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# Miscellaneous Properties of Generalized Fubini Polynomials

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**Abstract**
This article attempts to present the generalized Fubini polynomials $F_n(x,y,z,q)$. The results obtained here include various families of multilinear and multilateral generating functions, various properties, as well as some special cases for these generalized Fubini polynomials $F_n(x,y,z,q)$. Finally, we get several interesting results of this generalized Fubini polynomials and obtain an integral representation.

**Keywords:** Generalized Fubini polynomials, Generating function, Multilinear and multilateral generating function, Recurrence relations.
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## 1. Introduction

Numerous studies on families of special polynomials, including the Bernoulli, Euler, Genocchi, and Fubini polynomials, as well as their generalizations and unifications (see, for example, the most recent works in [1]- [6], have gained significant popularity due to the wide range of their applications in various branches of mathematics, including p-adic analytic number theory, umbral calculus, special functions, and mathematical analysis. The special functions of mathematical physics have undergone a major evolution in recent years, especially in their generalized and multivariable forms. Thus, research on the multivariate Fubini polynomials was done for this work. Now let's go through the fundamental terms and theories that we will be using for the duration of the entire study.

For $n \geq 0$, let

$$F_n = \sum_{k=0}^{n} k!S(n,k),$$

where $S(n,k)$ denotes the Stirling numbers of the second kind [11]. In [12], the Fubini numbers $F_n$ were connected with preference arrangements and the recursion for $F_n$ was derived. In [12], [13], the exponential generating function

$$\frac{1}{2-e^t} = \sum_{n=0}^{\infty} F_n \frac{t^n}{n!}$$

and an asymptotic estimate for $F_n$ were established. In [14], the Fubini polynomials $F_n(y)$ were defined by

$$F_n(y) = \sum_{k=0}^{n} k!S(n,k)y^k$$

and generated by

$$\frac{1}{1-y(e^t-1)} = \sum_{n=0}^{\infty} F_n(y)\frac{t^n}{n!}.$$

It is clear that $F_n(1) = F_n$. Due to the relation

$$\left(y\frac{d}{dy}\right)^m \frac{1}{1-y} = \sum_{k=0}^{\infty} k^m y^k = \frac{1}{1-y}F_m\left(\frac{y}{1-y}\right), \quad |y| < 1$$

in [15], one also calls $F_n(y)$ the geometric polynomials. In [16], the Fubini polynomials $F_n(x,y)$ of two variables $x$, $y$ are defined by means of the generating function

$$\frac{e^{xt}}{1-y(e^t-1)} = \sum_{n=0}^{\infty} F_n(x,y)\frac{t^n}{n!}.$$

It is apparent that $F_n(0,y) = F_n(y)$. In Particular, the special polynomials of two variables provided new means of analysis for the solution of large classes of partial differential equations often encountered in physical problems. Most of the special function of mathematical physics and their generalization has been suggested by physical problems (see, e.g., [7]-[10] and the references therein). In [17], the bivariate Fubini polynomials $F_n^{(r)}(x,y)$ of order $r$, generated by

$$\frac{e^{xt}}{[1-y(e^t-1)]^r} = \sum_{n=0}^{\infty} F_n^{(r)}(x,y)\frac{t^n}{n!}, \; r \in \mathbb{N}$$

were studied. It is obvious that $F_n^{(1)}(x,y) = F_n(x,y)$. The generating functions of $F_n$, $F_n(y)$, $F_n(x,y)$ and $F_n^{(r)}(x,y)$ remind us to consider the generating function

$$\frac{e^{xt}}{[z-y(e^t-1)]^q} = \sum_{n=0}^{\infty} F_n(x,y,z,q)\frac{t^n}{n!}, \; x,q \in \mathbb{R} \tag{1.1}$$

and the generalized Fubini polynomials $F_n(x,y,z,q)$ of four variables $x$, $y$, $z$, $q$ [18]. It is clear that, since

$$\frac{e^{xt}}{[z-y(e^t-1)]^q} = \frac{1}{z^q}\frac{e^{xt}}{[1-(y/z)(e^t-1)]^q},$$

we have

$$F_n(x,y,z,q) = \frac{F_n^{(r)}(x,y/z)}{z^r}.$$

The aim of this paper is to derive various families of multilinear and multilateral generating functions for the polynomials $F_n(x,y,z,q)$ given by (1.1). We present some special cases of our results and also obtain some other properties for these special cases.

## 2. Multilinear and Multilateral Generating Functions

The goal of this section is to derive several families of multilinear and multilateral generating functions for a class of polynomials in four variables given by equation (1.1) with the help of the method considered in refs. [20], [21].

**Lemma 2.1.** *The following addition formula holds for the generalized Fubini polynomials $F_n(x,y,z,q)$ :*

$$F_n(x_1+x_2,y,q_1+q_2) = \sum_{m=0}^{n} \binom{n}{m} F_{n-m}(x_1,y,z,q_1)F_m(x_2,y,z,q_2). \tag{2.1}$$

*Proof.* Replacing $x$ by $x = x_1 + x_2$ and $q$ by $q = q_1 + q_2$ in (1.1), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} F_n(x_1+x_2, y, q_1+q_2)\frac{t^n}{n!} &= \frac{e^{x_1 t + x_2 t}}{[z - y(e^t-1)]^{q_1+q_2}} \\
&= \frac{e^{x_1 t}}{[z - y(e^t-1)]^{q_1}} \frac{e^{x_2 t}}{[z - y(e^t-1)]^{q_2}} \\
&= \sum_{n=0}^{\infty} F_n(x_1, y, z, q_1)\frac{t^n}{n!} \sum_{m=0}^{\infty} F_m(x_2, y, z, q_2)\frac{t^m}{m!} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} F_n(x_1, y, z, q_1) F_m(x_2, y, z, q_2)\frac{t^{n+m}}{n!.m!} \\
&= \sum_{n=0}^{\infty}\sum_{m=0}^{n} \binom{n}{m} F_{n-m}(x_1, y, z, q_1) F_m(x_2, y, z, q_2)\frac{t^n}{n!}.
\end{aligned}$$

From the coefficients of $t^n$ on the both sides of the last equality, one can get the desired result. □

**Theorem 2.2.** *Corresponding to an identically non-vanishing function $\Omega_\mu(s_1, ..., s_r)$ of $r$ complex variables $s_1, ..., s_r$ $(r \in \mathbb{N})$ and of complex order $\mu$, $\psi$, let*

$$\Lambda_{\mu,\psi}(s_1, ..., s_r; \zeta) := \sum_{k=0}^{\infty} a_k \Omega_{\mu+\psi k}(s_1, ..., s_r)\zeta^k,$$

$$\theta_{n,p}^{\mu,\psi}(x, y, z, q; s_1, ..., s_r; \xi) := \sum_{k=0}^{[n/p]} a_k F_{n-pk}(x, y, z, q)\Omega_{\mu+\psi k}(s_1, ..., s_r)\frac{\xi^k}{(n-pk)!}.$$

*where $a_k \neq 0$, $n$, $p \in \mathbb{N}$ and the notation $[n/p]$ means the greatest integer less than or equal $p \in \mathbb{N}$. Then, for $p \in \mathbb{N}$ we have*

$$\sum_{n=0}^{\infty} \theta_{n,p}^{\mu,\psi}(x, y, z, q; s_1, ..., s_r; \frac{\eta}{t^p})t^n = \frac{e^{xt}}{[z - y(e^t-1)]^q}\Lambda_{\mu,\psi}(s_1, ..., s_r; \eta), \tag{2.2}$$

*provided that each member of (2.2) exists.*

*Proof.* For convenience, let $S$ denote the first member of the assertion of Theorem 2.2. Then,

$$S = \sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k F_{n-pk}(x, y, z, q)\Omega_{\mu+\psi k}(s_1, ..., s_r)\eta^k \frac{t^{n-pk}}{(n-pk)!}.$$

Replacing $n$ by $n + pk$; we may write that

$$\begin{aligned}
S &= \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} a_k F_n(x, y, z, q)\Omega_{\mu+\psi k}(s_1, ..., s_r)\eta^k \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} F_n(x, y, z, q)\frac{t^n}{n!}\sum_{k=0}^{\infty} a_k\, \Omega_{\mu+\psi k}(s_1, ..., s_r)\eta^k \\
&= \frac{e^{xt}}{[z - y(e^t-1)]^q}\Lambda_{\mu,\psi}(s_1, ..., s_r; \eta),
\end{aligned}$$

which completes the proof. □

Using Lemma 1, we have the following theorem.

**Theorem 2.3.** *Corresponding to an identically non-vanishing function $\Omega_\mu$ $(s_1, ..., s_r)$ of $r$ complex variables $s_1, ..., s_r$ $(r \in \mathbb{N})$ and of complex order $\mu$, $\psi$, let*

$$\Lambda_{\mu,\psi}^{n,p}(x_1+x_2, y, z, q_1+q_2; s_1, ..., s_r; t) := \sum_{k=0}^{[n/p]} a_k F_{n-pk}(x_1+x_2, y, q_1+q_2)\Omega_{\mu+\psi k}(s_1, ..., s_r)t^k$$

*where* $a_k \neq 0$, $n$, $p \in \mathbb{N}$. *Then, for* $p \in \mathbb{N}$, *we have*

$$\begin{aligned} & \sum_{k=0}^{n}\sum_{l=0}^{[k/p]} a_l \binom{n-pl}{k-pl} F_{n-k}(x_1,y,z,q_1)F_{k-pl}(x_2,y,z,q_2)\Omega_{\mu+\psi l}(s_1,\dots,s_r)t^l \\ = \; & \Lambda^{n,p}_{\mu,\psi}(x_1+x_2,y,z,q_1+q_2;s_1,\dots,s_r;t), \end{aligned} \tag{2.3}$$

*provided that each member of* (2.3) *exists.*

*Proof.* For convenience, let T denote the first member of the assertion of Theorem 2.3. Then, upon substituting for the polynomials $F_n(x_1+x_2,y,z,q_1+q_2)$ from the (2.3) into the left-hand side of (2.1), we obtain

$$\begin{aligned} T &= \sum_{l=0}^{[n/p]}\sum_{k=0}^{n-pl} a_l \binom{n-pl}{k} F_{n-k-pl}(x_1,y,z,q_1)F_k(x_2,y,z,q_2)\Omega_{\mu+\psi l}(s_1,\dots,s_r)t^l \\ &= \sum_{l=0}^{[n/p]} a_l \left(\sum_{k=0}^{n-pl}\binom{n-pl}{k} F_{n-k-pl}(x_1,y,z,q_1)F_k(x_2,y,z,q_2)\right)\Omega_{\mu+\psi l}(s_1,\dots,s_r)t^l \\ &= \sum_{l=0}^{[n/p]} a_l F_{n-pl}(x_1+x_2,y,q_1+q_2)\Omega_{\mu+\psi l}(s_1,\dots,s_r)t^l \\ &= \Lambda^{n,p}_{\mu,\psi}(x_1+x_2;s_1,\dots,s_r;t), \end{aligned}$$

which completes the proof. □

## 3. Special Cases

When the multivariable function $\Omega_{\mu+\psi k}(s_1,\dots,s_r)$, $k \in \mathbb{N}_0$, $r \in \mathbb{N}_0$ is expressed in terms of simpler functions of one and more variables, then we can give further applications of the above theorems. We first set

$$\Omega_{\mu+\psi k}(s_1,\dots,s_r) = T^{(\alpha_1,\alpha_2,\dots,\alpha_r;\alpha)}_{\mu+\psi k,\lambda,l}(s_1,\dots,s_r;s)$$

in Theorem 2.2, where the Lagrange-based Apostol- type polynomials $T^{(\alpha_1,\alpha_2,\dots,\alpha_r;\alpha)}_{n,\lambda,k}(x_1,\dots,x_r;x)$ [19], generated by

$$\sum_{n=0}^{\infty} T^{(\alpha_1,\alpha_2,\dots,\alpha_r;\alpha)}_{n,\lambda,l}(x_1,\dots,x_r;x)t^n = \left(\prod_{j=1}^{r}(1-x_jt)^{-\alpha_j}\right)\left(\frac{2^l t}{\lambda e^t+(-1)^{l+1}}\right)^{\alpha} e^{xt} \quad (\lambda;\, \alpha_j \in \mathbb{C}) \tag{3.1}$$

We are thus led to the following result which provides a class of bilateral generating functions for the Lagrange-based Apostol- type polynomials $T^{(\alpha_1,\alpha_2,\dots,\alpha_r;\alpha)}_{n,\lambda,l}(x_1,\dots,x_r;x)$ and the generalized Fubini polynomials $F_n(x,y,z,q)$.

**Corollary 3.1.** *If*

$$\Lambda_{\mu,\psi}(s_1,\dots,s_r;s;\zeta) := \sum_{k=0}^{\infty} a_k T^{(\alpha_1,\alpha_2,\dots,\alpha_r;\alpha)}_{\mu+\psi k,\lambda,l}(s_1,\dots,s_r;s)\zeta^k \quad (a_k \neq 0,\ \mu,\psi \in C)$$

*then, we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} a_k F_{n-pk}(x,y,z,q) T^{(\alpha_1,\alpha_2,\dots,\alpha_r;\alpha)}_{\mu+\psi k,\lambda,l}(s_1,\dots,s_r;s)\frac{\eta^k}{t^{pk}}\frac{t^n}{(n-pk)!} = \frac{e^{xt}}{[z-y(e^t-1)]^q}\Lambda_{\mu,\psi}(s_1,\dots,s_r;s;\eta), \tag{3.2}$$

*provided that each member of* (3.2) *exists.*

**Remark 3.2.** *Using the generating relation* (3.1) *for the Lagrange-based Apostol-type polynomials* $T^{(\alpha_1,\alpha_2,\dots,\alpha_r;\alpha)}_{n,\lambda,l}(s_1,\dots,s_r;s)$ *and getting* $a_k = 1$, $\mu = 0$, $\psi = 1$ *in Corollary 1, we find that*

$$\begin{aligned} & \sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]} F_{n-pk}(x,y,z,q)\, T^{(\alpha_1,\alpha_2,\dots,\alpha_r;\alpha)}_{k,\lambda,l}(s_1,\dots,s_r;s)\eta^k \frac{t^{n-pk}}{(n-pk)!} \\ = \; & \frac{e^{xt}}{[z-y(e^t-1)]^q}\left(\prod_{j=1}^{r}(1-s_j\eta)^{-\alpha_j}\right)\left(\frac{2^l\eta}{\lambda e^{\eta}+(-1)^{l+1}}\right)^{\alpha} e^{s\eta},\ (\lambda \in \mathbb{C};\ \alpha_j \in \mathbb{C}). \end{aligned}$$

In the particular cases when $l=0$, $l=1$ in the Corollary 1 and Remak 1, we have bilateral generating functions the Lagrange-based Apostol-Bernoulli polynomials $B_{k,\lambda}^{(\alpha_1,\alpha_2,\ldots,\alpha_r;\alpha)}(s_1,\ldots,s_r;s)$, the Lagrange-based Apostol-Genocchi polynomials $G_{k,\lambda}^{(\alpha_1,\alpha_2,\ldots,\alpha_r;\alpha)}(s_1,\ldots,s_r;s)$ and the generalized Fubini polynomials [28].

If we set $r=4$ and

$$\Omega_{\mu+\psi k}(s_1,s_2,s_3,s_4)=F_{\mu+\psi k}(s_1,s_2,s_3,s_4)$$

in Theorem 2.2, we have the following bilinear generating functions for the generalized Fubini polynomils.

**Corollary 3.3.** *If*

$$\Lambda_{\mu,\psi}(s_1,s_2,s_3,s_4;\zeta):=\sum_{k=0}^{\infty}a_kF_{\mu+\psi k}(s_1,s_2,s_3,s_4)\zeta^k,\ \ (a_k\neq 0\ \mu,\psi\in\mathbb{C})$$

*then, we have*

$$\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}a_kF_{n-pk}(x,y,z,q)F_{\mu+\psi k}(s_1,s_2,s_3,s_4)\frac{\eta^k}{t^{pk}}\frac{t^n}{(n-pk)!}=\frac{e^{xt}}{[z-y(e^t-1)]^q}\Lambda_{\mu,\psi}(s_1,s_2,s_3,s_4;\eta), \tag{3.3}$$

*provided that each member of (3.3) exists.*

**Remark 3.4.** *Using the generating relation (1.1) for the generalized Fubini polynomials $F_n(x,y,z,q)$ and getting*

$$a_k=\frac{1}{k!},\ \mu=0,\ \psi=1$$

*in Corollary 2, we find that*

$$\begin{aligned}&\sum_{n=0}^{\infty}\sum_{k=0}^{[n/p]}\frac{1}{k!}F_{n-pk}(x,y,z,q)\,F_k(s_1,s_2,s_3,s_4)\eta^k\frac{t^{n-pk}}{(n-pk)!}\\ =\ &\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}\sum_{k=0}^{\infty}F_k(s_1,s_2,s_3,s_4)\frac{\eta^k}{k!}\\ =\ &\frac{e^{xt}}{[z-y(e^t-1)]^q}\frac{e^{s_1t}}{[s_3-s_2(e^t-1)]^{s_4}}.\end{aligned}$$

If we set $r=1$ and

$$\Omega_{\mu+\psi k}(s_1)=F_{\mu+\psi k}(x_3,y,z,q_3)$$

in Theorem 2.3, we have the following summation formula for the generalized Fubini polynomials.

**Corollary 3.5.** *If*

$$\Lambda_{\mu,\psi}^{n,p}(x_1+x_2,y,z,q_1+q_2;x_3,y,z,q_3;\eta):=\sum_{k=0}^{[n/p]}a_kF_{n-pk}(x_1+x_2,y,z,q_1+q_2)F_{\mu+\psi k}(x_3,y,z,q_3)\eta^k,$$
$$(a_k\neq 0,\mu,\psi\in\mathbb{C}),$$

*then, we have*

$$\begin{aligned}&\sum_{k=0}^{n}\sum_{l=0}^{[k/p]}a_l\binom{n}{k}F_{n-k}(x_1,y,z,q_1)F_{k-pl}(x_2,y,z,q_2)F_{\mu+\psi l}(x_3,y,z,q_3)\eta^l\\ =\ &\Lambda_{\mu,\psi}^{n,p}(x_1+x_2,y,z,q_1+q_2;x_3,y,z,q_3;\eta),\end{aligned} \tag{3.4}$$

*provided that each member of (3.4) exists.*

**Remark 3.6.** *Using (2.1) and taking*

$$a_l=1,\ \mu=0,\ \psi=1,\ p=1,\ \eta^l=\binom{k}{l}$$

*in Corollary 3, we have*

$$\sum_{k=0}^{n}\sum_{l=0}^{k}\binom{n}{k}\binom{k}{l}F_{n-k}(x_1;y,z,q_1)F_{k-l}(x_2,y,z,q_2)F_l(x_3,y,z,q_3)=F_n(x_1+x_2+x_3,y,z,q_1+q_2+q_3).$$

Furthermore, for every suitable choice of the coefficients $a_k$ $(k \in \mathbb{N}_0)$, if the multivariable functions $\Omega_{\mu+\psi k}(s_1, ..., s_r)$, $r \in \mathbb{N}$, are expressed as an appropriate product of several simpler functions, the assertions of Theorem 2.2, Theorem 2.3 can be applied in order to derive various families of multilinear and multilateral generating functions for the family of the generalized Fubini polynomials given explicitly by (1.1).

## 4. Miscellaneous Properties

In this section, we give some properties for the generalized Fubini polynomials $F_n(x,y,z,q)$ given by (1.1).

Firstly, recall that the classical Frobenius-Euler polynomials $H_n^{(r)}(u;x)$ of order $r$ are generated by (see, e.g., [22]-[26])

$$\left(\frac{1-u}{e^t-u}\right)^r e^{xt} = \sum_{n=0}^{\infty} H_n^{(r)}(u;x)\frac{t^n}{n!}, \tag{4.1}$$

where $u \neq 1$.

We note that, for $r = 1$ in (4.1), the $H_n^{(1)}(u;x) = H_n(u;x)$, which denotes the Frobenius-Euler polynomials and for $u = 0$ in (4.1), the $H_n^{(r)}(0;x) = H_n^{(r)}(x)$, which denotes the Frobenius-Euler numbers of order $r$. For $x = -1$ in (4.1), the $H_n^{(r)}(u;-1) = E_n(u)$, which denotes the Euler polynomials (cf. [27]).

**Theorem 4.1.** *For $n \geq 0$, $y$, $z \neq 0$; we have*

$$F_n(x,y,z,q) = \frac{H_n^{(q)}(\frac{z+y}{y};x)}{z^q}. \tag{4.2}$$

*Proof.* Using (1.1) and (4.1), we obtain

$$\begin{aligned}
\sum_{n=0}^{\infty} F_n(x,y,z,q)\frac{t^n}{n!} &= \frac{e^{xt}}{[z-y(e^t-1)]^q} \\
&= \left[\frac{1-\frac{z+y}{y}}{e^t-\frac{z+y}{y}}\right]^q e^{xt} \\
&= z^{-q}\sum_{n=0}^{\infty} H_n^{(q)}(\frac{z+y}{y};x)\frac{t^n}{n!}.
\end{aligned}$$

Hence, we have

$$F_n(x,y,z,q) = \frac{H_n^{(q)}(\frac{z+y}{y};x)}{z^q}, \ (y,\ z \neq 0),$$

or

$$H_n^{(q)}(\frac{z+y}{y};x) = z^q F_n(x,y,z,q).$$

□

Some special cases of Theorem 4.1 are examined below.

**Corollary 4.2.** *For $n \geq 0$, $q = 1$, $z$, $y \neq 0$; we have*

$$H_n^{(1)}(\frac{z+y}{y};x) = H_n(\frac{z+y}{y};x) = zF_n(x,y,z,1).$$

**Corollary 4.3.** *For $n \geq 0$, $z = -y \neq 0$; we have*

$$H_n^{(q)}(0;x) = H_n^{(r)}(x) = (-y)^q F_n(x,y,-y,q).$$

**Corollary 4.4.** *For $n \geq 0$, $z$, $y \neq 0$, $x = -1$; we have*

$$H_n^{(q)}(\frac{z+y}{y};-1) = E_n(\frac{z+y}{y}) = z^q F_n(-1,y,z,q).$$

We now discuss some miscellaneous recurrence relations of the generalized Fubini polynomials.

**Theorem 4.5.** *The following (differential) recurrence relation for the generalized Fubini polynomials holds:*

$$\frac{\partial}{\partial x}F_n(x,y,z,q) = n.F_{n-1}(x,y,z,q) \tag{4.3}$$

*and* $\deg F_n(x,y,z,q) = n$.

*Proof.* If we take the derivative of (1.1) with respect to $x$ both sides of the expression, we have

$$\begin{aligned}
\frac{\partial}{\partial x}\left(\sum_{n=0}^{\infty} F_n(x,y,z,q)\frac{t^n}{n!}\right) &= \frac{\partial}{\partial x}\left[\frac{e^{xt}}{[z-y(e^t-1)]^q}\right], \\
\sum_{n=0}^{\infty}\frac{\partial}{\partial x}F_n(x,y,z,q)\frac{t^n}{n!} &= \frac{t\,e^{xt}}{[z-y(e^t-1)]^q}, \\
\sum_{n=0}^{\infty}\frac{\partial}{\partial x}F_n(x,y,z,q)\frac{t^n}{n!} &= t\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}, \\
\sum_{n=0}^{\infty}\frac{\partial}{\partial x}F_n(x,y,z,q)\frac{t^n}{n!} &= \sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^{n+1}}{n!} \\
\sum_{n=1}^{\infty}\frac{\partial}{\partial x}F_n(x,y,z,q)\frac{t^n}{n!} &= \sum_{n=1}^{\infty}F_{n-1}(x,y,z,q)\frac{t^n}{(n-1)!}.
\end{aligned}$$

On equating like powers of $t^n$ in the above expression, which completes the proof. □

**Theorem 4.6.** *The following (differential) recurrence relation for the generalized Fubini polynomials holds:*

$$(z+y)\frac{\partial}{\partial y}F_n(x,y,z,q) + q\sum_{n=0}^{\infty}F_n(x,y,z,q) = y\sum_{p=0}^{n}\binom{n}{p}\frac{\partial}{\partial y}F_{n-p}(x,y,z,q) + q\sum_{p=0}^{n}\binom{n}{p}\frac{\partial}{\partial y}F_{n-p}(x,y,z,q) \tag{4.4}$$

*and* $\deg F_n(x,y,z,q) = n$.

*Proof.* If we take the derivative of (1.1) with respect to $y$ both sides of the expression, we have

$$\frac{\partial}{\partial y}\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!} = \frac{\partial}{\partial y}\left[\frac{e^{xt}}{[z-y(e^t-1)]^q}\right],$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial y}F_n(x,y,z,q)\frac{t^n}{n!} = e^{xt}\left[-q\left(z-y\left(e^t-1\right)\right)\right]^{-q-1}(-1)(e^t-1),$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial y}F_n(x,y,z,q)\frac{t^n}{n!} = \frac{e^{xt}}{[z-y(e^t-1)]^q}\,\frac{q(e^t-1)}{z-y(e^t-1)},$$

$$(z+y)\sum_{n=0}^{\infty}\frac{\partial}{\partial y}F_n(x,y,z,q)\frac{t^n}{n!} - y\sum_{n=0}^{\infty}\frac{\partial}{\partial y}F_n(x,y,z,q)\frac{t^n}{n!}\sum_{p=0}^{\infty}\frac{t^p}{p!} = q\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}F_n(x,y,z,q)\frac{t^{n+p}}{n!p!} - q\sum_{n=0}^{\infty}\frac{\partial}{\partial y}F_n(x,y,z,q)\frac{t^n}{n!},$$

$$(z+y)\sum_{n=0}^{\infty}\frac{\partial}{\partial y}F_n(x,y,z,q)\frac{t^n}{n!} - y\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{\partial}{\partial y}F_n(x,y,z,q)\frac{t^{n+p}}{n!p!} = q\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{\partial}{\partial y}F_n(x,y,z,q)\frac{t^{n+p}}{n!p!} - q\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!},$$

$$\begin{aligned}
&(z+y)\sum_{n=0}^{\infty}\frac{\partial}{\partial y}F_n(x,y,z,q)\frac{t^n}{n!} - y\sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{\partial}{\partial y}F_{n-p}(x,y,z,q)\frac{t^n}{(n-p)!p!} \\
=\;& q\sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{\partial}{\partial y}F_{n-p}(x,y,z,q)\frac{t^n}{(n-p)!p!} - q\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!},
\end{aligned}$$

$$(z+y)\sum_{n=0}^{\infty}\frac{\partial}{\partial y}F_n(x,y,z,q)\frac{t^n}{n!}+q\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}$$
$$= \quad y\sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{\partial}{\partial y}F_{n-p}(x,y,z,q)\frac{t^n}{(n-p)!p!}+q\sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{\partial}{\partial y}F_{n-p}(x,y,z,q)\frac{t^n}{(n-p)!p!}$$

$$(z+y)\sum_{n=0}^{\infty}\frac{\partial}{\partial y}F_n(x,y,z,q)\frac{t^n}{n!}+q\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}$$
$$= \quad y\sum_{n=0}^{\infty}\sum_{p=0}^{n}\binom{n}{p}\frac{\partial}{\partial y}F_{n-p}(x,y,z,q)\frac{t^n}{n!}+q\sum_{n=0}^{\infty}\sum_{p=0}^{n}\binom{n}{p}\frac{\partial}{\partial y}F_{n-p}(x,y,z,q)\frac{t^n}{n!}$$

which upon comparison of the coefficients of $\frac{t^n}{n!}$ yields our stated result (4.4). □

**Theorem 4.7.** *The following (differential) recurrence relation for the generalized Fubini polynomials holds:*

$$(z+y)\frac{\partial}{\partial z}F_n(x,y,z,q)=y\sum_{p=0}^{n}\binom{n}{p}\frac{\partial}{\partial z}F_{n-p}(x,y,z,q)-qF_n(x,y,z,q)$$

*and* $\deg F_n(x,y,z,q)=n$.

*Proof.* If we take the derivative of (1.1) with respect to $z$ both sides of the expression, we have

$$\frac{\partial}{\partial z}\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}=\frac{\partial}{\partial z}\left[\frac{e^{xt}}{[z-y(e^t-1)]^q}\right],$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial z}F_n(x,y,z,q)\frac{t^n}{n!}=\left[e^{xt}\left(-q\left[z-y(e^t-1)\right]^{-q-1}\right)\right],$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial z}F_n(x,y,z,q)\frac{t^n}{n!}=-q\frac{e^{xt}}{[z-y(e^t-1)]^q\,(z-y(e^t-1))},$$

$$(z-y(e^t-1))\sum_{n=0}^{\infty}\frac{\partial}{\partial z}F_n(x,y,z,q)\frac{t^n}{n!}=-q\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!},$$

$$-q\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}=z\sum_{n=0}^{\infty}\frac{\partial}{\partial z}F_n(x,y,z,q)\frac{t^n}{n!}-y\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\frac{\partial}{\partial z}F_n(x,y,z,q)\frac{t^{n+p}}{n!p!}+y\sum_{n=0}^{\infty}\frac{\partial}{\partial z}F_n(x,y,z,q)\frac{t^n}{n!},$$

$$-q\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}=z\sum_{n=0}^{\infty}\frac{\partial}{\partial z}F_n(x,y,z,q)\frac{t^n}{n!}-y\sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{\partial}{\partial z}F_{n-p}(x,y,z,q)\frac{t^n}{(n-p)!p!}+y\sum_{n=0}^{\infty}\frac{\partial}{\partial z}F_n(x,y,z,q)\frac{t^n}{n!}.$$

$$(z+y)\sum_{n=0}^{\infty}\frac{\partial}{\partial z}F_n(x,y,z,q)\frac{t^n}{n!}=y\sum_{n=0}^{\infty}\sum_{p=0}^{n}\frac{\partial}{\partial z}F_{n-p}(x,y,z,q)\frac{t^n}{(n-p)!p!}-q\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!},$$

$$(z+y)\sum_{n=0}^{\infty}\frac{\partial}{\partial z}F_n(x,y,z,q)\frac{t^n}{n!}=y\sum_{n=0}^{\infty}\sum_{p=0}^{n}\binom{n}{p}\frac{\partial}{\partial z}F_{n-p}(x,y,z,q)\frac{t^n}{n!}-q\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}.$$

From the coefficients of $\frac{t^n}{n!}$ on the both sides of the last equality, one can get the desired result. □

**Theorem 4.8.** *The following (differential) recurrence relation for the generalized Fubini polynomials holds:*

$$\frac{\partial}{\partial q}F_n(x,y,z,q)=\sum_{m=0}^{\infty}\sum_{p=0}^{n}\binom{n}{p}\left(\frac{y}{z+y}\right)^{m+1}(m+1)^{p-1}F_{n-p}(x,y,z,q)-\ln(z+y)F_n(x,y,z,q)$$

*and* $\deg F_n(x,y,z,q)=n$.

*Proof.* If we take the derivative of (1.1) with respect to $q$ both sides of the expression, we have

$$\frac{\partial}{\partial q}\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}=\frac{\partial}{\partial q}\left[\frac{e^{xt}}{[z-y(e^t-1)]^q}\right],$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial q}F_n(x,y,z,q)\frac{t^n}{n!}=e^{xt}((-1)\left[z-y(e^t-1)\right]^{-q}\ln(z-y(e^t-1)),$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial q}F_n(x,y,z,q)\frac{t^n}{n!}=\frac{-e^{xt}}{[z-y(e^t-1)]^q}\ln(z+y)(1-\frac{ye^t}{z+y}),$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial q}F_n(x,y,z,q)\frac{t^n}{n!}=-\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}\left[\ln(z+y)+\ln(1-\frac{ye^t}{z+y})\right],$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial q}F_n(x,y,z,q)\frac{t^n}{n!}=-\ln(z+y)\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}-\ln(1-\frac{ye^t}{z+y})\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!},$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial q}F_n(x,y,z,q)\frac{t^n}{n!}=-\ln(z+y)\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}-\left[-\frac{ye^t}{z+y}F(1,1;2;\frac{ye^t}{z+y})\right]\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!},$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial q}F_n(x,y,z,q)\frac{t^n}{n!}=-\ln(z+y)\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}+\frac{y}{z+y}e^t\sum_{n=0}^{\infty}\frac{(1)_m(1)_m}{(2)_m}\frac{(\frac{ye^t}{z+y})^m}{m!}\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!},$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial q}F_n(x,y,z,q)\frac{t^n}{n!}=-\ln(z+y)\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}F_n(x,y,z,q)(\frac{y}{z+y})^{m+1}\frac{(e^t)^{m+1}}{m+1}\frac{t^n}{n!},$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial q}F_n(x,y,z,q)\frac{t^n}{n!}=-\ln(z+y)\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}\frac{F_n(x,y,z,q)}{m+1}(\frac{y}{z+y})^{m+1}\frac{t^p(m+1)^p}{p!}\frac{t^n}{n!},$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial q}F_n(x,y,z,q)\frac{t^n}{n!}=-\ln(z+y)\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{\infty}F_n(x,y,z,q)(\frac{y}{z+y})^{m+1}\frac{(m+1)^{p-1}}{p!}\frac{t^{n+p}}{n!},$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial q}F_n(x,y,z,q)\frac{t^n}{n!}=-\ln(z+y)\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{n}F_{n-p}(x,y,z,q)(\frac{y}{z+y})^{m+1}\frac{(m+1)^{p-1}}{p!}\frac{t^n}{(n-p)!},$$

$$\sum_{n=0}^{\infty}\frac{\partial}{\partial q}F_n(x,y,z,q)\frac{t^n}{n!}=-\ln(z+y)\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}+\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{p=0}^{n}\binom{n}{p}F_{n-p}(x,y,z,q)(\frac{y}{z+y})^{m+1}(m+1)^{p-1}\frac{t^n}{n!}.$$

On equating like powers of $\frac{t^n}{n!}$ on both sides in the above expression and after some simplification, we arrive at our desired result. □

**Theorem 4.9.** *The following recurrence relation for the generalized Fubini polynomials holds:*

$$(z+y)F_{n+1}(x,y,z,q)-x(z+y)F_n(x,y,z,q)=y\sum_{m=0}^{n+1}F_{n-m+1}(x,y,z,q)+(q-x)y\sum_{m=0}^{n}\binom{n}{m}F_{n-m}(x,y,z,q).$$

*Proof.* If we take the derivative of (1.1) with respect to $t$ both sides of the expression, we have

$$\frac{\partial}{\partial t}\left[\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}\right]=\frac{\partial}{\partial t}\left[\frac{e^{xt}}{[z-y(e^t-1)]^q}\right],$$

$$\left[\sum_{n=1}^{\infty}nF_n(x,y,z,q)\frac{t^{n-1}}{n!}\right]=xe^{xt}\left[\frac{1}{[z-y(e^t-1)]^q}\right]-q\left[z-y\left(e^t-1\right)\right]^{q-1}\left[-ye^t\right]e^{xt},$$

$$\left[\sum_{n=1}^{\infty}nF_n(x,y,z,q)\frac{t^{n-1}}{n!}\right]=x\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}+\frac{qy\sum_{m=0}^{\infty}\frac{t^m}{m!}\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}}{z-y(e^t-1)},$$

$$[z-y\left(e^t-1\right)]\sum_{n=1}^{\infty}F_n(x,y,z,q)\frac{t^{n-1}}{n!}=x\left[z-y\left(e^t-1\right)\right]\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}+qy\sum_{n=0}^{\infty}\sum_{m=0}^{n}F_{n-m}(x,y,z,q)\frac{t^n}{(n-m)!m!},$$

$$\begin{aligned}&\left(z-y\left(e^t-1\right)\right)\sum_{n=1}^{\infty}F_n(x,y,z,q)\frac{t^{n-1}}{n!}\\ =\;& x(z+y)\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}-xy\sum_{n=0}^{\infty}\sum_{m=0}^{n}F_{n-m}(x,y,z,q)\frac{t^n}{(n-m)!m!}+qy\sum_{n=0}^{\infty}\sum_{m=0}^{n}F_{n-m}(x,y,z,q)\frac{t^n}{(n-m)!m!},\end{aligned}$$

$$\begin{aligned}&(z+y)\sum_{n=o}^{\infty}F_{n+1}(x,y,z,q)\frac{t^n}{n!}-y\sum_{n=0}^{\infty}\sum_{m=0}^{n+1}F_{n+1-m}(x,y,z,q)\frac{t^n}{n!}\\ =\;& x(z+y)\sum_{n=0}^{\infty}F_n(x,y,z,q)\frac{t^n}{n!}-xy\sum_{n=0}^{\infty}\sum_{m=0}^{n}F_{n-m}(x,y,z,q)\frac{t^n}{(n-m)!m!}+qy\sum_{n=0}^{\infty}\sum_{m=0}^{n}F_{n-m}(x,y,z,q)\frac{t^n}{(n-m)!m!},\end{aligned}$$

which yields our stated result. □

**Theorem 4.10.** *The following integral representation*

$$\int_{\alpha}^{\beta}F_n(x,y,z,q)dx=\frac{F_{n+1}(\beta,y,z,q)-F_{n+1}(\alpha,y,z,q)}{n+1}\tag{4.5}$$

*holds for* $n\geq 0$.

*Proof.* From (4.3), we derive that

$$\begin{aligned}\int_{\alpha}^{\beta}F_n(x,y,z,q)dx &= \frac{1}{n+1}\int_{\alpha}^{\beta}\frac{\partial}{\partial x}F_{n+1}(x,y,z,q)dx\\ &= \frac{F_{n+1}(\beta,y,z,q)-F_{n+1}(\alpha,y,z,q)}{n+1},\end{aligned}$$

which means the asserted result (4.5). □

## 5. Conclusion

In this paper, we have established some generating functions for the generalized Fubini polynomials by using series rearrangement techniques. Also, some summation formulae for that polynomials are derived by using certain operational techniques and by using different analytical means on its generating function. Further, many generating functions and summation formulae for the polynomials related to generalized Fubini polynomials are obtained as applications of main results. The approach presented in this paper is general and can be extended to establish other properties of special polynomials.

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