ for any $(\nu, \mathscr{S}_{\mathfrak{a}}) \in I_3^0 \times \mathfrak{T}_{\mathfrak{g}}$. Hence, this diagram, which is to be read horizontally, from left to right and vertically, from top to bottom, follows:

$$\begin{array}{ccccccc} \mathscr{K}_{\mathfrak{a}} & = & \mathscr{K}_{\mathfrak{a}} & = & \mathscr{K}_{\mathfrak{a}} & = & \mathscr{K}_{\mathfrak{a}} \\ \cup\!| & & \cup\!| & & \cup\!| & & \cup\!| \\ \neg\,\text{op}_{\mathfrak{o},0}(\mathscr{K}_{\mathfrak{a}}) & \supseteq & \neg\,\text{op}_{\mathfrak{o},1}(\mathscr{K}_{\mathfrak{a}}) & \supseteq & \neg\,\text{op}_{\mathfrak{o},3}(\mathscr{K}_{\mathfrak{a}}) & \subseteq & \neg\,\text{op}_{\mathfrak{o},2}(\mathscr{K}_{\mathfrak{a}}) \\ \cup\!| & & \cup\!| & & \cup\!| & & \cup\!| \\ \neg\,\text{op}_{\mathfrak{g},0}(\mathscr{K}_{\mathfrak{a}}) & \supseteq & \neg\,\text{op}_{\mathfrak{g},1}(\mathscr{K}_{\mathfrak{a}}) & \supseteq & \neg\,\text{op}_{\mathfrak{g},3}(\mathscr{K}_{\mathfrak{a}}) & \subseteq & \neg\,\text{op}_{\mathfrak{g},2}(\mathscr{K}_{\mathfrak{a}}). \end{array} \tag{4.2}$$

In FIG. 2, we present the relationships between the elements of the collections $\{\mathfrak{g}\text{-Cl}_{\mathfrak{o},\nu}(\mathscr{S}_{\mathfrak{a}}) : \nu \in I_3^0\}$ in the $\mathscr{T}_{\mathfrak{o}}$-space $\mathfrak{T}_{\mathfrak{o}}$ and $\{\mathfrak{g}\text{-Cl}_{\mathfrak{g},\nu}(\mathscr{S}_{\mathfrak{a}}) : \nu \in I_3^0\}$ in the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}}$; FIG. 2 may well be called a $(\mathfrak{g}\text{-Cl}_{\mathfrak{o}}, \mathfrak{g}\text{-Cl}_{\mathfrak{g}})$-*valued diagram.*

As in the works of other authors [44, 45, 46, 47], the manner we have positioned the arrows in the $(\mathfrak{g}\text{-Int}_{\mathfrak{o}}, \mathfrak{g}\text{-Int}_{\mathfrak{g}})$, $(\mathfrak{g}\text{-Cl}, \mathfrak{g}\text{-Cl}_{\mathfrak{g}})$-*valued diagrams* (FIGS 1, 2) is solely to stress that, in general, the implications in FIGS 1, 2 are irreversible.

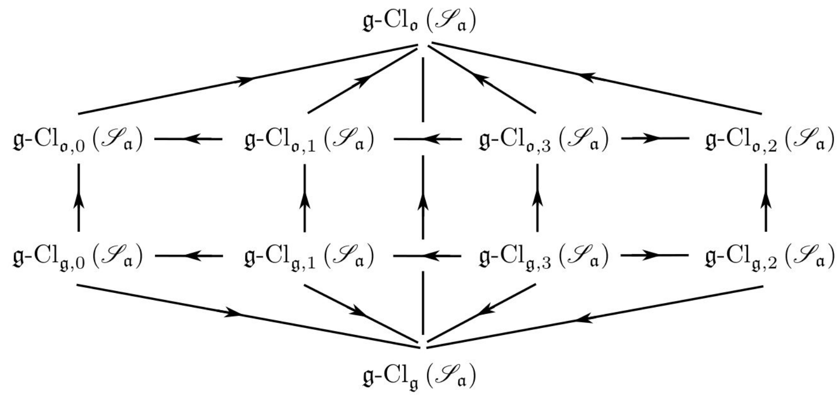


FIGURE 2. Relationships: $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{o}}$-closure operators in $\mathscr{T}_{\mathfrak{o}}$-spaces and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}_{\mathfrak{g}}$-spaces.

4.2. **A Nice Application.** The focus is on essential concepts from the standpoint of the theory of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in an attempt to shed lights on the essential properties established in the earlier sections. Let $\Omega = \big\{\xi_\nu : \nu \in I_5^*\big\}$ denotes the underlying set and consider the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$, where $\Omega$ is topologized by the choice:

$$\begin{aligned} \mathscr{T}_{\mathfrak{g}}(\Omega) &= \big\{\emptyset,\ \{\xi_1\},\ \{\xi_1,\xi_3,\xi_5\},\ \Omega\big\} && (4.3)\\ &= \big\{\mathscr{O}_{\mathfrak{g},1},\ \mathscr{O}_{\mathfrak{g},2},\ \mathscr{O}_{\mathfrak{g},3},\ \mathscr{O}_{\mathfrak{g},4}\big\};\\ \neg\mathscr{T}_{\mathfrak{g}}(\Omega) &= \big\{\Omega,\ \{\xi_2,\xi_3,\xi_4,\xi_5\},\ \{\xi_2,\xi_4\},\emptyset\big\} && (4.4)\\ &= \big\{\mathscr{K}_{\mathfrak{g},1},\ \mathscr{K}_{\mathfrak{g},2},\ \mathscr{K}_{\mathfrak{g},3},\ \mathscr{K}_{\mathfrak{g},4}\big\}. \end{aligned}$$

Evidently, $\mathscr{T}_{\mathfrak{g}}, \neg\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}\big(\{\xi_\nu : \nu \in I_5^*\}\big)$ establish the classes of $\mathscr{T}_{\mathfrak{g}}$-open and $\mathscr{T}_{\mathfrak{g}}$-closed sets, respectively. Since conditions $\mathscr{T}_{\mathfrak{g}}(\emptyset) = \emptyset$, $\mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu}) \subseteq \mathscr{O}_{\mathfrak{g},\nu}$ for every $\nu \in I_4^*$, $\mathscr{T}_{\mathfrak{g}}(\Omega) = \Omega$, and $\mathscr{T}_{\mathfrak{g}}\big(\bigcup_{\nu\in I_4^*}\mathscr{O}_{\mathfrak{g},\nu}\big) = \bigcup_{\nu\in I_4^*}\mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})$ are satisfied, $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}\big(\{\xi_\nu : \nu \in I_5^*\}\big)$ is a strong $\mathfrak{g}$-topology and hence, $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is a strong $\mathscr{T}_{\mathfrak{g}}$-space. Because $\mathscr{T}_{\mathfrak{g}}\big(\bigcap_{\nu\in I_4^*}\mathscr{O}_{\mathfrak{g},\nu}\big) = \bigcap_{\nu\in I_4^*}\mathscr{T}_{\mathfrak{g}}(\mathscr{O}_{\mathfrak{g},\nu})$ is satisfied, $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}\big(\{\xi_\nu : \nu \in I_5^*\}\big)$ is also an $\mathfrak{o}$-topology and thus, $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ is a $\mathscr{T}_{\mathfrak{o}}$-space $\mathfrak{T}_{\mathfrak{o}} = (\Omega, \mathscr{T}_{\mathfrak{o}})$. Moreover, $\mathscr{O}_{\mathfrak{g},\mu} \in \mathfrak{g}$-$\nu$-$\mathrm{O}\big[\mathfrak{T}_{\mathfrak{o}}\big]$ for every $(\nu,\mu) \in I_3^0 \times I_4^*$. Thus, the $\mathscr{T}_{\mathfrak{g}}$-open sets forming the $\mathfrak{g}$-topology $\mathscr{T}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}\big(\{\xi_\nu : \nu \in I_5^*\}\big)$ of the $\mathscr{T}_{\mathfrak{g}}$-space $\mathfrak{T}_{\mathfrak{g}} = (\Omega, \mathscr{T}_{\mathfrak{g}})$ are $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{o}}$-open sets relative to the $\mathscr{T}_{\mathfrak{o}}$-space $\mathfrak{T}_{\mathfrak{o}} = (\Omega, \mathscr{T}_{\mathfrak{o}})$.

For convenience of notation, express $\mathscr{P}(\Omega)$ in set-builder notation as a collection indexed by the Cartesian product $I^*_{\mathrm{card}(\mathscr{P}(\Omega))} \times I^0_{\mathrm{card}(\Omega)}$:

$$\mathscr{P}(\Omega) \;=\; \big\{\mathscr{S}_{\mathfrak{g},(\nu,\mu)} \in \mathscr{P}(\Omega) :\ (\nu,\mu) \in I^*_{\mathrm{card}(\mathscr{P}(\Omega))} \times I^0_{\mathrm{card}(\Omega)}\big\}, \qquad (4.5)$$

where $\mathscr{S}_{\mathfrak{g},(\nu,\mu)} \in \mathscr{P}(\Omega)$ denotes a $\mathfrak{T}_{\mathfrak{g}}$-set labeled $\nu \in I^*_{\mathrm{card}(\mathscr{P}(\Omega))}$ and containing $\mu \in I^0_{\mathrm{card}(\Omega)}$ elements. Below is established the indexing by the Cartesian product $I^*_{\mathrm{card}(\mathscr{P}(\Omega))} \times I^0_{\mathrm{card}(\Omega)}$ by the choice: $\mathscr{S}_{\mathfrak{g},(1,0)} = \emptyset$, …, $\mathscr{S}_{\mathfrak{g},(\nu,\mu)} = \big\{\xi_1,\xi_2,\ldots,\xi_\mu\big\}$, …, $\mathscr{S}_{\mathfrak{g},(32,5)} = \Omega$.

For $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ such that $\mathrm{card}(\mathscr{S}_{\mathfrak{g}}) \in \{0,5\}$, let $\mathscr{S}_{\mathfrak{g},(1,0)} = \emptyset$ and $\mathscr{S}_{\mathfrak{g},(32,5)} = \Omega$. For $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ such that $\mathrm{card}(\mathscr{S}_{\mathfrak{g}}) \in \{1,4\}$, let $\mathscr{S}_{\mathfrak{g},(2,1)} = \{\xi_1\}$, $\mathscr{S}_{\mathfrak{g},(3,1)} = \{\xi_2\}$,

$\mathscr{S}_{\mathfrak{g},(4,1)} = \{\xi_3\}$, $\mathscr{S}_{\mathfrak{g},(5,1)} = \{\xi_4\}$, and $\mathscr{S}_{\mathfrak{g},(6,1)} = \{\xi_5\}$; $\mathscr{S}_{\mathfrak{g},(27,4)} = \{\xi_1, \xi_2, \xi_3, \xi_4\}$, $\mathscr{S}_{\mathfrak{g},(28,4)} = \{\xi_2, \xi_3, \xi_4, \xi_5\}$, $\mathscr{S}_{\mathfrak{g},(29,4)} = \{\xi_1, \xi_3, \xi_4, \xi_5\}$, $\mathscr{S}_{\mathfrak{g},(30,4)} = \{\xi_1, \xi_2, \xi_3, \xi_5\}$, and $\mathscr{S}_{\mathfrak{g},(31,4)} = \{\xi_1, \xi_2, \xi_4, \xi_5\}$. For $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ such that $\operatorname{card}(\mathscr{S}_{\mathfrak{g}}) \in \{2, 3\}$, let $\mathscr{S}_{\mathfrak{g},(7,2)} = \{\xi_1, \xi_2\}$, $\mathscr{S}_{\mathfrak{g},(8,2)} = \{\xi_1, \xi_3\}$, $\mathscr{S}_{\mathfrak{g},(9,2)} = \{\xi_1, \xi_4\}$, $\mathscr{S}_{\mathfrak{g},(10,2)} = \{\xi_1, \xi_5\}$, $\mathscr{S}_{\mathfrak{g},(11,2)} = \{\xi_2, \xi_3\}$, $\mathscr{S}_{\mathfrak{g},(12,2)} = \{\xi_2, \xi_4\}$, $\mathscr{S}_{\mathfrak{g},(13,2)} = \{\xi_2, \xi_5\}$, $\mathscr{S}_{\mathfrak{g},(14,2)} = \{\xi_3, \xi_4\}$, $\mathscr{S}_{\mathfrak{g},(15,2)} = \{\xi_3, \xi_5\}$, and $\mathscr{S}_{\mathfrak{g},(16,2)} = \{\xi_4, \xi_5\}$; $\mathscr{S}_{\mathfrak{g},(17,3)} = \{\xi_1, \xi_2, \xi_3\}$, $\mathscr{S}_{\mathfrak{g},(18,3)} = \{\xi_1, \xi_3, \xi_4\}$, $\mathscr{S}_{\mathfrak{g},(19,3)} = \{\xi_1, \xi_4, \xi_5\}$, $\mathscr{S}_{\mathfrak{g},(20,3)} = \{\xi_1, \xi_2, \xi_4\}$, $\mathscr{S}_{\mathfrak{g},(21,3)} = \{\xi_1, \xi_2, \xi_5\}$, $\mathscr{S}_{\mathfrak{g},(22,3)} = \{\xi_1, \xi_3, \xi_5\}$, $\mathscr{S}_{\mathfrak{g},(23,3)} = \{\xi_2, \xi_3, \xi_4\}$, $\mathscr{S}_{\mathfrak{g},(24,3)} = \{\xi_2, \xi_3, \xi_5\}$, $\mathscr{S}_{\mathfrak{g},(25,3)} = \{\xi_3, \xi_4, \xi_5\}$, and $\mathscr{S}_{\mathfrak{g},(26,3)} = \{\xi_2, \xi_4, \xi_5\}$.

Then, from a series of calculations it results that

$$\begin{aligned} \operatorname{int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\big) \subseteq \mathfrak{g}\text{-}\operatorname{Int}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\big) &= \mathscr{S}_{\mathfrak{g},(\nu,\mu)} \\ &= \mathfrak{g}\text{-}\operatorname{Cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\big) \subseteq \operatorname{cl}_{\mathfrak{g}}\big(\mathscr{S}_{\mathfrak{g},(\nu,\mu)}\big) \end{aligned} \tag{4.6}$$

for every $(\nu, \mu) \in I^{*}_{\operatorname{card}(\mathscr{P}(\Omega))} \times I^{0}_{\operatorname{card}(\Omega)}$. On inspecting Eq. (4.6), some interesting features can be remarked and thus, some interesting conclusions can be drawn.

Having ordered the $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior operators $\operatorname{int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\operatorname{Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, by setting $\mathfrak{g}\text{-}\operatorname{Int}_{\mathfrak{g}} \succsim \operatorname{int}_{\mathfrak{g}}$ if and only if $\mathfrak{g}\text{-}\operatorname{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \operatorname{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and the $\mathfrak{T}_{\mathfrak{g}}$, $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $\operatorname{cl}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\operatorname{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively, by setting $\mathfrak{g}\text{-}\operatorname{Cl}_{\mathfrak{g}} \precsim \operatorname{cl}_{\mathfrak{g}}$ if and only if $\mathfrak{g}\text{-}\operatorname{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \operatorname{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, where $\mathscr{S}_{\mathfrak{g}} \in \mathscr{P}(\Omega)$ is arbitrary, Eq. (4.6), then, is but a result validating the following outstanding facts: $\mathfrak{g}\text{-}\operatorname{Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\operatorname{int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\operatorname{int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-}\operatorname{Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$; $\mathfrak{g}\text{-}\operatorname{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\operatorname{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\operatorname{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-}\operatorname{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$.

If the discussions of this nice application be explored a step further, other interesting conclusions can be drawn.

## 5. Conclusion

In this paper, the notions of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}_{\mathfrak{g}}$-spaces were presented in as general and unified a manner as possible and, their essential properties were discussed in such a way as to show that much of the fundamental structure of $\mathscr{T}_{\mathfrak{g}}$-spaces is better considered for $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators $\mathfrak{g}\text{-}\operatorname{Int}_{\mathfrak{g}}$, $\mathfrak{g}\text{-}\operatorname{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ than for the $\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{T}_{\mathfrak{g}}$-closure operators $\operatorname{int}_{\mathfrak{g}}$, $\operatorname{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$, respectively. If $\mathfrak{g}\text{-}\operatorname{Int}_{\mathfrak{g}} \succsim \operatorname{int}_{\mathfrak{g}}$ stands for $\mathfrak{g}\text{-}\operatorname{Int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \supseteq \operatorname{int}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$ and $\mathfrak{g}\text{-}\operatorname{Cl}_{\mathfrak{g}} \precsim \operatorname{cl}_{\mathfrak{g}}$, for $\mathfrak{g}\text{-}\operatorname{Cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}}) \subseteq \operatorname{cl}_{\mathfrak{g}}(\mathscr{S}_{\mathfrak{g}})$, then the outstanding facts are: $\mathfrak{g}\text{-}\operatorname{Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\operatorname{int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\operatorname{int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\mathfrak{g}\text{-}\operatorname{Int}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$; $\mathfrak{g}\text{-}\operatorname{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *coarser* (or, *smaller*, *weaker*) than $\operatorname{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ or, $\operatorname{cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$ is *finer* (or, *larger*, *stronger*) than $\mathfrak{g}\text{-}\operatorname{Cl}_{\mathfrak{g}} : \mathscr{P}(\Omega) \longrightarrow \mathscr{P}(\Omega)$.

Moreover, the paper offers very nice features for the passage from $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-(interior, closure) to $\mathfrak{T}_{\mathfrak{g}}$-(interior, closure) operators, respectively. Hence, several concepts and proven results it contained hold equally well when $(\Omega, \mathscr{T}_{\mathfrak{g}}) = (\Omega, \mathscr{T}_{\mathfrak{o}})$, while adapting other set-theoretic and topological features accordingly. For instance, the theoretical framework categorises $\big(\mathfrak{g}\text{-}\operatorname{Int}_{\mathfrak{a},\nu}(\mathscr{S}_{\mathfrak{a}}), \mathfrak{g}\text{-}\operatorname{Cl}_{\mathfrak{a},\nu}(\mathscr{S}_{\mathfrak{a}})\big)$ as a pair of $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-open and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{a}}$-closed sets of categories $\nu$, where $\mathscr{S}_{\mathfrak{a}} \subset \mathfrak{T}_{\mathfrak{a}}$ and $(\nu, \mathfrak{a}) \in I^{0}_{3} \times \{\mathfrak{o}, \mathfrak{g}\}$, and theorises the concepts in a unified way.

The study of the commutativity of the $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-interior and $\mathfrak{g}$-$\mathfrak{T}_{\mathfrak{g}}$-closure operators in $\mathscr{T}_{\mathfrak{g}}$-spaces will be presented in a subsequent paper, and the discussion of this paper ends here.

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