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WEAKLY NIL-CLEAN INDEX AND UNIQUELY WEAKLY NIL-CLEAN RINGS

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Abstract. We introduce and study *the weakly nil-clean index* associated to a ring. We also give some simple properties of this index and show that rings with the weakly nil-clean index 1 are precisely those rings that are abelian weakly nil-clean, thus showing that they coincide with uniquely weakly nil-clean rings. Next, we define certain types of nilpotent elements and weakly nil-clean decompositions by obtaining some results when the weakly nil-clean index is at most 2 and, moreover, we somewhat characterize rings with weakly nil-clean index 2. After that, we compute the weakly nil-clean index for $\mathbb{T}_2(\mathbb{Z}_p)$, $\mathbb{T}_3(\mathbb{Z}_p)$ and $\mathbb{M}_2(\mathbb{Z}_3)$, respectively, as well as we establish a result on the weakly nil-clean index of $\mathbb{M}_n(R)$ whenever R is a ring. Our results considerably extend and correct the corresponding ones from [Int. Electron. J. Algebra 15(2014), 145–156].

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1. Introduction and background

All rings R in this paper are associative with 1, but not necessarily commutative. The symbols $U(R)$, $J(R)$, $Id(R)$, and $Nil(R)$ will stand for the group of units, the Jacobson radical, the set of idempotents and the set of nilpotents of R, respectively. Also, for $e \in R$, we define $C(e) = \{x \in R \mid ex = xe\}$. All other unexplained explicitly below notion and notation are standard and follow essentially those from [9]. For instance, $\mathbb{M}_n(R)$ denotes the full $n \times n$ matrix ring and $\mathbb{T}_n(R)$ denotes the upper triangular $n \times n$ matrix ring.

In [8] a ring R is said to be *nil-clean* if each element $a \in R$ can be represented as $a = b + e$, where $b \in Nil(R)$ and $e \in Id(R)$; note that this is equivalent to the presentation that, for every $a \in R$, we have $a = b - e$. If this presentation is unique, the ring R is called *uniquely nil-clean*. This is tantamount to the requirement that the existing idempotent e is unique (see, e.g., [5,8]).

On the other vein, in [3] and [7] was stated the definition of a *weakly nil-clean* ring as such a ring R for which any element $a \in R$ is of the form $a = b + e$ or $a = b - e$, where $b \in Nil(R)$ and $e \in Id(R)$. Moreover, a ring R is said to be *uniquely weakly nil-clean* if the existing idempotent e is unique.

This work is motivated by the notions of unique nil-cleanness and weak nil-cleanness and we will combine them into a new concept. So, the aim of the current paper is to explore some variations of unique weak nil-cleanness in order to enlarge the principal known results on unique nil-cleanness from [5] and [6]. Although weakly nil-clean rings were recently completely characterized independently in [6] and [12], the full description of uniquely weakly nil-clean rings remains interesting and worthy of exploration. For any $a \in R$, let $\mathcal{E}(a) = \{e \in R \mid e^2 = e,\ a - e \in U(R)\}$ and then the *clean index* of R, denoted as $c(R)$, is defined in [10] by $c(R) = sup\{|\mathcal{E}(a)| : a \in R\}$. For any $a \in R$, set $\eta(a) = \{e \in R \mid e^2 = e$ and $a - e \in Nil(R)\}$ and then the *nil-clean index* of R, denoted as $Nin(R)$, is defined in [1] by $sup\{|\eta(a)| : a \in R\}$. In this way, for a more comprehensive investigation of these two notions and, especially, as a natural generalization of the nil-clean index, we also define the concept of *weakly nil-clean index* of a ring. Thereby, as it will be showed below, a ring is uniquely weakly nil-clean if and only if it is weakly nil-clean of weakly nil-clean index 1.

The paper is organized as follows: In the first section, we already have given the main definitions of the used concepts. In the second section, we set and explore in details the weakly nil-clean index of rings and discuss the original notion of uniquely weakly nil-clean rings stated in Problem 3 of [7]. We also investigate here some other aspects of unique weak nil-cleanness which arise from its specific definition. And we close the work in the final third section by stating certain open problems of some interest and importance.

2. Weakly nil-clean index of rings

In [10] and [11] the clean index $c(R)$ of a ring R was defined and studied. Imitating this, in [1] was introduced the nil-clean index $Nin(R)$ of R and a detailed study was given.

In parallel to these two notions, we proceed by stating the following concepts.

Definition 2.1. Let R be a ring and $a \in R$. We define the set

$$\alpha(a) = \{e \in R : e^2 = e \text{ and } a - e \text{ or } a + e \text{ is a nilpotent}\}.$$

Definition 2.2. For an element $a \in R$ the *weakly nil-clean index* of a, abbreviated as $wnc(a)$, is defined to be the cardinality of the set $\alpha(a)$.

Definition 2.3. We define the *weakly nil-clean index* of a ring R as follows:

$$wnc(R) = \sup\{|\alpha(a)| : a \in R\}.$$

We foremost start with a series of elementary but useful basic properties of the operator $wnc(R)$ which extend the analogous ones in [1].

Lemma 2.4. *For any ring R the inequality $wnc(R) \geq 1$ holds. In addition, if R is a ring which has at most n idempotents or at most n nilpotents, then $wnc(R) \leq n$.*

Proof. Straightforward. □

Example 2.5. *A direct check shows that $wnc(\mathbb{Z}_3) = 1$.*

Lemma 2.6. *If R is a ring with a subring S, then $wnc(R) \geq wnc(S)$.*

Proof. Follows in the same manner as [1, Lemma 2.2]. □

Lemma 2.7. *If R is a ring with a nil ideal I, then $wnc(R/I) \leq wnc(R)$.*

Proof. Letting $a \in R$ be an arbitrary element, then for any idempotent $b + I \in \alpha(a+I)$, so $b^2 - b \in I$ and there exists $e \in Id(R)$ with $b+I = e+I$, one may derive that $(a+I)-(b+I) = (a-e)+I$ with $(a-e)^t \in I$ or that $(a+I)+(b+I) = (a+e)+I$ with $(a+e)^t \in I$ for some $t \in \mathbb{N}$. Since I is nil, it follows that either $a-e \in Nil(R)$ or $a+e \in Nil(R)$. Consequently, $e \in \alpha(a)$ and thus $|\alpha(a)| \geq |\alpha(a+I)|$, as needed. □

Remark 2.8. *In* [1, Lemma 2.4 (1)] *the condition "If idempotents lift modulo I" is redundant, because I is a nil-ideal. Moreover, the inequality $Nin(R/I) \geq Nin(R)$ is not true and the purported there proof is errorneous. This can be subsumed via the following construction: set $R = \mathbb{Z}_p$ and $I = \{(a_{ij}) \in \mathbb{T}_n(R) : \forall a_{ii} = 0\}$. It is readily seen that this is a nil-ideal of $\mathbb{T}_n(R)$ with the property that $\mathbb{T}_n(R)/I \cong R \times \cdots \times R$, where the product is taken n times.*

Next, choosing $n = 2 = p$, we detect that $\mathbb{T}_2(\mathbb{Z}_2)/I \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, whence with the aid of [1, Lemma 2.3] *we derive $Nin(\mathbb{T}_2(\mathbb{Z}_2)/I) = Nin(\mathbb{Z}_2 \times \mathbb{Z}_2) = Nin(\mathbb{Z}_2)Nin(\mathbb{Z}_2) = 1 \cdot 1 = 1$. On the other hand,* [8, Theorem 4.1] *is a guarantor that $\mathbb{T}_2(\mathbb{Z}_2)$ is nil-clean, so that $Nin(\mathbb{T}_2(\mathbb{Z}_2)) = wnc(\mathbb{T}_2(\mathbb{Z}_2)) = 2$, owing to Example 2.23 listed below. Thus this contradiction demonstrates $Nin(R/I) < Nin(R)$.*

If now we choose $n = 3 = p$, then the same trick successfully works to manifestly illustrate with the help of Example 2.24 quoted below that $wnc(R) > wnc(R/I)$.

The next assertion improves [1, Lemma 2.8].

Lemma 2.9. *For any ring R the inequalities $c(R) \geq wnc(R) \geq Nin(R)$ hold.*

Proof. Since the second inequality is trivial, we will deal only with the first one. To that goal, for any $a \in R$, writing that $a = q + e$ or $a = q - e$ for some nilpotent q and idempotent e, we deduce that $a + 1 = (q + 1) + e$ with a unit $q + 1$ or that $a = (q - 1) + (1 - e)$ with a unit $q - 1$, so that both $a + 1$ and a are clean elements. Since $a + 1 \in R$, the further proof goes on as in [1, Lemma 2.8]. □

Remark 2.10. *Note that if R is a nil-clean ring, then $wnc(R) = Nin(R)$.*

The next assertion extends [1, Theorem 3.2].

Proposition 2.11. *Suppose R is a ring. Then $wnc(R) = 1$ if and only if R is abelian.*

Proof. First of all we will prove that $wnc(R) = 1$ if and only if R is abelian and for any non-zero idempotent $e \in R$, the relation $e \neq m + n$ holds for all $m, n \in Nil(R)$.

Since with Lemma 2.9 at hand we have $1 = wnc(R) \geq Nin(R) \geq 1$, it follows that $Nin(R) = 1$ and by [1, Lemma 3.1] we get that R is abelian and for any idempotent $0 \neq e \in R$, the ratio $e \neq m + n$ is valid for all $m, n \in Nil(R)$.

Now let R be abelian and, for any idempotent $e \in R \setminus \{0\}$, the inequality $e \neq m + n$ is true for all $m, n \in Nil(R)$. Suppose, for concreteness, $a \in R$ has two weakly nil-clean decompositions. We have three possible cases:

(1) $a = e_1 + n_1 = e_2 + n_2$, with e_1 and e_2 idempotents and $n_1, n_2 \in Nil(R)$. In this case the decompositions are actually nil-clean, so this situation was handled in [1, Lemma 3.1] and leaded to $e_1 = e_2$. It follows now that $wnc(R) = 1$.

(2) $a = -e_1 + n_1 = -e_2 + n_2$, with e_1 and e_2 idempotents and $n_1, n_2 \in Nil(R)$. Then $-e_1(1-e_1)+n_1(1-e_1) = -e_2(1-e_1)+n_2(1-e_1)$, so $e_2(1-e_1) = n_2(1-e_1) - n_1(1-e_1)$. Since R is abelian, the element $e_2(1-e_1)$ is an idempotent and both $n_2(1-e_1)$, $n_1(1-e_1)$ are nilpotents. So, by hypothesis, we get $e_2(1-e_1) = 0$, that is $e_2 = e_1e_2$. Consequently, $n_1 - n_2 = e_1 - e_2 = e_1(1-e_2)$, and hence by hypothesis we derive that $e_1(1-e_2) = 0$. Thus $e_1 = e_1e_2$ and $e_1 = e_2$. It again follows that $wnc(R) = 1$.

(3) $a = -e_1 + n_1 = e_2 + n_2$, with e_1 and e_2 idempotents and $n_1, n_2 \in Nil(R)$. Then $e_2(1-e_2) + n_2(1-e_2) = -e_1(1-e_2) + n_1(1-e_2)$ and so $e_1(1-e_2) = n_1(1-e_2) - n_2(1-e_2)$. Thus $e_1(1-e_2) = 0$, i.e., $e_1 = e_1e_2$.

Let $n_1^m = 0$ and $f = e_1 + e_2$. Now lifting $f + n_2 = n_1$ to the $m - th$ power, we obtain that $\sum_{k=0}^{m} \binom{m}{k} f^{m-k} n_2^k = 0$. But $f^k = (e_1 + e_2)^k = \sum_{l=0}^{m} \binom{m}{k} e_1^{k-l} e_2^k = e_1 + e_2 + e_1 e_2 (2^k - 2) = e_1 + e_2 + e_1 (2^k - 2) = (2^k - 1)e_1 + e_2$, so $\sum_{k=0}^{m} \binom{m}{k}((2^{m-k} - 1)e_1 + e_2)n_2^k = 0$, which gives $e_1 \sum_{k=0}^{m} \binom{m}{k} 2^{m-k} n_2^k + (e_2 - e_1) \sum_{k=0}^{m} \binom{m}{k} n_2^k = 0$. This is equivalent to $e_1(2+n_2)^k + (e_2 - e_1)(n_2 + 1)^k = 0$. Hence $e_2(n_2+1)^k + e_1((2+n_2)^k - (1+n_2)^k) = 0$. Multiplying by $(1-e_1)$ we get $(1-e_1)e_2(n_2+1)^k = 0$, but n_2+1 is a unit, so $e_2 = e_1 e_2$ and from $e_1 = e_1 e_2$ we have $e_1 = e_2$. It follows once again that $wnc(R) = 1$.

Knowing that $wnc(R) = 1$ if and only if R is abelian and for any non-zero idempotent $e \in R$, the relation $e \neq m + n$ holds for all $m, n \in Nil(R)$ and by Lemma 3.1 from [1], we infer that $wnc(R) = 1$ if and only if $Nin(R) = 1$ and now using Theorem 3.2 from [1] we get the desired result. □

We will now consider the special case of rings having the weakly nil-clean index one and shall completely characterize them. Notice once again that weakly nil-clean rings are independently classified in [6] and [12], respectively. So, we come now to one of our basic statements which does *not* follow directly by the cited result.

Theorem 2.12. *The following are equivalent for a ring R:*

(1) *R is uniquely weakly nil-clean;*

(2) *R is abelian weakly nil-clean;*

(3) *$R \cong R_1 \times R_2$, where R_1 is either 0 or an abelian nil-clean ring with $J(R_1)$ nil and $R_1/J(R_1) \cong B$, where B is a Boolean ring, and R_2 is either 0 or a local weakly nil-clean ring such that $J(R_2)$ is nil and $R_2/J(R_2) \cong \mathbb{Z}_3$.*

Proof. (1) ⇔ (2) This is a direct consequence of Proposition 2.11.

(2) ⇔ (3) It follows directly from [3]. □

We recall from [5, Theorem 5.4] that a ring R is uniquely nil-clean if and only if R is abelian nil-clean. So, with Theorem 2.12 at hand, one can deduce the following.

Corollary 2.13. *A ring R is uniquely nil-clean if and only if R is uniquely weakly nil-clean and $2 \in J(R)$.*

As a connection to strongly π-regular rings, one may state the following strengthening of results on unique nil-cleanness of rings from [5] and [8].

Corollary 2.14. *A ring R is uniquely weakly nil-clean if and only if R is abelian strongly π-regular such that $R/J(R)$ is isomorphic to either a Boolean ring, or to $\mathbb{Z}_3$, or to the direct product of two such rings.*

Proof. It is well known that strongly π-regular rings R have nil $J(R)$. We therefore employ [3] and Theorem 2.12 to get what we asserted. $\square$

Remark 2.15. *We shall now explore two various notions of unique weak nil-cleanness. At the beginning, if we use the "weak unicity" for a ring R, i.e., every element $r \in R$ can be written down in at most one way as a nil-clean element or $-r$ in at most one way as a nil-clean element, then we just obtain uniquely weakly nil-clean rings and vice versa.*

However, if we use the "strong unicity" for a ring R, i.e., every element $r \in R$ can be written down in a unique way as $n + f$, with n a nilpotent and f or $-f$ an idempotent, then such a ring is either uniquely nil-clean of characteristic 2 or uniquely weakly nil-clean but not nil-clean. This follows because we can write $-1 = 0 + (-1) = (-2) + 1$, so if $2 \neq 0$ we have that 2 is not a nilpotent.

Remark 2.16. *It is worthwhile noticing that indecomposable rings, and hence local rings, always have weakly nil-clean index one.*

Remark 2.17. *For any ring R and any $s \in R$, we set $P_s = es(1-e)$ and $P'_s = (1-e)se$.*

Let now R be a ring and $r \in R$. We then have the following weakly nil-clean decompositions for each idempotent e:

$$e = e + 0 = (e - P_r) + P_r = (e - P'_r) + P'_r = (e + P_r) - P_r = (e + P'_r) - P'_r.$$

Proposition 2.18. *Let R be a ring with $wnc(R) \leq 2$. Then, for any $s \in R$ and for any $e \in Id(R)$, we have $2es(1-e) = 0$.*

Proof. Let $e \in R$ be an idempotent and let $s \in R$.

If e is central, then $R = C(e)$, so for every $s \in R$ we obtain $es = se = ese$ and, therefore, $es(1-e) = 0$, hence $2es(1-e) = 0$. If e is not central, then there is $s \notin C(e)$ and so $P_s \neq 0$ or $P'_s \neq 0$. We have $e = e+0 = (e-P_s)+P_s = (e-P_{2s})+P_{2s}$ and by $wnc(R) \leq 2$ and $P_s \neq 0$ we get $P_{2s} = 0$ or $P_{2s} = P_s$. If $P_{2s} = P_s$, it follows $e2s(1-e) = es(1-e)$ and thus $es(1-e) = 0$, which is a contradiction because $P_s \neq 0$. Consequently, $P_{2s} = 0$, so $2es(1-e) = 0$. $\square$

Remark 2.19. *Another proof for Proposition 2.18 is as follows:*

Let e be an idempotent. We have

$$e = e + 0 = (e + er(1-e)) - er(1-e) = (e - er(1-e)) + er(1-e),$$

thus we get three weakly nil-clean decompositions of e. Therefore,

$$e = e \pm er(1-e) \text{ or } e + er(1-e) = e - er(1-e),$$

which is equivalent to

$$er(1-e) = 0 \text{ or } 2er(1-e) = 0.$$

Corollary 2.20. *Let R be a ring with $wnc(R) \leq 2$. Then, for any $s \in R$ and for any $e \in Id(R)$, we have $2(es - se) = 0$.*

Proof. Utilizing P_s as in Proposition 2.18, we obtain that $2es(1-e) = 0$. Now, considering P'_s, we have $2(1-e)se = 0$ and, therefore, $2es = 2ese = 2se$, so $2(es - se) = 0$. □

Proposition 2.21. *Let R be a ring with $wnc(R) \leq 2$ and $e \in Id(R)$. Then $|R/C(e)| \leq 2$.*

Proof. If we assume the contrary, $|R/C(e)| > 2$, then there are two different elements, say $s, t \notin C(e)$, such that $s - t \notin C(e)$. By using Remark 2.19 and $wnc(R) \leq 2$, we differ the following cases:

- $P_s = P_t$ and $P'_s = P'_t$, then $es(1-e) = et(1-e)$ and $(1-e)se = (1-e)te$, hence $e(s-t) = e(s-t)e$ and $(s-t)e = e(s-t)e$, so $e(s-t) = (s-t)e$, which is a contradiction.
- $P_s = 0$ and $P'_s = 0$, then $es = ese$ and $ese = se$, so $es = se$, which is a contradiction.
- $P_s = 0$ and $P'_t = 0$, then since s and t are not in $C(e)$, it follows $P'_s \neq 0$ and $P_t \neq 0$ and $P'_s = P_t$ and by this we get $e(1-e)se = eet(1-e)$, so $et(1-e) = 0$, which is a contradiction.
- $P_s = P_t$ and $P'_s = 0$, then $P'_t \neq 0$, so $P_s = P_t = P'_t$, which is a contradiction.

□

Proposition 2.22. *Let R be a ring and $e \in Id(R)$. Then*

$$|R/A(e)| \leq |\alpha(e)|,$$

where $A(e) = \{r \in R \mid er(1-e) = 0\}$.

Proof. Letting $n + 1 \leq |R/A(e)|$, then we can find an inclusion

$$\{A(e), r_1 + A(e), \ldots, r_n + A(e)\} \subseteq R/A(e).$$

So, for any r_i, r_j such that $i, j \in \{1, 2, \ldots, n\}$, we have $r_i + A(e) \neq r_j + A(e)$ and, therefore, $r_i - r_j \notin A(e)$. It follows that $P_{r_i - r_j} \neq 0$. Thus $P_{r_i} \neq P_{r_j}$. Also, for any

r_i, we have $r_i \notin A(e)$. Hence $P_{r_i} \neq 0$. So the set $\{0\} \cup \{P_{r_i} | i \in \{1, 2, \dots, n\}\}$ has $n+1$ elements and since for an idempotent e we get $e = e+0 = (e-P_{r_i})+P_{r_i}$ for any $i \in \{1, 2, \dots, n\}$, the desired inequality $|R/A(e)| \leq |\alpha(e)|$ follows, as asserted. □

We will now compute $wnc(R)$ for some concrete rings R. Specifically, we will show that the following equalities hold.

Example 2.23. *$wnc(\mathbb{T}_2(\mathbb{Z}_p)) = p$, where p is a prime number.*

Proof. It is a well-known fact that a matrix in $\mathbb{T}_2(\mathbb{Z}_p)$ is a nilpotent if and only if it has a zero principal diagonal. We are looking now for idempotents. In fact, $\begin{pmatrix} x_1 & a \\ \overline{0} & x_2 \end{pmatrix} \begin{pmatrix} x_1 & a \\ \overline{0} & x_2 \end{pmatrix} = \begin{pmatrix} x_1^2 & a(x_1+x_2) \\ \overline{0} & x_2^2 \end{pmatrix}$, hence $\begin{pmatrix} x_1 & a \\ \overline{0} & x_2 \end{pmatrix} = \begin{pmatrix} x_1^2 & a(x_1+x_2) \\ \overline{0} & x_2^2 \end{pmatrix}$, and thus $x_1, x_2 \in \{\overline{0}, \overline{1}\}$ and $a(x_1 + x_2 - \overline{1}) = \overline{0}$. Each pair (x_1, x_2) will give a set of solutions for the problem of idempotent matrices.

- case I : $x_1 = \overline{0}, x_2 = \overline{0}$, then $S_1 = \{\begin{pmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{0} \end{pmatrix}\}$;
- case II : $x_1 = \overline{0}, x_2 = \overline{1}$, then $S_2 = \{\begin{pmatrix} \overline{0} & \alpha \\ \overline{0} & \overline{1} \end{pmatrix}, \alpha \in \mathbb{Z}_p\}$;
- case III : $x_1 = \overline{0}, x_2 = \overline{1}$, then $S_3 = \{\begin{pmatrix} \overline{1} & \alpha \\ \overline{0} & \overline{0} \end{pmatrix}, \alpha \in \mathbb{Z}_p\}$;
- case IV : $x_1 = \overline{1}, x_2 = \overline{1}$, then $S_4 = \{\begin{pmatrix} \overline{1} & \overline{0} \\ \overline{0} & \overline{1} \end{pmatrix}\}$.

Let $A \in \mathbb{T}_2(\mathbb{Z}_p)$. Letting $A - E$ be a nilpotent, where E is an idempotent, then A has the main diagonal of the form of an idempotent diagonal (so it has $\overline{0}$ and/or $\overline{1}$). If $A+E$ is a nilpotent, with E an idempotent, then A has in the main diagonal an element from $\{\overline{0}, \overline{-1}\}$. Therefore, except for A with main zero diagonal, only one of the following can hold: $A+E$ or $A-E$ is a nilpotent, with E an idempotent.

Let A be with $\overline{0}$ or $-\overline{1}$ in the main diagonal. We look for m as big as possible such that $A + E_1, \cdots, A + E_m$ are nilpotents. Thus $E_1, \cdots, E_m$ share the same main diagonal, that is, they are in the same S_i. Hence the problem is reduced to finding the maximum cardinality of $S_i, i \in \{1, 2, 3, 4\}$. Also, trying to find out the maximum r such that $A - E_1, \cdots, A - E_r$ are nilpotents, with A having $\overline{0}$ and $\overline{1}$ in the main diagonal and $E_1, \cdots, E_r$ being idempotents leads to the same problem, finding the maximum cardinality of $S_i, i \in \{1, 2, 3, 4\}$. We finally conclude that

$|S_1| = |S_4| = 1$ and $|S_2| = |S_3| = p$, because the free variable α can take exactly the p values $\overline{0}, \overline{1}, \ldots, \overline{p-1}$. So, $wnc(\mathbb{T}_2(\mathbb{Z}_p)) = p$, as promised. □

Example 2.24. *$wnc(\mathbb{T}_3(\mathbb{Z}_p)) = p^2$, where p is a prime number.*

Proof. It is a well-known fact that a matrix in $\mathbb{T}_3(\mathbb{Z}_p)$ is a nilpotent if and only if it has a zero main diagonal. We are looking now for idempotents. In fact, $\begin{pmatrix} x_1 & a & b \\ \overline{0} & x_2 & c \\ \overline{0} & \overline{0} & x_3 \end{pmatrix}\begin{pmatrix} x_1 & a & b \\ \overline{0} & x_2 & c \\ \overline{0} & \overline{0} & x_3 \end{pmatrix} = \begin{pmatrix} x_1 & a & b \\ \overline{0} & x_2 & c \\ \overline{0} & \overline{0} & x_3 \end{pmatrix}$, which is equivalent to $\begin{pmatrix} x_1^2 & a(x_1+x_2) & b(x_1+x_3)+ac \\ \overline{0} & x_2^2 & c(x_2+x_3) \\ \overline{0} & \overline{0} & x_3^2 \end{pmatrix} = \begin{pmatrix} x_1 & a & b \\ \overline{0} & x_2 & c \\ \overline{0} & \overline{0} & x_3 \end{pmatrix}$, which is equivalent to

$$\begin{cases} x_1 \in \{\overline{0}, \overline{1}\} \\ x_2 \in \{\overline{0}, \overline{1}\} \\ x_3 \in \{\overline{0}, \overline{1}\} \\ a(x_1 + x_2 - \overline{1}) = \overline{0} \\ b(x_1 + x_3 - \overline{1}) = -ac \\ c(x_2 + x_3 - \overline{1}) = \overline{0} \end{cases}$$

For $x_1 = \overline{0}, x_2 = \overline{0}, x_3 = \overline{0}$, we have $S_1 = \{O_3\}$.

For $x_1 = \overline{0}, x_2 = \overline{0}, x_3 = \overline{1}$, we have $S_2 = \{\begin{pmatrix} \overline{0} & \overline{0} & \alpha \\ \overline{0} & \overline{0} & \gamma \\ \overline{0} & \overline{0} & \overline{1} \end{pmatrix} \mid \alpha, \gamma \in \mathbb{Z}_p\}$.

For $x_1 = \overline{0}, x_2 = \overline{1}, x_3 = \overline{0}$, we have $S_3 = \{\begin{pmatrix} \overline{0} & \alpha & \alpha\gamma \\ \overline{0} & \overline{1} & \gamma \\ \overline{0} & \overline{0} & \overline{0} \end{pmatrix} \mid \alpha, \gamma \in \mathbb{Z}_p\}$.

For $x_1 = \overline{0}, x_2 = \overline{1}, x_3 = \overline{1}$, we have $S_4 = \{\begin{pmatrix} \overline{0} & \alpha & \beta \\ \overline{0} & \overline{1} & \overline{0} \\ \overline{0} & \overline{0} & \overline{1} \end{pmatrix} \mid \alpha, \beta \in \mathbb{Z}_p\}$.

For $x_1 = \overline{1}, x_2 = \overline{0}, x_3 = \overline{0}$, we have $S_5 = \{\begin{pmatrix} \overline{0} & \alpha & \beta \\ \overline{0} & \overline{0} & \overline{0} \\ \overline{0} & \overline{0} & \overline{0} \end{pmatrix} \mid \alpha, \beta \in \mathbb{Z}_p\}$.

For $x_1 = \overline{1}, x_2 = \overline{0}, x_3 = \overline{1}$, we have $S_6 = \{\begin{pmatrix} \overline{1} & \alpha & -\alpha\gamma \\ \overline{0} & \overline{0} & \gamma \\ \overline{0} & \overline{0} & \overline{1} \end{pmatrix} \mid \alpha, \gamma \in \mathbb{Z}_p\}$.

For $x_1 = \overline{1}, x_2 = \overline{1}, x_3 = \overline{0}$, we have $S_7 = \{\begin{pmatrix} \overline{1} & \overline{0} & \alpha \\ \overline{0} & \overline{1} & \gamma \\ \overline{0} & \overline{0} & \overline{0} \end{pmatrix} \mid \alpha, \gamma \in \mathbb{Z}_p\}$.

For $x_1 = \overline{1}, x_2 = \overline{1}, x_3 = \overline{1}$, we have $S_8 = \{I_3\}$. Following the same argument as in Example 2.23, we derive that $wnc(\mathbb{T}_3(\mathbb{Z}_3))$ is the maximum cardinality of S_i, $i \in \{1, 2, \ldots, 8\}$. Since $|S_1| = |S_8| = 1$ and $|S_2| = |S_3| = \ldots = |S_7| = p^2$ (2 free variables and $|\mathbb{Z}_p| = p$), it finally follows that $wnc(\mathbb{T}_3(\mathbb{Z}_p)) = p^2$, as stated. $\square$

Remark 2.25. *When studying weakly nil-clean matrices, it is not enough to study companion matrices which are (or are not) blocks of other companion matrices. In fact, note that not all matrices are similar to a companion matrix (see the proof of the main result in* [2] *or* [4]*).*

Example 2.26. $wnc(\mathbb{M}_2(\mathbb{Z}_3)) = 5$.

Proof. Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathbb{M}_2(\mathbb{Z}_3)$. Then $A^2 = \begin{pmatrix} a^2 + bc & b(a+d) \\ c(a+d) & d^2 + bc \end{pmatrix}$. We claim $A^2 = A$ in order to find idempotents. They are the following:

$$\begin{pmatrix} \overline{0} & s \\ \overline{0} & \overline{1} \end{pmatrix}, \begin{pmatrix} \overline{1} & \overline{0} \\ s & 0 \end{pmatrix}, \begin{pmatrix} \overline{1} & \overline{0} \\ s & \overline{1} \end{pmatrix}, \begin{pmatrix} \overline{0} & \overline{0} \\ s & \overline{1} \end{pmatrix},$$

where $s \in \mathbb{Z}_3$ and also

$$\begin{pmatrix} \overline{2} & \overline{2} \\ \overline{2} & \overline{2} \end{pmatrix}, \begin{pmatrix} \overline{2} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix}.$$

Next, we claim $A^2 = O_2$ to find out nilpotents. They are the following:

$$\begin{pmatrix} \overline{1} & \overline{1} \\ \overline{2} & \overline{2} \end{pmatrix}, \begin{pmatrix} \overline{2} & \overline{1} \\ \overline{2} & \overline{1} \end{pmatrix}, \begin{pmatrix} \overline{2} & \overline{2} \\ \overline{1} & \overline{1} \end{pmatrix}, \begin{pmatrix} \overline{1} & \overline{2} \\ \overline{1} & \overline{2} \end{pmatrix}$$

and also

$$\begin{pmatrix} \overline{0} & s \\ \overline{0} & \overline{0} \end{pmatrix}, \begin{pmatrix} \overline{0} & \overline{0} \\ s & \overline{0} \end{pmatrix},$$

where $s \in \mathbb{Z}_3$.

If $A + E$, with E an idempotent, is nilpotent, then $tr(A + E) = 0$, whence $trA = -trE$. If $A - E$, with E an idempotent, is nilpotent, then $tr(A - E) = 0$, whence $trA = trE$.

For an idempotent E, we deduce:

- $trE = \overline{1}$ if and only if $E \neq O_2$, $E \neq I_2$;
- $trE = \overline{2}$ if and only if $E = I_2$;
- $trE = \overline{0}$ if and only if $E = O_2$.

Let $A = \begin{pmatrix} \overline{0} & y \\ \overline{1} & \overline{0} \end{pmatrix}$. Then, for an idempotent E, if $A+E$ is a nilpotent, then $tr(E) = 0$, and thus $E = O_2$. Also, if $A-E$ is a nilpotent, then $tr(E) = 0$ and hence $E = O_2$. Therefore, if $A = \begin{pmatrix} \overline{0} & y \\ \overline{1} & \overline{0} \end{pmatrix}$, we have $\alpha(A) = \{O_2\}$, and it follows that

$$wnc(\begin{pmatrix} \overline{0} & y \\ \overline{1} & \overline{0} \end{pmatrix}) \leq 1$$

such that $wnc(\begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{0} \end{pmatrix}) = 1$ and $wnc(\begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{0} \end{pmatrix}) = wnc(\begin{pmatrix} \overline{0} & \overline{2} \\ \overline{1} & \overline{0} \end{pmatrix}) = 0$.

Let $A = \begin{pmatrix} \overline{0} & y \\ \overline{1} & \overline{1} \end{pmatrix}$. Furthermore, for an idempotent E, if $A+E$ is a nilpotent, then $tr(E) = \overline{2}$, and so $E = I_2$. But $\begin{pmatrix} \overline{0} & y \\ \overline{1} & \overline{1} \end{pmatrix} + \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{0} & \overline{1} \end{pmatrix} = \begin{pmatrix} \overline{1} & y \\ \overline{1} & \overline{2} \end{pmatrix}$ is a nilpotent if and only if $y = \overline{2}$. Also, if $A-E$ is a nilpotent, then $tr(E) = 1$ and hence $E \neq O_2, I_2$.

We infer that

- $\begin{pmatrix} \overline{0} & y \\ \overline{1} & \overline{1} \end{pmatrix} - \begin{pmatrix} \overline{0} & s \\ \overline{0} & \overline{1} \end{pmatrix} = \begin{pmatrix} \overline{0} & y-s \\ \overline{1} & \overline{0} \end{pmatrix}$ is a nilpotent if and only if $s = y$;
- $\begin{pmatrix} \overline{0} & y \\ \overline{1} & \overline{1} \end{pmatrix} - \begin{pmatrix} \overline{1} & \overline{0} \\ s & \overline{0} \end{pmatrix} = \begin{pmatrix} \overline{2} & y \\ \overline{1}-s & \overline{1} \end{pmatrix}$ is a nilpotent if and only if ($y = \overline{2}$ and $s = \overline{0}$) or ($y = \overline{1}$ and $s = \overline{2}$);
- $\begin{pmatrix} \overline{0} & y \\ \overline{1} & \overline{1} \end{pmatrix} - \begin{pmatrix} \overline{1} & s \\ \overline{0} & \overline{0} \end{pmatrix} = \begin{pmatrix} \overline{1} & y-s \\ \overline{1} & \overline{1} \end{pmatrix}$, which is not a nilpotent;
- $\begin{pmatrix} \overline{0} & y \\ \overline{1} & \overline{1} \end{pmatrix} - \begin{pmatrix} \overline{0} & \overline{0} \\ s & \overline{1} \end{pmatrix} = \begin{pmatrix} \overline{0} & y \\ \overline{1}-s & \overline{1} \end{pmatrix}$, is a nilpotent if and only if ($y = \overline{0}$ and $s \in \mathbb{Z}_3$) or ($y \in \mathbb{Z}_3$ and $s = \overline{1}$);
- $\begin{pmatrix} \overline{0} & y \\ \overline{1} & \overline{1} \end{pmatrix} - \begin{pmatrix} \overline{2} & \overline{2} \\ \overline{2} & \overline{2} \end{pmatrix} = \begin{pmatrix} \overline{1} & y-\overline{2} \\ \overline{2} & \overline{2} \end{pmatrix}$, which is a nilpotent if and only if $y = \overline{0}$;
- $\begin{pmatrix} \overline{0} & y \\ \overline{1} & \overline{1} \end{pmatrix} - \begin{pmatrix} \overline{2} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix} = \begin{pmatrix} \overline{2} & y-\overline{1} \\ \overline{0} & \overline{2} \end{pmatrix}$, which is not a nilpotent.

By virtue of the above results, we get the following:

For $A = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{1} \end{pmatrix}$ we have $E_1 = \begin{pmatrix} \overline{0} & s \\ \overline{0} & \overline{1} \end{pmatrix}$, $E_2 = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{1} \end{pmatrix}$, $E_3 = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{1} \end{pmatrix}$, $E_4 = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{2} & \overline{1} \end{pmatrix}$, $E_5 = \begin{pmatrix} \overline{2} & \overline{2} \\ \overline{2} & \overline{2} \end{pmatrix}$ such that $A - E_i$ is a nilpotent ($i \in \{1,2,3,4,5\}$)

and there are no idempotents E such that $A+E$ is a nilpotent. So

$$\text{wnc }(\begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{1} \end{pmatrix}) = 5.$$

For $A = \begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{1} \end{pmatrix}$ we obtain the idempotents $E_1 = \begin{pmatrix} \overline{0} & \overline{1} \\ \overline{0} & \overline{1} \end{pmatrix}$, $E_2 = \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{2} & \overline{0} \end{pmatrix}$, $E_3 = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{1} \end{pmatrix}$ such that $A - E_i$ is a nilpotent, $i \in \{1,2,3\}$ and there are no idempotents E such that $A+E$ is a nilpotent. So

$$\text{wnc }(\begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{1} \end{pmatrix}) = 3.$$

For $A = \begin{pmatrix} \overline{0} & \overline{2} \\ \overline{1} & \overline{1} \end{pmatrix}$ we obtain the idempotents $E_1 = \begin{pmatrix} \overline{0} & \overline{2} \\ \overline{0} & \overline{1} \end{pmatrix}$, $E_2 = \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{0} & \overline{0} \end{pmatrix}$, $E_3 = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{1} \end{pmatrix}$ such that $A - E_i$ is a nilpotent, $i \in \{1,2,3\}$ and there is one idempotent, namely $E = I_2$ such that $A+E$ is a nilpotent. So

$$\text{wnc }(\begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{1} \end{pmatrix}) = 4.$$

Let $A = \begin{pmatrix} \overline{0} & y \\ \overline{1} & \overline{2} \end{pmatrix}$. Furthermore, for an idempotent E, if $A+E$ is a nilpotent, then $tr(E) = \overline{1}$, and thus $E \neq I_2, O_2$. Also, if $A - E$ is a nilpotent, then $tr(E) = \overline{2}$ and hence $E = I_2$.

We derive
$\begin{pmatrix} \overline{0} & y \\ \overline{1} & \overline{2} \end{pmatrix} - \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{0} & \overline{1} \end{pmatrix} = \begin{pmatrix} \overline{2} & y \\ \overline{1} & \overline{1} \end{pmatrix}$, which is a nilpotent if and only if $y = \overline{2}$

- Let $A = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{2} \end{pmatrix}$. Then $\begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{0} & s \\ \overline{0} & \overline{1} \end{pmatrix} = \begin{pmatrix} \overline{0} & s \\ \overline{1} & \overline{0} \end{pmatrix}$ is a nilpotent if and only if $s = 0$;
- $\begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{1} & \overline{0} \\ s & \overline{0} \end{pmatrix} = \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{1}+s & \overline{2} \end{pmatrix}$, which is not a nilpotent;
- $\begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{1} & s \\ \overline{0} & \overline{0} \end{pmatrix} = \begin{pmatrix} \overline{1} & s \\ \overline{1} & \overline{2} \end{pmatrix}$, which is a nilpotent if and only if $s = \overline{2}$;
- $\begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{0} & \overline{0} \\ s & \overline{1} \end{pmatrix} = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1}+s & \overline{0} \end{pmatrix}$, is a nilpotent;

- $\begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{2} & \overline{2} \\ \overline{2} & \overline{2} \end{pmatrix} = \begin{pmatrix} \overline{2} & \overline{2} \\ \overline{0} & \overline{1} \end{pmatrix}$, which is not a nilpotent;
- $\begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{2} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix} = \begin{pmatrix} \overline{2} & \overline{1} \\ \overline{2} & \overline{1} \end{pmatrix}$, which is a nilpotent.

For $A = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{2} \end{pmatrix}$ we have the idempotents $E_1 = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{0} & \overline{1} \end{pmatrix}$, $E_2 = \begin{pmatrix} \overline{1} & \overline{2} \\ \overline{0} & \overline{0} \end{pmatrix}$, $E_3 = \begin{pmatrix} \overline{0} & \overline{0} \\ s & \overline{1} \end{pmatrix}$, $E_4 = \begin{pmatrix} \overline{2} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix}$ such that $A + E_i$ is a nilpotent, $i \in \{1, 2, 3, 4\}$ and there are no idempotents E such that $A - E$ is a nilpotent. So

$$\text{wnc }(\begin{pmatrix} \overline{0} & \overline{0} \\ \overline{1} & \overline{2} \end{pmatrix}) = 4.$$

Let $A = \begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix}$.

- $\begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{0} & s \\ \overline{0} & \overline{1} \end{pmatrix} = \begin{pmatrix} \overline{0} & s+1 \\ \overline{1} & \overline{0} \end{pmatrix}$ is a nilpotent if and only if $s = \overline{2}$;
- $\begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{1} & \overline{0} \\ s & \overline{0} \end{pmatrix} = \begin{pmatrix} \overline{1} & \overline{1} \\ \overline{1}+s & \overline{2} \end{pmatrix}$, which is a nilpotent if and only if $s = \overline{1}$;
- $\begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{1} & s \\ \overline{0} & \overline{0} \end{pmatrix} = \begin{pmatrix} \overline{1} & s+1 \\ \overline{1} & \overline{2} \end{pmatrix}$, which is a nilpotent if and only if $s = \overline{1}$;
- $\begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{0} & \overline{1} \\ s+1 & \overline{1} \end{pmatrix} = \begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1}+s & \overline{0} \end{pmatrix}$, is a nilpotent if and only if $s = \overline{2}$;
- $\begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{2} & \overline{2} \\ \overline{2} & \overline{2} \end{pmatrix} = \begin{pmatrix} \overline{2} & \overline{0} \\ \overline{0} & \overline{1} \end{pmatrix}$, which is not a nilpotent;
- $\begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{2} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix} = \begin{pmatrix} \overline{2} & \overline{2} \\ \overline{2} & \overline{1} \end{pmatrix}$, which is not a nilpotent.

For $A = \begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix}$ we obtain the idempotents $E_1 = \begin{pmatrix} \overline{0} & \overline{2} \\ \overline{0} & \overline{1} \end{pmatrix}$, $E_2 = \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{1} & \overline{0} \end{pmatrix}$, $E_3 = \begin{pmatrix} \overline{1} & \overline{1} \\ \overline{0} & \overline{0} \end{pmatrix}$, $E_4 = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{2} & \overline{1} \end{pmatrix}$ such that $A + E_i$ is a nilpotent, $i \in \{1, 2, 3, 4\}$ and there are no idempotents E such that $A - E$ is a nilpotent. So

$$\text{wnc }(\begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix}) = 4.$$

Let $A = \begin{pmatrix} \overline{0} & \overline{2} \\ \overline{1} & \overline{2} \end{pmatrix}$. We have

- $\begin{pmatrix} \overline{0} & \overline{2} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{0} & s \\ \overline{0} & \overline{1} \end{pmatrix} = \begin{pmatrix} \overline{0} & s+\overline{2} \\ \overline{1} & \overline{0} \end{pmatrix}$ is a nilpotent if and only if $s = \overline{1}$;
- $\begin{pmatrix} \overline{0} & \overline{2} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{1} & \overline{0} \\ s & \overline{0} \end{pmatrix} = \begin{pmatrix} \overline{1} & \overline{2} \\ \overline{1}+s & \overline{2} \end{pmatrix}$, which is a nilpotent if and only if $s = \overline{1}$.
- $\begin{pmatrix} \overline{0} & \overline{2} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{1} & s \\ \overline{0} & \overline{0} \end{pmatrix} = \begin{pmatrix} \overline{1} & s+\overline{2} \\ \overline{1} & \overline{2} \end{pmatrix}$, which is a nilpotent if and only if $s = \overline{0}$;
- $\begin{pmatrix} \overline{0} & \overline{2} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{0} & \overline{1} \\ s+1 & \overline{1} \end{pmatrix} = \begin{pmatrix} \overline{0} & \overline{2} \\ \overline{1}+s & \overline{0} \end{pmatrix}$, is a nilpotent if and only if $s = \overline{2}$;
- $\begin{pmatrix} \overline{0} & \overline{2} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{2} & \overline{2} \\ \overline{2} & \overline{2} \end{pmatrix} = \begin{pmatrix} \overline{2} & \overline{1} \\ \overline{0} & \overline{1} \end{pmatrix}$, which is not a nilpotent;
- $\begin{pmatrix} \overline{0} & \overline{2} \\ \overline{1} & \overline{2} \end{pmatrix} + \begin{pmatrix} \overline{2} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix} = \begin{pmatrix} \overline{2} & \overline{0} \\ \overline{2} & \overline{1} \end{pmatrix}$, which is not a nilpotent.

For $A = \begin{pmatrix} \overline{0} & \overline{2} \\ \overline{1} & \overline{2} \end{pmatrix}$ we obtain the idempotents $E_1 = \begin{pmatrix} \overline{0} & \overline{1} \\ \overline{0} & \overline{1} \end{pmatrix}$, $E_2 = \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{0} & \overline{1} \end{pmatrix}$, $E_3 = \begin{pmatrix} \overline{1} & \overline{0} \\ \overline{0} & \overline{0} \end{pmatrix}$, $E_4 = \begin{pmatrix} \overline{0} & \overline{0} \\ \overline{2} & \overline{2} \end{pmatrix}$ such that $A + E_i$ is a nilpotent, $i \in \{1, 2, 3, 4\}$ and there is one idempotent $E = I_2$ such that $A - E$ is a nilpotent. So

$$\text{wnc}\,(\begin{pmatrix} \overline{0} & \overline{1} \\ \overline{1} & \overline{2} \end{pmatrix}) = 5.$$

In conclusion, $\text{wnc}(\mathbb{M}_2(\mathbb{Z}_3)) = 5$, as expected. □

For rings A and B and for a bimodule ${}_AM_B$, we denote by $\begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$ the formal triangular matrix ring.

The next statement strengthens [1, Theorem 4.1].

Proposition 2.27. *Let R be a ring. The following statements are equivalent:*

(1) *wnc(R)=2;*

(2) $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, *where A and B are abelian rings, and ${}_AM_B$ is a bimodule with $|M| = 2$.*

Proof. (1) $\Rightarrow$ (2):
If $wnc(R) = 2$, since $wnc(R) \geq Nin(R)$, then $Nin(R) = 1$ or $Nin(R) = 2$.

- If $Nin(R) = 1$, then R is abelian and so $wnc(R) = 1$, which is a contradiction.
- If $Nin(R) = 2$, then by Theorem 4.1 in [1] we get the desired form of R.

(2) $\Rightarrow$ (1):
Nilpotent elements in R are $\begin{pmatrix} n_A & w \\ 0 & n_B \end{pmatrix}$, where n_A is a nilpotent in A, n_B is a nilpotent in B and w is any element in $M = \{0, x\}$.
Idempotent elements in R are $\begin{pmatrix} e_A & w \\ 0 & e_B \end{pmatrix}$, where e_A is an idempotent in A, e_B is an idempotent in B and $w \in M$ which satisfies the condition $e_A w + w e_B = w$. Since $wnc(A) = wnc(B) = 1$ and $x = x + 0 = 0 + x = x - 0 = 0 - x$ are the only decompositions of x, we have at most four weakly nil clean decompositions for $\begin{pmatrix} a & x \\ 0 & b \end{pmatrix}$ as follows:

$$\begin{pmatrix} a & w \\ 0 & b \end{pmatrix} = \begin{pmatrix} n_A & x \\ 0 & n_B \end{pmatrix} + \begin{pmatrix} e_A & 0 \\ 0 & e_B \end{pmatrix};$$

$$\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} n_A & 0 \\ 0 & n_B \end{pmatrix} + \begin{pmatrix} e_A & x \\ 0 & e_B \end{pmatrix} \text{ with } e_A x + x e_B = 0;$$

$$\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} n'_A & x \\ 0 & n'_B \end{pmatrix} - \begin{pmatrix} e_A & 0 \\ 0 & e_B \end{pmatrix};$$

$$\begin{pmatrix} a & x \\ 0 & b \end{pmatrix} = \begin{pmatrix} n'_A & 0 \\ 0 & n'_B \end{pmatrix} - \begin{pmatrix} e_A & x \\ 0 & e_B \end{pmatrix} \text{ with } e_A x + x e_B = x.$$

Hence we get at most two idempotents in $\alpha(\begin{pmatrix} a & x \\ 0 & b \end{pmatrix})$.

Since $wnc(A) = wnc(B) = 1$ and $0 = 0 + 0 = x + x = 0 - 0 = x - x$ are the only decompositions of x, we have at most four weakly nil clean decompositions for $\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ as follows:

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} n_A & 0 \\ 0 & n_B \end{pmatrix} + \begin{pmatrix} e_A & 0 \\ 0 & e_B \end{pmatrix};$$

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} n_A & x \\ 0 & n_B \end{pmatrix} + \begin{pmatrix} e_A & x \\ 0 & e_B \end{pmatrix} \text{ with } e_Ax + xe_B = x;$$

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} n'_A & 0 \\ 0 & n'_B \end{pmatrix} - \begin{pmatrix} e_A & 0 \\ 0 & e_B \end{pmatrix};$$

$$\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} = \begin{pmatrix} n'_A & x \\ 0 & n'_B \end{pmatrix} - \begin{pmatrix} e_A & x \\ 0 & e_B \end{pmatrix} \text{ with } e_Ax + xe_B = x.$$

Hence we got at most 2 idempotents in $\alpha(\begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix})$.

Therefore, $wnc(R) \leq 2$, and so if we find q in R such that we can get two idempotents in $\alpha(q)$, then $wnc(R) = 2$. Thus q is $\begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix}$ and the idempotents are $\begin{pmatrix} 0 & 0 \\ 0 & 1_B \end{pmatrix}$ and $\begin{pmatrix} 0 & x \\ 0 & 1_B \end{pmatrix}$. □

We continue by showing that the next assertion is *not* an analogue of [1, Proposition 4.2].

Example 2.28. *If* $R = \begin{pmatrix} A & M \\ 0 & B \end{pmatrix}$, *where* $wnc(A) = wnc(B) = 1$ *and* ${}_AM_B$ *is a bimodule with* $|M| = 3$, *then* $wnc(R) = 3$ *cannot be happen in general. In fact, in accordance with Example 2.26,* $R = \begin{pmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ \mathbb{Z}_3 & \mathbb{Z}_3 \end{pmatrix} = \begin{pmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ \mathbb{Z}_3 & 0 \end{pmatrix}$ *is a ring with* $wnc(R) = 5 > 3$.

Note that if $P = \begin{pmatrix} \mathbb{Z}_3 & \mathbb{Z}_3 \\ 0 & \mathbb{Z}_3 \end{pmatrix}$, *then* $P/J(P) \cong \mathbb{Z}_3 \times \mathbb{Z}_3$.

We now proceed by extending [1, Proposition 4.4] in the following manner.

Proposition 2.29. *Let* R *be a ring and let* $n \geq 1$ *be an integer. Then*

(i) $wnc(\mathbb{M}_n(R)) \geq 3$, *provided* $n \geq 2$.

(ii) $wnc(\mathbb{M}_n(R)) = 3$ *if and only if* $n = 2$ *and* $R \cong \mathbb{Z}_2$.

Proof. (i) Applying Lemma 2.9, it follows that $wnc(\mathbb{M}_n(R)) \geq Nin(\mathbb{M}_n(R))$. Furthermore, [1, Proposition 4.4 (1)] applies to get the wanted inequality.

(ii) Referring again to Lemma 2.9, $Nin(\mathbb{M}_n(R)) \leq wnc(\mathbb{M}_n(R))$ so that either $Nin(\mathbb{M}_n(R)) = 1$, or $Nin(\mathbb{M}_n(R)) = 2$, or $Nin(\mathbb{M}_n(R)) = 3$. The first two cases are impossible appealing to [1, Theorem 3.2] or to [1, Theorem 4.1], respectively. The third case is handled in [1, Proposition 4.4 (2)], which gives our claim. □

Remark 2.30. *It is noteworthy that by virtue of* [2] *the ring* $\mathbb{M}_2(R) \cong \mathbb{M}_2(\mathbb{Z}_2)$ *is nil-clean and consequently* $wnc(\mathbb{M}_2(\mathbb{Z}_2)) = Nin(\mathbb{M}_2(\mathbb{Z}_2))$.

3. Open questions

We finish the paper with a series of left-answered problems:

Problem 1. For a ring R find a criterion when the equality $c(R) = wnc(R)$ holds.

Problem 2. For a ring R find a criterion when the equality $c(R) = Nin(R)$ holds.

Problem 3. For a ring R find a criterion when the equality $wnc(R) = Nin(R)$ holds.

Problem 4. If $R = S \times T$ is a direct decomposition of a ring R, does it follow that $wnc(R) = Nin(S)wnc(T) = wnc(S)Nin(T)$?

In that direction, this is related to the existence of such rings R satisfying the inequality $wnc(R) > Nin(R)$.

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