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Differential Geometry of 1-type Submanifolds and Submanifolds with 1-type Gauss Map

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(Dedicated to the memory of Prof. Dr. Krishan Lal DUGGAL (1929 - 2022))

ABSTRACT

The theory of finite type submanifolds was introduced by the first author in late 1970s and it has become a useful tool for investigation of submanifolds. Later, the first author and P. Piccinni extended the notion of finite type submanifolds to finite type maps of submanifolds; in particular, to submanifolds with finite type Gauss map. Since then, there have been rapid developments in the theory of finite type. The simplest finite type submanifolds and submanifolds with finite type Gauss maps are those which are of 1-type. The classes of such submanifolds constitute very special and interesting families in the finite type theory.
The purpose of this paper is thus to provide a comprehensive survey on 1-type submanifolds and submanifolds with 1-type Gauss maps done during the last forty years.

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Contents

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1. Introduction

The main objects of studies in algebraic geometry are algebraic varieties. Since algebraic varieties were defined via algebraic equations, one can define the degree of an algebraic variety using its algebraic structure. On the other hand, the famous Nash imbedding theorem stated that every Riemannian manifold can be realized as a Euclidean submanifold with sufficiently high codimension. In contrast, one lacks the notion of "degree" for arbitrary Euclidean submanifolds. Inspired by this simple observations, the first author introduced in the late 1970s the notions of "order" and "type" for submanifolds of Euclidean spaces. By applying these notions, the first author was able to establish some sharp estimates of the total mean curvature of closed Euclidean submanifolds in terms of their orders. Further, by utilizing the notion of type, he introduced the concept of finite type submanifolds. Just like minimal submanifolds, it was shown in [45, 46] that finite type submanifolds can be characterized by a spectral variation principle; namely as critical points of directional deformations. It was well-known that the study of finite type submanifolds were extended to the study of finite type maps in a natural way (see, e.g., [22, 33, 38]).

The study of finite type submanifold and finite type maps provides a natural way to link the geometry of Riemannian manifolds with the spectral behaviors of the Riemannian manifolds via their immersions (see, e.g., [22, 33, 38]). In such way, one may obtain some useful information on eigenvalues of a Riemannian manifold which can always be isometrically imbedded in some Euclidean spaces due to Nash's imbedding theorem. For instance, by applying the notion of order the first author was able to provide an optimal estimate of the first non-zero eigenvalue of Laplacian for compact homogeneous spaces in terms of his δ-invariants (see [34, 36]) which extended a well-known result of T. Nagano obtained in [132].

The simplest finite type submanifolds and submanifolds with finite type Gauss maps are those which are of 1-type. The class of such submanifolds constitute a very special and very interesting family in the finite type theory. The purpose of this paper is thus to provide a comprehensive survey on 1-type submanifolds and submanifolds with 1-type Gauss maps done during the last forty years.

2. Preliminaries

In the following, by a manifold we mean a connected differentiable manifold and by a closed manifold we mean a compact manifold without boundary. Manifolds, maps, functions and vector fields are assumed to differentiable and of class C^∞. Let $\mathfrak{X}(M)$ denote the space of all vector fields on a manifold M and $\mathcal{F}(M)$ be the space of all functions on M.

2.1. *Basic definitions and notations*

We denote by $\mathbb{E}^m_t$ the pseudo-Euclidean m-space of index i equipped with the pseudo-Euclidean metric given by

$$g_0 = -\sum_{i=1}^{t} dx_i^2 + \sum_{j=t+1}^{n} dx_j^2, \tag{2.1}$$

where $(x_1, \ldots, x_m)$ is a rectangular coordinate system on $\mathbb{E}^m_t$. Let $\langle\ ,\ \rangle$ denote the inner product on $\mathbb{E}^m_t$ associated with the metric g_0.

Put

$$S^k_s(c) = \left\{ x \in \mathbb{E}^{k+1}_s : \langle x, x\rangle = \frac{1}{c} > 0 \right\}, \tag{2.2}$$

$$H^k_s(c) = \left\{ x \in \mathbb{E}^{k+1}_{s+1} : \langle x, x\rangle = \frac{1}{c} < 0 \right\}. \tag{2.3}$$

Then it is well-known that $S^k_s(c)$ and $H^k_s(c)$ are complete pseudo-Riemannian manifolds of constant curvature c with index s, which are called a *pseudo-Riemannian k-sphere* and a *pseudo-hyperbolic k-space*, respectively.

$\mathbb{E}^k_s, S^k_s(c)$ and $H^k_s(-c)$ are called *pseudo-Riemannian space forms*. In particular, the Lorentzian manifolds $\mathbb{E}^k_1$, $S^k_1(c)$ and $H^k_1(c)$ are called *Minkowski, de Sitter*, and *anti-de Sitter space-times*, respectively. These three Lorentzian manifolds of constant curvatures are known as the *Lorentzian space forms*. A non-zero vector v on a pseudo-Riemannian manifold is called *space-like* (respectively, *time-like*) if $\langle v, v\rangle > 0$ (respectively, $\langle v, v\rangle < 0$). A non-zero vector v is called *light-like* if it satisfies $\langle v, v\rangle = 0$ identically. A pseudo-Riemannian submanifold M of a pseudo-Riemannian manifold is called *space-like* if every non-zero tangent vector on M is space-like.

2.2. *Gradient, divergence and Laplacian*

If (M, g) is a pseudo-Riemannian n-manifold and $f \in \mathcal{F}(M)$, then the *gradient* of f, denote by ∇f or by grad f, is the vector field dual to the differential df, i.e., ∇f is defined by

$$\langle \nabla f, X\rangle = df(X) = Xf, \quad \forall X \in \mathfrak{X}(M). \tag{2.4}$$

In terms of a local coordinate system $\{x_1, \ldots, x_n\}$ on M, df and ∇f are given respectively by

$$df = \sum_{j=1}^{n} \frac{\partial f}{\partial x_j} dx_j \ \text{ and } \ \nabla f = \sum_{i,j} g^{ij} \frac{\partial f}{\partial x_i} \partial_j \tag{2.5}$$

where $\partial_j = \frac{\partial}{\partial x_j}$. Let $X \in \mathfrak{X}(M)$ and $\{e_1, \ldots, e_n\}$ be an orthonormal local frame on M, then the *divergence* of X, denoted by div X, is given by

$$\operatorname{div} X = \sum_{j=1}^{n} \epsilon_j \left\langle \nabla_{e_i} X, e_i \right\rangle, \tag{2.6}$$

where $g(e_i, e_j) = \epsilon_i \delta_{ij}$ and $\epsilon_i = g(e_i, e_i) = \pm 1$.

The *Laplacian* of a function $f \in \mathcal{F}(M)$, denoted by Δf, is given by

$$\Delta f = -\operatorname{div}(\nabla f). \tag{2.7}$$

In terms of a local coordinate system $\{x_1, \ldots, x_n\}$, we have

$$\Delta f = -\sum_{i,j=1}^{n} \left\{ \frac{\partial^2 f}{\partial x_i \partial x_j} - \sum_{k=1}^{n} \Gamma_{ij}^{k} \frac{\partial f}{\partial x_k} \right\}, \tag{2.8}$$

where $\Gamma_{ij}^k, i, j, k = 1, \ldots, n,$ are the Christoffel symbols of (M, g). In terms of the metric tensor $g = \sum_{i,j=1}^{n} g_{ij} dx_i dx_j$, the Laplacian Δ can be expressed as

$$\Delta f = -\frac{1}{\sqrt{\mathfrak{g}}} \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i} \left(\sqrt{\mathfrak{g}} g^{ij} \frac{\partial f}{\partial x_j} \right),$$

where $(g^{ij}) = (g_{k\ell})^{-1}$ and $\mathfrak{g} = \det(g_{ij})$.

2.3. *Basic and Beltrami formulas*

Let $\phi : M \to \widetilde{M}$ be an isometric immersion of an n-dimensional pseudo-Riemannian manifold into another pseudo-Riemannian manifold. Denote by ∇ and $\tilde{\nabla}$ the Levi-Civita connection on M and $\widetilde{M}$, respectively. Then the formulas of Gauss and Weingarten are given respectively by (cf. [36, 136])

$$\widetilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \tag{2.9}$$

$$\widetilde{\nabla}_X \xi = -A_\xi X + D_X \xi \tag{2.10}$$

for vector fields $X, Y \in \mathfrak{X}(M)$ and a normal vector field ξ of M, where h, A and D are the *second fundamental form*, the *shape operator* and the *normal connection*. For each normal vector ξ of M at $p \in M$, the shape operator A_ξ is a symmetric endomorphism of the tangent space T_pM. The *mean curvature vector* of M in $\widetilde{M}$ is given by

$$H = \frac{1}{n} \operatorname{Trace} h. \tag{2.11}$$

A submanifold is called *minimal* if its mean curvature vector H vanishes identically. The shape operator and the second fundamental form are related simply by

$$\left\langle h(X, Y), \xi \right\rangle = \left\langle A_\xi X, Y \right\rangle. \tag{2.12}$$

For an isometric immersion ϕ of a pseudo-Riemannian submanifold M into a pseudo-Euclidean space $\mathbb{E}_t^m$, we have the following formula of Beltrami (see, e.g., [36, page 41, Proposition 2.6]):

$$\Delta \phi = -nH. \tag{2.13}$$

2.4. *Submanifolds of Kähler manifolds*

For a submanifold M of a Kähler manifold $\widetilde{M}$, there exist three important classes of submanifolds; namely the classes of complex, totally real and slant submanifolds based on the action of the almost complex structure J of $\widetilde{M}$ on the tangent bundle TM of M defined as follows.

A submanifold M of a Kähler manifold $\widetilde{M}$ is called a *complex submanifold* if the tangent bundle TM of M invariant under the action of J, i.e., $J(T_pM) = T_pM$ for every $p \in N$. A submanifold M of $\widetilde{M}$ is called *totally real* if J maps each tangent vector of M into the corresponding normal space, i.e., $J(T_pM) \subseteq T_p^{\perp}M$ for any point $p \in M$. In particular, a totally real submanifold M of $\widetilde{M}$ is called *Lagrangian* if it satisfies $\dim_{\mathbf{R}} M = \dim_{\mathbf{C}} \widetilde{M}$. We refer to [35] for general results on Lagrangian submanifolds.

For a non-zero vector $X \in T_pM$ of a submanifold M of a Kähler manifold $\widetilde{M}$, the angle $\theta(X)$ between JX and T_pN is called the *Wirtinger angle* of X. According to [28, 29], a submanifold M of $\widetilde{M}$ is called a *slant submanifold* if the Wirtinger angle $\theta(X)$ is independent of the choice of $X \in T_pM$ and also independent of $p \in M$. In this case, the angle θ is called the *slant angle* of the slant submanifold. Obviously, complex and totally real submanifolds are exactly slant submanifolds with slant angle $\theta = 0$ and $\theta = \frac{\pi}{2}$, respectively. From J-action points of view, slant submanifolds (including complex and totally real submanifolds) are the simplest and the most natural submanifolds of a Kähler manifold.

Besides complex, totally real and slant submanifolds, there is another important class of submanifolds, called CR-submanifolds [10]. More precisely, a submanifold M of a Kähler manifold $\widetilde{M}$ is called a *CR-submanifold* if there exist a holomorphic distribution $\mathcal{D}$ (i.e., $J\mathcal{D} = \mathcal{D}$) and a totally real distribution $\mathcal{D}^{\perp}$ (i.e., $J\mathcal{D}^{\perp} \subset T^{\perp}M$) on M such that $TM = \mathcal{D} \oplus \mathcal{D}^{\perp}$, where $T^{\perp}M$ denotes the normal bundle of M.

3. Maps and submanifolds of finite type

We refer to the books [36, 38, 39, 40, 137] for the basic notations, definitions and formulas for submanifolds of Riemannian and pseudo-Riemannian manifolds.

Let us recall the notions of order and finite type of submanifolds and of maps which were introduced in the late of 1970s by the first author through his attempts to find the best possible estimates of total mean curvature (see [20, 21, 33]).

If M is a closed Riemannian n-manifold with metric g, then the Laplacian Δ of M is an elliptic differential operator acting on $\mathcal{F}(M)$. It is known that the eigenvalues of Δ form a discrete infinite sequence:

$$0 = \lambda_0 < \lambda_1 < \lambda_2 < \ldots \nearrow \infty.$$

Let $V_t = \{f \in \mathcal{F}(M) : \Delta f = \lambda_t f\}$ be the eigenspace of Δ with eigenvalue λ_t. Then V_0 is 1-dimensional and each $V_t\ (t \geq 1)$ is finite-dimensional.

Define an inner product (,) on $\mathcal{F}(M)$ by

$$(f, h) = \int_M fh\, dV.$$

Then $\sum_{t=0}^{\infty} V_t$ is dense in $\mathcal{F}(M)$ (in L^2-sense) so that we have $\mathcal{F}(M) = \hat{\oplus}_k V_k$, where $\hat{\oplus}_{t=0}^{\infty} V_t$ (in L^2-sense) is the completion of $\sum_{t=0}^{\infty} V_t$.

For a function $f \in \mathcal{F}(M)$, let f_t denote the projection of f onto V_t. Then we have the following spectral decomposition of f:

$$f = \textstyle\sum_{t=0}^{\infty} f_t \quad (\text{in } L^2\text{-sense}). \tag{3.1}$$

Since $\dim V_0 = 1$ for each non-constant $f \in \mathcal{F}(M)$, f_0 is a constant. Thus, there exists a natural number p such that $f_p \neq 0$ and

$$f = f_0 + \textstyle\sum_{t \geq p} f_t.$$

If there exist infinite many non-zero f_t's, then we simply put $q = \infty$. Otherwise, there exists a natural number $q \geq p$ such that

$$f_q \neq 0 \ \text{ and } \ f = f_0 + \textstyle\sum_{t=p}^{q} f_t. \tag{3.2}$$

If we allow q to be ∞, then we have (3.2) in general. We call the set

$$T(f) = \{t > 0 : f_t \neq 0\} \tag{3.3}$$

the *order* of f. The smallest element p in $T(f)$ is called the *lower order* of f and the supremum of $T(f)$ is called the *upper order*. A function $f \in \mathcal{F}(M)$ is said to be of *finite type* if $T(f)$ is a finite set. Otherwise, f is *of infinite type*. The function f is said to be of *k-type* if $T(f)$ contains k elements. In a natural way, the above notions can be extended to maps $\phi : M \to \mathbb{E}^m$ of a closed Riemannian manifold M into a Euclidean m-space $\mathbb{E}^m$ or a pseudo-Euclidean m-space $\mathbb{E}^m_s$ with index m.

For a map $\phi : M \to \mathbb{E}^m_s$ of a non-closed Riemannian manifold M into $\mathbb{E}^m_s$, we do not have the spectral decomposition (3.1) for ϕ in general. However, in this case, a map $\phi : M \to \mathbb{E}^m_s$ is simply said to be of *finite type* if ϕ can be decomposed into a finite sum of vector eigenfunctions of the Laplacian of M; namely,

$$\phi = c_0 + \phi_1 + \cdots + \phi_k, \tag{3.4}$$

where c_0 is a constant vector and $\phi_1, \ldots, \phi_k$ are non-constant $\mathbb{E}^m_s$-valued eigenfunctions of Δ. In particular, if the eigenvalues associated with $\phi_1, \ldots, \phi_k$ are mutual distinct, then ϕ is said to be of k-type. In particular, if one of the eigenvalues associated with $\phi_1, \ldots, \phi_k$ is zero, then M is said to be of *null k-type* (see, e.g., [22, 38]).

Let $\phi : M \to \mathbb{E}^m$ be a k-type map with spectral decomposition given by (3.4). We put $\mathcal{E}_i = \mathrm{Span}\{\phi_i(x) : x \in M\} \subset \mathbb{E}^m$ for $i = 1, \ldots, k$. Then ϕ is called *linearly independent* if $\mathcal{E}_1, \ldots, \mathcal{E}_k$ are linearly independent subspaces, i.e., if

$$\dim(\mathrm{Span}\{\mathcal{E}_1 \cup \ldots \cup \mathcal{E}_k\}) = \dim \mathcal{E}_1 + \cdots + \dim \mathcal{E}_k.$$

And ϕ is called *orthogonal* if $\mathcal{E}_1, \ldots, \mathcal{E}_k$ are mutually orthogonal subspaces (see [31]). Clearly, 1-type immersions are both orthogonal and linearly independent.

The following results were proved in [54].

Theorem 3.1. *A finite type immersion $\phi : M \to \mathbb{E}^m$ is linearly independent if and only if it satisfies $\Delta\phi = A\phi + B$ for some constant $m \times m$ matrix A and vector $B \in \mathbb{R}^m$.*

Theorem 3.2. *A finite type immersion $\phi : M \to \mathbb{E}^m$ is orthogonal if and only if it satisfies $\Delta\phi = A\phi + B$ for some constant $m \times m$ symmetric matrix A and vector $B \in \mathbb{R}^m$.*

When A is a diagonal matrix, then an immersion ϕ of M into a Euclidean space is said to be of *coordinate 1-type* if it satisfies $\Delta\phi = A\phi$ (cf. [87, 64]).

4. 1-type submanifolds in Euclidean and pseudo-Euclidean spaces

T. Takahashi proved in [147] that an isometric immersion ϕ of a Riemannian manifold M into a Euclidean space satisfies

$$\Delta\phi = \lambda\phi \tag{4.1}$$

for some constant λ if and only if ϕ is either a minimal immersion or a minimal immersion in a hypersphere. In the same paper, T. Takahashi also proved that every irreducible compact homogeneous space admits a minimal immersion into a hypersphere.

In terms of finite type theory, Takahashi's theorem can be stated as follows: A submanifold of a Euclidean space $\mathbb{E}^m$ is of 1-type if and only if it is either a minimal submanifold of $\mathbb{E}^m$ or a minimal submanifold of a hypersphere of $\mathbb{E}^m$.

4.1. 1-type submanifolds in $\mathbb{E}^m_s$

The next result of the first author classifies all 1-type submanifolds of pseudo-Euclidean spaces.

Theorem 4.1. [24] *If M is a pseudo-Riemannian manifold of $\mathbb{E}^m_s$, then M is of 1-type if and only if it is one of the following:*

(a) *A minimal submanifold of $\mathbb{E}^m_s$;*
(b) *A minimal submanifold of a pseudo-sphere $S^{m-1}_s(c) \subset \mathbb{E}^m_s$, $c > 0$;*
(c) *A minimal submanifold of a pseudo-hyperbolic space $H^{m-1}_{s-1}(c) \subset \mathbb{E}^m_s$, $c < 0$.*

An isometric immersion $\phi : M \to \mathbb{E}^m$ of a Riemannian manifold M into Euclidean space is called *biharmonic* if the immersion satisfies $\Delta^2\phi = 0$ identically (see , e.g., [47, 48, 137]).

The next theorem is due to I. Dimitrić [65, 68].

Theorem 4.2. *The only finite type biharmonic submanifolds of $\mathbb{E}^m$ are minimal submanifolds; hence there are of 1-type.*

4.2. Curves and surfaces in $\mathbb{E}^3$

Obviously, the simplest Euclidean submanifolds are planar curves. For planar curves we have the following.

Theorem 4.3. [21] *The only closed planar curve of finite type are circles which is of 1-type.*

For non-closed planar curves, we have the following results.

Theorem 4.4. [26, 43] *Curves of finite type in $\mathbb{E}^2$ are 1-type curves given by open portions of circles or open portions of lines.*

The next result shown that there exist ample examples of finite type curves in $\mathbb{E}^3$ which are not of 1-type.

Theorem 4.5. [44] *For every integer $k \geq 2$ there are infinitely many non-equivalent curves of k-type in $\mathbb{E}^3$.*

For finite type curves of a 2-sphere in a Euclidean 3-space, we have

Theorem 4.6. [43] *Every curve of finite type in a 2-sphere is a circle, which is of 1-type.*

The following conjectures were made by the first author in [25, 30, 33] which remain open till now.

Conjecture 1. *The only finite type closed hypersurfaces of a Euclidean space are the hyperspheres, which are of 1-type.*

Conjecture 2. *The only finite type surfaces in $\mathbb{E}^3$ are either of 1-type or open portion of circular cylinders.*

The first classification result for surfaces of finite type in a Euclidean 3-space is the following which provided a partial solution to Conjecture 2.

Theorem 4.7. [25] *A tube in $\mathbb{E}^3$ is of finite type if and only if it is an open portion of a circular cylinder.*

A surface in $\mathbb{E}^3$ is called a *spiral surface* if it is generated by rotating a plane curve γ about an axis L contained in the plane of the curve γ and simultaneously transforming γ homothetically relative to a point of L.

The next result from [8] provided another partial support for Conjecture 2.

Theorem 4.8. *A spiral surface in $\mathbb{E}^3$ is of finite type if and only if it is a minimal surface which is of 1-type.*

4.3. 1-type surfaces in $\mathbb{E}^3$ with respect to $\Delta^J\,(J = II, III)$

S. Stamatakis and H. Al-Zoubi studied finite type surfaces M in $\mathbb{E}^3$ via the Beltrami–Laplace operators $\Delta^J\,(J = II, III)$ corresponding to the second and the third fundamental form respectively of a surface M in $\mathbb{E}^3$. In defining the operators Δ^{II} and Δ^{III} on M, they assumed that the surface M consists of only of elliptic points. Analogous to the finite type submanifolds as introduced by the first author earlier, they called a surface M in $\mathbb{E}^3$ is of finite J-type if the position vector of M can be decomposed as a finite sum of eigenvectors of Δ^J ($J = II$ or $J = III$). Also, they proved the following three results in [145].

Theorem 4.9. *A surface in $\mathbb{E}^3$ is of II-type 1 if and only if it is an open portion of a sphere.*

Theorem 4.10. *A surface in $\mathbb{E}^3$ is of null III-type 1 if and only if it is minimal.*

Theorem 4.11. *A surface in $\mathbb{E}^3$ is of III-type 1 if and only if it is an open portion of a sphere.*

5. 1-type integral submanifolds of Sasakian $\mathbb{R}^{2n+1}(-3)$

For general references to almost contact metric and Sasakian manifolds, we refer to D. E. Blair's book [16]. An odd-dimensional manifold M is called an *almost contact metric manifold* if there exist a tensor field ϕ of type $(1,1)$, a vector field ξ, 1-form η, and a metric g on M such that

$$\phi^2 = -I + \eta \otimes \xi, \quad \eta(\xi) = 1, \tag{5.1}$$

$$g(\phi X, \phi Y) = g(X,Y) - \eta(X)\eta(Y), \quad g(\xi, X) = \eta(X), \tag{5.2}$$

for all $X, Y \in \mathfrak{X}(M)$. The structure (ϕ, ξ, η, g) is called an *almost contact metric structure*. Further, if an almost contact metric manifold satisfies

$$d\eta(X,Y) = g(X, \phi Y), \tag{5.3}$$

then M is called a *contact metric manifold*. The vector field ϕ on a contact metric manifold is called the *characteristic vector field*. A contact metric manifold is called a *Sasakian manifold* if it satisfies

$$[\phi, \phi] + 2d\eta \otimes \phi = 0,$$

where $[\phi, \phi]$ denotes the Nijenhuis torsion of ϕ. A plane section in a tangent space T_xM of M is called a *ϕ-section* if there exists a vector $X \in T_xM$ orthogonal to ξ such that $\{X, \phi(X)\}$ span the section. The sectional curvature $K(X, \phi(X))$ is called ϕ-sectional curvature.

Consider $\mathbb{R}^{2n+1}$ with Cartesian coordinates $(x_1, y_1, \ldots, x_n, y_n, z)$ and the contact form given by

$$\eta = \frac{1}{2}\Big(dz - \sum_{i=1}^{n} y_i dx_i\Big). \tag{5.4}$$

The characteristic vector field ξ of $\mathbb{R}^{2n+1}$ is given by $2\frac{\partial}{\partial z}$ and its Sasakian metric and ϕ are given respectively by

$$g = \eta \otimes \eta + \frac{1}{4}\sum_{i=1}^{n}\{(dx_i)^2 + (dy_i)^2\}, \tag{5.5}$$

$$\phi = \begin{pmatrix} 0 & \delta_{ij} & 0 \\ -\delta_{ij} & 0 & 0 \\ 0 & y_i & 0 \end{pmatrix}, \quad i = 1, \ldots, n. \tag{5.6}$$

It is known that with this contact metric structure (ϕ, ξ, η, g), $\mathbb{R}^{2n+1}$ is a Sasakian manifold with constant ϕ-sectional curvature -3, simply denoted by $\mathbb{R}^{2n+1}(-3)$ (see, e.g., [6, 38]). For $\mathbb{R}^{2n+1}(-3)$, we have

$$\begin{aligned} &\phi\xi = 0, \quad \eta \circ \phi = 0, \quad \bar{\nabla}_X \xi = -\phi X, \\ &g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \\ &(\bar{\nabla}_X \phi)Y = g(X, Y)\xi - \eta(Y)X, \end{aligned} \tag{5.7}$$

where $\bar{\nabla}$ is the Levi-Civita connection. The vector fields

$$e_i = 2\frac{\partial}{\partial y_i}, \quad \phi e_i = 2\Big(\frac{\partial}{\partial x_i} + y_i \frac{\partial}{\partial z}\Big), \quad \xi = 2\frac{\partial}{\partial z}, \quad i = 1, \ldots, n, \tag{5.8}$$

form an orthonormal basis of $\mathbb{R}^{2n+1}(-3)$, called a *ϕ-basis.*

Let $x : M \to \mathbb{R}^{2n+1}(-3)$ be an isometric immersion. We put

$$x = \sum_{i=1}^{n} (\bar{x}_i e_i + \bar{x}_{n+i} e_{n+i}) + \bar{x}_{2n+1}\xi, \quad e_{n+i} = \phi e_i. \tag{5.9}$$

The *ϕ-position vector* x_ϕ of the immersion ϕ is defined by $(\bar{x}_1, \ldots, \bar{x}_{2n})$. Now, since one has the spectral decomposition for each coordinate function $\bar{x}^A$, we also have the spectral decomposition for the ϕ-position vector field as follows:

$$x_\phi = (x_\phi)_0 + \sum_{t+p}^{q} (x_\phi)_t, \tag{5.10}$$

where $(x_\phi)_0$ is a constant vector and p and q denote the lower and upper orders of ϕ-position vector x_ϕ.

Similar to what we did earlier, an immersion of x of M in $\mathbb{R}^{2n+1}(-3)$ is called *finite type* if q is finite; and it is *of k-type* if there are exactly k non-zero terms in the spectral decomposition (5.10). Hence, we can also study finite type submanifolds of $\mathbb{R}^{2n+1}(-3)$ in the same way as for submanifolds in Euclidean spaces. A Riemannian manifold M isometrically immersed in the Sasakian $\mathbb{R}^{2n+1}(-3)$ is called an *integral submanifold* if η restricted to M vanishes. Further, if $\dim M = n$, then M is called a *Legendre submanifold*.

Consider the following cylinder $N^{2n}(c) \subset \mathbb{R}^{2n+1}(-3)$ defined by

$$N^{2n}(c) = \Big\{ x \in \mathbb{R}^{2n+1}(-3) : \sum_{i=1}^{2n} (x^i - x_0^i)^2 = 4c^2 \Big\}, \tag{5.11}$$

where c is a constant and $\phi^2 x_0 \in \mathbb{R}^{2n+1}(-3)$.

The next result on 1-type Legendre curves in $\mathbb{R}^3(-3)$ was due to C. Baikoussis and D. E. Blair.

Theorem 5.1. [6] *Every Legendre curve of* $\mathbb{R}^3(-3)$ *lying in* $N^2(c)$ *defined by* (5.11) *is of 1-type and it is of the form*

$$x(t) = x_0 + (c_1, c_2, 0) + \big(2c\cos t, 2c\sin t, -c^2(2t - \sin 2t) + 2cc_2\cos t + c_0\big)$$

for some constants c_1 *and* c_2.

Baikoussis and Blair also proved the following theorem.

Theorem 5.2. [6] *Let* $x : M \to \mathbb{R}^{2n+1}(-3)$ *be an* r*-dimensional integral closed submanifold whose* ϕ*-position vector field satisfies* (5.10)*. Then* M *is of 1-type with order* p *if and only if it is a minimal submanifold of the cylinder* $N^{2n}(c)$ *with* $c^2 = r/\lambda_p$.

6. Standard imbedding of compact irreducible symmetric spaces

An isometry s of a Riemannian manifold is called involutive if $s^2 = s \circ s$ is the identity map. A Riemannian manifold M is called a *symmetric space* if, for each $p \in M$, there is an involutive isometry s_p such that p is an isolated fixed point of s_p on M. We call s_p the point symmetry at p and denote by G_M or by G the closure of the group of isometries generated by $\{s_p : p \in M\}$ in the compact-open topology. Then G is a Lie group acting transitively on M; hence the typical isotropy subgroup K at a point o (i.e., K is the stabilizer of o) is compact and $M = G/K$. A compact symmetric space is called isotropy-irreducible if the linear isotropy representation $K \to O(T_oM)$ of K is irreducible. A compact symmetric space is called an *irreducible* if it is isotropy-irreducible.

6.1. Standard immersions of compact symmetric spaces

Let $M = G/K$ be a compact irreducible symmetric space. For an eigenvalue λ of the Laplacian on M we denote the corresponding eigenspace by V_λ. Let m_λ be the multiplicity of λ. Assume that $\phi_1, \ldots, \phi_{m_\lambda}$ is an orthonormal basis of the eigenspace V_λ. Then there exists a constant $c_\lambda > 0$ such that the map $\Psi_\lambda : M \to \mathbb{E}^{m_\lambda}$ by

$$\Psi_\lambda(u) = c_\lambda(\phi_1(u), \ldots, \phi_{m_\lambda}(u)) \tag{6.1}$$

is a minimal immersion of M into a hypersphere of a Euclidean (m_λ)-space $\mathbb{E}^{m_\lambda}$ (see, e.g., [147]). When λ is the i-th non-zero eigenvalue of Laplacian of a compact irreducible symmetric space M, then Ψ_λ is called the *i-th standard immersion* of M. Obviously, every standard immersion of a compact irreducible symmetric space is of 1-type.

If M is an n-sphere S^n, then the first standard immersion is the usual imbedding of S^n in $\mathbb{E}^{n+1}$ as an ordinary hypersphere. The second standard immersion of S^n is called the *Veronese immersion*, which is the first standard imbedding of the real projective n-space RP^n into $S^{\binom{n+2}{2}-1} \subset \mathbb{E}^{\binom{n+2}{2}}$. Notice that not every 1-type isometric immersion of order $\{k\}$ of a compact irreducible symmetric space in a Euclidean space is a standard immersion. This is not true even for a 3-sphere. In fact, N. Ejiri constructed in [85] a 1-type immersion of the 3-sphere $S^3(4)$ of radius 4 into a unit hypersphere S^6 of $\mathbb{E}^7$ which is not a standard immersion.

6.2. First standard imbedding of projective spaces

Let $\mathbb{F}$ denote the field of $\mathbb{R}$ of real numbers, $\mathbb{C}$ of complex numbers, or $\mathbb{Q}$ of quaternions. Then we have the following canonical inclusions $\mathbb{R} \subset \mathbb{C} \subset \mathbb{Q}$. Let us put $d = d(\mathbb{F})$ which is equal to $1, 2$ or 4 according to $\mathbb{F}$ is $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{Q}$, respectively. For an element in $\mathbb{Q}$,

$$z = z_0 + z_1\mathrm{i} + z_2\mathrm{j} + z_3\mathrm{k} \in \mathbb{Q}, \quad z_0, z_1, z_2, z_3 \in \mathbb{R},$$

the conjugate $\bar{z}$ of z is given by $\bar{z} = z_0 - z_1\mathrm{i} - z_2\mathrm{j} - z_3\mathrm{k}$. For a complex number z, the conjugate $\bar{z}$ is the ordinary complex conjugate of z; and if z is real, we have $\bar{z} = z$. The column space $\mathbb{F}^{m+1}$ is an $d(m+1)$-dimensional vector space over $\mathbb{R}$ with the usual Euclidean inner product $\langle z, w\rangle = Re(z^*w)$, where $z^* = \bar{z}^T$ for $z \in \mathbb{F}^{m+1}$. Let $S^{(m+1)d-1}(1)$ be the unit hypersphere of $\mathbb{F}^{m+1}$ defined by $\{z \in \mathbb{F}^{m+1} : z^*z = 1\}$.

For a matrix A over $\mathbb{F}$, let A^T and $\bar{A}$ denote the transpose and the conjugate of A, respectively. Put

$$\begin{aligned} &M(m+1;\mathbb{F}) = \text{the space of } (m+1)\times(m+1) \text{ matrices over } \mathbb{F},\\ &U(m+1;\mathbb{F}) = \{A \in M(m+1;\mathbb{F}) : A^*A = I\} \ \text{ (unitary matrices over } \mathbb{F}),\\ &H(m+1;\mathbb{F}) = \{A \in M(m+1;\mathbb{F}) : A^* = A\} \ \text{ (Hermitian matrices over } \mathbb{F}), \end{aligned}$$

where $A^* = \bar{A}^T$ and I is the identity matrix. Then with the inner product:

$$\langle A, B\rangle = \frac{1}{2} Re \, \mathrm{Tr}(AB^*), \tag{6.2}$$

the space $M(m+1;\mathbb{F})$ is an $d(m+1)^2$-dimensional Euclidean space. Clearly, we have $\langle A, B\rangle = \frac{1}{2}\mathrm{Tr}(AB)$ for any $A, B \in H(m+1;\mathbb{F})$.

The standard way to construct the projective space $\mathbb{F}P^m$ is using Hopf's fibration $\pi : S^{(m+1)d-1}(1) \to \mathbb{F}P^m$, where $\mathbb{F}P^m$ is obtained as the quotient space of $S^{(m+1)d-1}(1) \subset \mathbb{F}^{m+1}$ via the action group of elements in $\mathbb{F}$ of norm 1 as follows: Consider the map $\tilde{\psi}_{\mathbb{F}} : S^{(m+1)d-1}(1) \to H(m+1;\mathbb{F})$ defined by

$$\tilde{\psi}_{\mathbb{F}}(z) = zz^* = \begin{pmatrix} |z_0|^2 & z_0\bar{z}_1 & \cdots & z_0\bar{z}_m \\ & \cdots & \cdots & \\ z_m\bar{z}_0 & z_m\bar{z}_1 & \cdots & |z_m|^2 \end{pmatrix}, \; z = (z_i) \in S^{(m+1)d-1}(1). \tag{6.3}$$

Then $\tilde{\psi}$ induces a map $\psi_{\mathbb{F}} : \mathbb{F}P^m \to (m+1;\mathbb{F})$ defined by

$$\psi_{\mathbb{F}}(\pi(z)) = \tilde{\psi}_{\mathbb{F}}(z) = zz^*. \tag{6.4}$$

We simply denote $\psi_{\mathbb{F}}(\pi(z))$ by $\psi_{\mathbb{F}}(z)$ if there is no confusion.

The following result is well-known (see, e.g., [38, 142, 146])

Theorem 6.1. *The map $\psi_{\mathbb{F}} : \mathbb{F}P^m \to H(m+1;\mathbb{F})$ defined by (6.4) is the first standard imbedding of $\mathbb{F}P^m$ into $H(m+1;\mathbb{F})$ which is equivariant and invariant under the action of $U(m+1;\mathbb{F})$. Also, $\psi_{\mathbb{F}}(\mathbb{F}P^m)$ lies in a hypersphere of $H(m+1;\mathbb{F})$ centered at $I/(m+1)$ with radius $r = \sqrt{m/2(m+1)}$.*

7. 1-type submanifolds via projective spaces

Let $\psi : \mathbb{F}P^m \to H(m+1;\mathbb{F})$ be the imbedding of $\mathbb{F}P^m$ into $H(m+1;\mathbb{F})$ defined by (6.4). If $\phi : M \to FP^m$ is an isometric immersion of a compact Riemannian n-manifold M into FP^m, we have the following associated immersion:

$$\hat{\phi}_{\mathbb{F}} = \psi_{\mathbb{F}} \circ \phi : M \to H(m+1;\mathbb{F}) = \mathbb{E}^{d(m+1)}. \tag{7.1}$$

The immersion given by the composition (7.1) is called the *quadratic representation* of ϕ.

Now, we present several classification theorems for which the immersions $\hat{\phi}_{\mathbb{F}} : M \to H(m+1;\mathbb{F}) = \mathbb{E}^{d(m+1)}$ is a 1-type submanifold via (6.4).

7.1. 1-type submanifolds via real projective spaces

The following result of I. Dimitrić [72] completely classified 1-type submanifolds of RP^m.

Theorem 7.1. *Let $\psi_{\mathbb{R}} : \mathbb{R}P^m \to H(m+1;\mathbb{R})$ be the first standard imbedding of $\mathbb{R}P^m$ into $H(m+1;\mathbb{R})$ via (6.4) and $\phi : M \to \mathbb{R}P^m$ an isometric immersion of a Riemannian n-manifold. Then the quadratic representation of $\phi : M \to \mathbb{R}P^m$ is of 1-type if and only if M is a totally geodesic $\mathbb{R}P^n$ in $\mathbb{R}P^m$ via ϕ.*

Remark 7.1. If $\phi : M \to \mathbb{R}P^n$ is a minimal immersion in RP^m, this theorem is due to A. Ros [142].

7.2. 1-type submanifolds via complex projective spaces

Let CP^m be the complex projective m-space equipped with the Fubini-Study metric of constant holomorphic sectional curvature 4. Let

$$\psi_{\mathbb{C}} : CP^m \to \mathbb{E}^{2(m+1)} = H(m+1,\mathbb{C})$$

be the first standard isometric imbedding of $CP^m(4)$ given by (6.4).

Submanifolds of $\mathbb{C}P^m$ with 1-type quadratic representations were first studied by A. Ros in [142]. More precisely, he proved the following.

Theorem 7.2. *Let M be an n-dimensional closed minimal CR-submanifold of $\mathbb{C}P^m$. Then the quadratic representation of M via $\psi_{\mathbb{C}}$ is of 1-type in $H(m+1,\mathbb{C})$ if and only if one of the following two cases occurs:*

(1) *n is even and M is a totally geodesic $\mathbb{C}P^{\frac{n}{2}} \subset \mathbb{C}P^m$.*

(2) *M is a totally real minimal submanifold of a totally geodesic $\mathbb{C}P^n \subset \mathbb{C}P^m$.*

Real hypersurfaces of $\mathbb{C}P^m$ with 1-type quadratic representation were classified by A. Ros in [143]. The complete classification of submanifolds of CP^m with 1-type quadratic representation was achieved by I. Dimitrić in [67, 70] as the following.

Theorem 7.3. *Let $\phi : M \to \mathbb{C}P^m$ be an isometric immersion of a Riemannian n-manifold M into $\mathbb{C}P^m(4)$. Then the quadratic representation of ϕ via $\psi_{\mathbb{C}}$ is of 1-type if and only if one of the following three cases occurs:*

(1) *n is even and M is an open portion of $\mathbb{C}P^{n/2}$ immersed in $\mathbb{C}P^m$ as a totally geodesic complex submanifold.*

(2) *M is a totally real minimal submanifold of a totally geodesic $\mathbb{C}P^n \subset \mathbb{C}P^m$.*

(3) *n is odd and M is an open portion of the geodesic hypersphere*

$$\pi\left(S^1\left(\tfrac{1}{\sqrt{n+3}}\right) \times S^n\left(\sqrt{\tfrac{n+2}{n+3}}\right)\right)$$

of radius $r = \cot^{-1}(\frac{1}{\sqrt{n+2}})$ of a canonically imbedded complex projective space $\mathbb{C}P^{\frac{1}{2}(n+1)}$ immersed in $\mathbb{C}P^m(4)$ as a totally geodesic submanifold, where $\pi : S^{2m+1} \to \mathbb{C}P^m$ is the Hopf's fibration.

7.3. 1-type submanifolds via quaternion projective spaces

Assume that $(\bar{M}, \bar{g})$ is a Riemannian manifold such that there exists a rank 3-subbundle σ of $End(T\bar{M})$ with local basis $\{J_1, J_2, J_3\}$ satisfying

$$\bar{g}(J_\alpha X, J_\alpha Y) = \bar{g}(X, Y), \;\; J_\alpha^2 = -I, \;\; \textit{and} \;\; J_\alpha J_{\alpha+1} = -J_{\alpha+1} J_\alpha = J_{\alpha+2},$$

for $\alpha \in \{1, 2, 3\}$, where I is the identity tensor of type (1, 1) on $\bar{M}$ and the indices are taken from $\{1, 2, 3\}$ modulo 3. Then $(\bar{M}, \sigma, \bar{g})$ is called an almost quaternionic Hermitian manifold. Further, if the bundle σ is parallel with respect to the Levi-Civita connection $\bar{\nabla}$ of $\bar{g}$, then $(\bar{M}, \sigma, \bar{g})$ is called a quaternionic Kähler manifold.

The notion of quaternion CR-submanifolds of a quaternion Kähler manifold was introduced in [9] as follows. A submanifold M of a quaternionic Kähler manifold $(\bar{M}, \sigma, \bar{g})$ is said to be a quaternionic CR-submanifold if there exists two orthogonal complementary distributions $\mathcal{D}$ and $\mathcal{D}^\perp$ on M such that $\mathcal{D}$ is invariant under quaternionic structure and $\mathcal{D}^\perp$ is totally real.

The following classification result for quaternion CR-submanifolds were proved by I. Dimitrić in [71].

Theorem 7.4. *Let $\phi : M \to \mathbb{Q}P^m$ be an isometric immersion of a complete Riemannian n-manifold M into $\mathbb{Q}P^m$ as a quaternion CR-submanifold of $\mathbb{Q}P^m$. Then the quadratic representation of ϕ via $\psi_{\mathbb{Q}}$ is of 1-type in $H(m+1; \mathbb{Q})$ if and only if one of the following two cases occurs:*

(1) *$n \equiv 0$ (mod 4) and M is $\mathbb{Q}^{n/4}$ which is imbedded canonically in $\mathbb{Q}P^m$ as a totally geodesic quaternion submanifold.*

(2) *M is a totally real minimal submanifold of a canonically imbedded $\mathbb{Q}P^n \subset \mathbb{Q}P^m$.*

A submanifold of quaternion Kähler manifold $(\bar{M}, \sigma, \bar{g})$ is called an anti-CR submanifold if its normal bundle splits as $T^\perp M = \mathcal{K} \oplus \mathcal{L}$ with $J_\alpha \mathcal{K} \subset TM$ and $J_\alpha \mathcal{L}^L \subset \mathcal{L}$ for every $\alpha = 1, 2, 3$.

For anti-CR submanifolds, Dimitrić proved the following classification in [71].

Theorem 7.5. *Let $\phi : M \to \mathbb{Q}P^m$ $(n, m \geq 2)$ be an isometric immersion of a complete Riemannian n-manifold M into $\mathbb{Q}P^m$ as a quaternion anti-CR submanifold of $\mathbb{Q}P^m$. Then the quadratic representation of ϕ via $\psi_{\mathbb{Q}}$ is a 1-type submanifold in $H(m+1; \mathbb{Q})$ if and only if one of the following four cases occurs:*

(1) *$n \equiv 0$ (mod 4) and M is $\mathbb{Q}^{n/4}$ which is imbedded canonically in $\mathbb{Q}P^m$ as a totally geodesic quaternion submanifold.*

(2) *$n \equiv 0$ (mod 3) and M is immersed as a minimal anti-Lagrangian submanifold of a canonically imbedded $\mathbb{Q}P^{n/3} \subset \mathbb{Q}P^m$.*

(3) *$n \equiv 0$ (mod 4) and M is imbedded as a geodesic hypersphere of radius $r = \cot^{-1}(\sqrt{3}/(n+2))$ of a canonical $\mathbb{Q}P^{(n+1)/4} \subset \mathbb{Q}P^m$.*

(4) *$n = 3$ and M is imbedded as an arbitrary geodesic hypersphere of a canonical $S^4(\frac{1}{2}) \subset QP^1 \subset \mathbb{Q}P^m$.*

7.4. 1-type submanifolds via Cayley plane

Analogous to the first standard imbedding of projective spaces $\mathbb{F}P^m$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{Q}$ into $H(m+1;\mathbb{F})$, there exist a similar standard imbedding $\psi_{\mathcal{O}} : \mathcal{O}P^2 \to H(3, \mathcal{O})$ of the Cayley projective plane $\mathcal{O}P^2$ into the space of 3×3 Hermitian Cayley matrices $H(3, \mathcal{O})$ (see [125]).

For real hypersurface in the Cayley projective plane $\mathcal{O}P^2$, I. Dimitrić proved the following.

Theorem 7.6. [71] *Let M be a real hypersurface of the Cayley projective plane $\mathcal{O}P^2$. Then M has 1-type quadratic representation via $\psi_{\mathcal{O}}$ if and only if M is a geodesic hypersphere of radius $r = \cot^{-1}\sqrt{7/17}$.*

8. 1-type submanifolds in Grassmann manifolds

Let $U(n, \mathbb{F})$ be the group of unitary matrices of degree n over $\mathbb{F}$ defined as before. Then we have

$$O(n) = U(n, \mathbb{R}), \quad U(n) = U(n, \mathbb{C}) \ \ and \ \ Sp(n) = U(n, \mathbb{Q}).$$

Let $G(p, q)$ be the real (unoriented) Grassmann manifold $O(p+q)/(O(p) \times O(q))$ and $L(n)$ be the Lagrangian Grassmannian $U(n)/O(n)$. Define the mapping:

$$\varphi : G(p, q) \to U(n)/O(n), \quad n = p + q, \tag{8.1}$$

which carries a linear p-subspace $L \subset \mathbb{E}^n$ to $L \oplus L^{\perp} \in \mathbb{C}^n \cong \mathbb{E}^n \oplus (i\mathbb{E}^n)$. Let $S(n, \mathbb{C})$ be the vector space of symmetric matrices of order n over the complex field $\mathbb{C}$. Then $S(n, \mathbb{C})$ can be identified with $\mathbb{C}^{\frac{1}{2}n(n+1)}$ equipped with the following Hermitian inner product:

$$\langle X, Y \rangle = \frac{1}{8}\text{Trace}(X \cdot \bar{Y}), \quad X, Y \in S(n, \mathbb{C}).$$

The map $A \mapsto AA^t$ from $U(n)$ into $S(n, \mathbb{C})$ induces an isometric equivariant imbedding:

$$\mu : U(n)/O(n) \to S(n, \mathbb{C}). \tag{8.2}$$

The composition map

$$\hat{\varphi} = \mu \circ \varphi : G(k, h) \to S(n, \mathbb{C}) \tag{8.3}$$

is a minimal imbedding into a hypersphere of a $\frac{1}{2}n(n+1)$-dimensional subspace $\mathbb{E}^{\frac{1}{2}n(n+1)} \subset S(n, \mathbb{C})$.

For minimal submanifolds of $G(p, q)$ and of $U(n)/O(n)$, C. Brada and L. Niglio proved the next two classification results in [17].

Theorem 8.1. *Let M be a minimal submanifold of the Grassmann manifold $G(p, q)$ with $n = p + q$. If M is a 1-type submanifold of $\mathbb{E}^{\frac{1}{2}n(n+1)} \subset S(n, \mathbb{C})$ via (8.3), then M is isometric to one of the following Riemannian products:*

$$G(r, s_1) \times \cdots \times G(r_k, s_k), \tag{8.4}$$

where $r_1, \ldots, r_k, s_1, \ldots, s_k$ are positive integers satisfying

$$r_1 + s_1 = \cdots = r_k + s_k, \quad r_1 + \cdots + r_k \leq p, \quad s_1 + \cdots + s_k \leq q.$$

Moreover, the immersion is given in a natural way.

Theorem 8.2. *Let M be a minimal submanifold of $U(n)/O(n)$. If M is a 1-type submanifold of $\mathbb{E}^{\frac{1}{2}n(n+1)} \subset S(n, \mathbb{C})$ via (8.3), then M is isometric to one of the following Riemannian products:*

(1) *$G(r_1, s_1) \times \cdots \times G(r_k, s_k)$, where $r_1, \ldots, r_k, s_1, \ldots, s_k$ are positive integers satisfying $r_1 + s_1 = \ldots = r_k + s_k$.*
(2) *$U(r_0)/O(r_0) \times G(r_1, s_1) \times \cdots \times G(r_k, s_k)$ where $r_1, \ldots, r_p, s_1, \ldots, s_k$ are positive integers satisfying $r_0 + 1 = \frac{1}{2}(r_1 + s_1) = \cdots = \frac{1}{2}(r_k + s_k)$.*

Moreover, the immersion is given in a natural way.

Remark 8.1. B.-Y. Wu considered in [153], an isometric imbedding ψ of the complex Grassmann manifold $G^{\mathbb{C}}(p, q) = U(p+q)/(U(p) \times U(q))$ into a suitable Euclidean space $\mathbb{E}^N$ (see [125] for details). He then applied this isometric imbedding ψ to prove that an isometric minimal immersion $\phi : M \to G^{\mathbb{C}}(p, q)$ is of 1-type in $\mathbb{E}^N$ via ψ if and only if the Gauss map ν_ϕ of ϕ is a harmonic map.

9. 1-type submanifolds in hyperbolic spaces

Let $\mathbb{C}_1^{m+1}$ denote the complex pseudo-Euclidean space with complex index one. Put

$$H_1^{2m+1}(-1) = \{z = (z_1, z_2, \ldots, z_{m+1}) \in \mathbb{C}_1^{m+1} : \langle z, z\rangle = -1\}, \tag{9.1}$$

where $\langle\ ,\ \rangle$ is the inner product on $\mathbb{C}_1^{m+1}$ associated with the metric of $\mathbb{C}_1^{m+1}$. Then $H_1^{2m+1}(-1)$ is a complete Lorentzian manifold of constant sectional curvature -1, which is known as the $(2m+1)$-dimensional *anti-de Sitter spacetime.*

Let us put $H_1^1 = \{\lambda \in \mathbb{C} : \lambda\bar{\lambda} = 1\}$. Then there is a natural H_1^1-action on $H_1^{2m+1}(-1)$ which maps $z \mapsto \lambda z$. Under the identification induced from this action, the quotient space $H_1^{2m+1}/\sim$ becomes the complex hyperbolic space $\mathbb{C}H^m$ with constant holomorphic sectional curvature -4. This gives rise to the well-known Hopf fiberation:

$$\pi : H_1^{2m+1}(-1) \to \mathbb{C}H^m. \tag{9.2}$$

By identifying a complex line $L = [z]$, $z \in H_1^{2m+1}$, with the operator of the orthogonal projection P onto L, we have an isometric imbedding (see, e.g., [38, 70, 143])

$$\tilde{\psi} : \mathbb{C}H^m \to M(m+1;\mathbb{C}). \tag{9.3}$$

Using this imbedding, O. J. Garay and A. Romero studied in 1-type real hypersurfaces of complex hyperbolic space and obtained the following result.

Theorem 9.1. [88] *There exist no real hypersurfaces of $\mathbb{C}H^m$ for any $m \geq 2$ which are of 1-type via* (9.3)*.*

The complete classification of 1-type submanifolds of $\mathbb{C}H^m$ via (9.3) was obtained by I. Dimitrić in [70].

Theorem 9.2. *Let $\phi : M \to \mathbb{C}H^m$ be an isometric immersion of an n-dimensional Riemannian manifold M into complex hyperbolic space $\mathbb{C}H^m$. Then M is of 1-type via* (9.3) *if and only if one of the following two cases occurs:*

(1) *n is even and M is a complex-space-form of constant holomorphic sectional curvature -4 which is immersed as a totally geodesic complex submanifold of $\mathbb{C}H^m$.*
(2) *M is a totally real minimal submanifold of a complex totally geodesic $\mathbb{C}H^n \subset \mathbb{C}H^m$.*

10. Classical, spherical and hyperbolic Gauss maps

For a given Riemannian manifold (M, g), let $V(\lambda)$ denote the space of all eigenfunctions of M associated with a given eigenvalue λ of the Laplacian Δ as before. Then for any m elements $f_1, \ldots, f_m \in V(\lambda)$, the map $\phi = (f_1, \ldots, f_m) : M \to \mathbb{E}^m$ is always a 1-type map. Hence, in order to study 1-type maps into Euclidean spaces, one needs to impose some suitable conditions on the map, for instance, to study submanifolds with 1-type Gauss map, etc.

10.1. Classical Gauss map

Consider be a linear n-subspace W of the Euclidean m-space $\mathbb{E}^m$. Suppose that $\{e_1, \ldots, e_n\}$ is an oriented orthonormal basis of W, then the wedge product $e_1 \wedge \cdots \wedge e_n$ is a decomposable n-vector of norm 1. Further, $e_1 \wedge \cdots \wedge e_n$ defines an orientation of W in a natural way. Conversely, every decomposable n-vector of norm 1 determines a unique oriented linear n-subspace of $\mathbb{E}^m$.

Let $G(n, m-n)$ be the Grassmann manifold of oriented linear n-subspaces in $\mathbb{E}^m$. Then we may identify the Grassmannian $G(n, m-n)$ with the set of decomposable n-vectors of norm 1. Hence, $G(n, m-n)$ can be considered to be a submanifold of the unit hypersphere $S^{\binom{m}{n}-1}$ in $\mathbb{E}^{\binom{m}{n}} = \wedge^n\mathbb{E}^m$ centered at the origin. Therefore, we have the canonical inclusions:

$$\psi : G(n, m-n) \subset S^{\binom{m}{n}-1} \subset \mathbb{E}^{\binom{m}{n}}. \tag{10.1}$$

For an n-dimensional submanifold of $\mathbb{E}^m$, the (classical) *Gauss map* ν of M is the map:

$$\nu : M \to G(n, m-n) \subset S^{\binom{m}{n}-1} \subset \mathbb{E}^{\binom{m}{n}} \tag{10.2}$$

which carries a point $p \in M$ to the linear n-subspace of $\mathbb{E}^m$ obtained from the tangent space T_pM at p via parallel displacement. More precisely, if $\{e_1, \ldots, e_n\}$ is an oriented local orthonormal frame of TM, then the *Gauss map* of M is given by

$$\nu(p) = (e_1 \wedge \cdots \wedge e_n)(p) \in \mathbb{E}^{\binom{m}{n}}, \quad p \in M. \tag{10.3}$$

10.2. Spherical and hyperbolic Gauss maps

An isometric immersion φ of an oriented Riemannian n-manifold M into a sphere $S^m(1)$ can be also considered as an isometric immersion into a Euclidean space $\mathbb{E}^{m+1}$. Hence, the Gauss map associated with such an immersion can be determined in the classical sense as above. On the other hand, M. Obata modified the definition of the Gauss map in [135] to better capture the properties of the immersion into the sphere, rather than into the Euclidean space, as follows:

Let $\phi : M \to \widetilde{M}$ be an isometric immersion from a Riemannian n-manifold M into an m-dimensional real space form $\widetilde{M}$. Then Obata defined the *generalized Gauss map* as a map which assigns to each $x \in M$ the totally geodesic n-dimensional submanifold of $\widetilde{M}$ tangent to $\phi(M)$ at $\phi(x)$. In the case $\widetilde{M} = S^m(1)$, the generalized Gauss map carries $x \in M$ to a totally geodesic n-sphere of S^m which is uniquely determined by a linear $(n+1)$-dimensional subspace of $\mathbb{E}^{m+1}$, by intersecting S^m with this subspace. Hence, this map can be seen as a map from M to $G(n+1, m+1)$, called the *spherical Gauss map* of ϕ. For the case $\widetilde{M} = H^m(-1)$, the corresponding generalized Gauss map of an isometric immersion $\varphi : M \to H^m(-1)$ is called the *hyperbolic Gauss map* of φ.

10.3. Pseudo-spherical and pseudo-hyperbolic Gauss maps

The classical, generalized, spherical, and hyperbolic Gauss map can be extended naturally from Euclidean ambient space $\mathbb{E}^m$ to a pseudo-Euclidean ambient space $\mathbb{E}^m_s$ as follows:

Let $\phi : M^n_t \to \widetilde{M}^m_s$ be an oriented isometric immersion from a pseudo-Riemannian n-manifold M^n_t with index t into the complete pseudo-Riemannian space $\widetilde{M}^m_s$ of constant curvature. Then the *classical Gauss map* is a map associated to ϕ which carries each $x \in M^n_t$ to a the totally geodesic n-subspace of $\widetilde{M}^m_s$ tangent to $\phi(M^n_t)$ at $\phi(x)$. In the case, $\widetilde{M} = S^m_t(1)$, the totally geodesic n-subspace of $\widetilde{M}^m_s$ tangent to $\phi(M^n_t)$ at $\phi(x)$ is the pseudo-sphere $S^n_t(1)$, it gives rise to a unique oriented $(n+1)$-plane containing $S^n_t(1)$. Thus, the generalized Gauss map can be extended to a map $\hat{\nu}$ of M^n_t into the Grassmann manifold $G_s(n+1, m+1)$ in the natural way, and the composition $\tilde{\nu}$ of $\hat{\nu}$ followed by the natural inclusion of $G_s(n+1, m+1)$ into a pseudo-Euclidean space $\mathbb{E}^N_q$, $N = \binom{m}{n+1}$, for some integer q is called the the *pseudo-spherical Gauss map*.

For the case $\widetilde{M} = H^m_t(-1)$, the totally geodesic n-subspace of $\widetilde{M}^m_s$ tangent to $\phi(M^n_t)$ at $\phi(x)$ is the pseudo-hyperbolic space $H^n_t(-1)$. And it gives rise to a unique oriented $(n+1)$-plane containing $H^n_t(-1)$. Thus, the generalized Gauss map can be extended to a map $\hat{\nu}$ of M^n_t into the Grassmann manifold $G_s(n+1, m+1)$ in a natural way, and the composition $\tilde{\nu}$ of $\hat{\nu}$ followed by the natural inclusion of $G_s(n+1, m+1)$ into a pseudo-Euclidean space $\mathbb{E}^N_q$, $N = \binom{m}{n+1}$, for some integer q is called the *pseudo-hyperbolic Gauss map*.

11. Submanifolds of $\mathbb{E}^m$ with 1-type Gauss maps

The study of Euclidean submanifolds with finite type Gauss map was initiated by the first author and P. Piccinni in [55]. In particular, they characterized closed submanifolds of Euclidean space with 1-type and 2-type Gauss maps. Further, they proved that the Gauss map of a closed submanifold of a Euclidean space is always mass-symmetric, i.e., the center of the Gauss map is at the origin of the ambient Euclidean space.

From their definition, an oriented submanifold M of a Euclidean space $\mathbb{E}^m$ has 1-type Gauss map if and only if its Gauss map ν satisfies

$$\Delta\nu = \lambda(\nu + C) \tag{11.1}$$

for a constant λ and a constant vector C.

For curves in a Euclidean space, we have the following results of the first author P. Piccinni from [55].

Theorem 11.1. *Let γ be a curve in $\mathbb{E}^m$. Then γ is of finite-type if and only if its Gauss map is of finite type. In particular, a closed curve in $\mathbb{E}^m$ is of 1-type if and only its Gauss map is of 1-type.*

In contrast, for a closed submanifold M of $\mathbb{E}^m$ with $\dim M \geq 2$, the type number of M and the type number of its Gauss map are not necessary the same.
The following results are also proved in [55].

Theorem 11.2. *Let M be a closed submanifold of $\mathbb{E}^m$. Then M has 1-type Gauss map if and only if it has parallel mean curvature vector, flat normal connection, and constant scalar curvature.*

Theorem 11.2 implies the following.

Corollary 11.1. *Let M be an isoparametric hypersurface of a hypersphere $S_o^{n+1} \subset \mathbb{E}^{n+2}$. Then M has 1-type Gauss map.*

Corollary 11.2. *The only closed hypersurfaces of Euclidean space with 1-type Gauss map are hyperspheres.*

Since hypersurfaces of a hypersphere $S_o^{n+1} \subset \mathbb{E}^{n+2}$ have flat normal connection in $\mathbb{E}^{n+2}$ automatically , Theorem 11.2 also implies the following.

Corollary 11.3. *A hypersurface M of a hypersphere $S_o^{n+1} \subset \mathbb{E}^{n+2}$ has 1-type Gauss map if and only if one of the following three cases occurs:*

(1) *M is a small hypersphere of S_o^{n+1},*

(2) *M is a hypersurface of S_o^{n+1} which is 2-type in $\mathbb{E}^{n+2}$,*

(3) *M is a minimal hypersurface of S_o^{n+1} with constant scalar curvature.*

The next result also from [55] classified closed surfaces of $\mathbb{E}^m$ with 1-type Gauss map.

Theorem 11.3. [55] *Let M be a closed surface of $\mathbb{E}^m$. Then M has 1-type Gauss map if and only if it is either*

(1) *an ordinary sphere $S^2 \subset \mathbb{E}^3 \subset \mathbb{E}^m$ or*

(2) *the product of two circles $S^1(a) \times S^1(b) \subset \mathbb{E}^4 \subset \mathbb{E}^m$.*

The next result was obtained by C. Jang and K. Park in [102].

Theorem 11.4. *The only surfaces with flat normal connection in $\mathbb{E}^m$ whose Gauss map is of 1-type are ordinary spheres, products of two plane circles and helical cylinders (i.e. products of a straight line and a circular helix).*

Remark 11.1. Without "closed" condition on the surfaces, K.-O. Jang and Y. H. Kim proved in [103] that every 2-type surface in $\mathbb{E}^3$ with 1-type Gauss map is of null 2-type. Hence, it is an open part of a circular cylinder according to a result given in [26]. Also, C. Jang proved in [101] that an orientable, connected surface M in $\mathbb{E}^3$ has 1-type Gauss map if and only if it is an open part of a sphere or an open part of a circular cylinder.
First, we mention that Theorem 11.2 was extended in [61] to the following.

Theorem 11.5. *Let M be a space-like submanifold of $\mathbb{E}_s^m$. Then M has 1-type Gauss map if and only if it has parallel mean curvature vector, flat normal connection, and constant scalar curvature.*

Theorem 11.3 was also extended in [61] to the following.

Theorem 11.6. *Let M be a space-like surface of $\mathbb{E}_s^m$. Then M has 1-type Gauss map if and only if it is M is locally one of the following:*

(1) *a Euclidean plane $\mathbb{E}^2$, a hyperbolic plane H^2, or a hyperbolic cylinder $H^1 \times \mathbb{R}$ in $\mathbb{E}_1^3 \subset \mathbb{E}^m$,*

(2) *the product $H^1 \times H^1$ of two hyperbolic curves in $\mathbb{E}_2^4 \subset \mathbb{E}_s^m$.*

The following results were obtained by U. Dursun in [77].

Theorem 11.7. *f M is an oriented hypersurface of the hyperbolic space $H^{n+1}(-1) \subset \mathbb{E}_1^{n+2}$ with at most two distinct principal curvatures, then M has 1-type Gauss map if and only if it is congruent to one of the surfaces given by*

(1) *$H^n(-c) \subset H^{n+1}(-1)$ with $0 < c \leq 1$,*
(2) *$S^n(c) \subset H^{n+1}(-1)$ with $c > 1$,*
(3) *$H^k(-a) \times S^{n-k}(b)$ with $\frac{1}{b} - \frac{1}{a} = -1$.*

12. Topology and submanifolds with 1-type Gauss map

The following result obtained by the first author, J.-M. Morvan and T. Nore provided a link between topology and type number of Gauss map of a Euclidean closed submanifold.

Theorem 12.1. [50, 51] *Let M be an oriented closed submanifold of $\mathbb{E}^m$. If the Euler class $e(T^{\perp}M)$ of the normal bundle of M is non-trivial, then the Gauss map ν of M is of k-type with $k > m/2$.*

In particular, Theorem 12.1 implies immediately the following.

Theorem 12.2. [50, 51] *Let M be an oriented closed Lagrangian submanifold of $\mathbb{C}^n$ with $n \geq 2$. If Gauss map of M is of 1-type, then the Euler number of M is trivial.*

A result of R. K. Lashof and S. Smale [126] states that if M is an oriented closed n-dimensional submanifold of a Euclidean $2n$-space, then the Euler number of the normal bundle of M equals to twice of the self-intersection number of M in $\mathbb{E}^{2n}$. Hence, Theorem 11.2 implies the following.

Corollary 12.1. [50, 51] *Let M be an n-dimensional closed submanifold of $\mathbb{E}^{2n}$ with $n \geq 2$. If the Gauss map of M is of 1-type, then M has trivial self-intersection number.*

Since the self-intersection number of M in $\mathbb{E}^{2n}$ is a regular homotopic invariant according to a result of S. Smale [144], we also have the following result.

Corollary 12.2. [50, 51] *Let $\phi : M \to \mathbb{E}^{2n}$ be an n-dimensional oriented closed submanifold of $\mathbb{E}^{2n}$. If the Euler number of the normal bundle is non-trivial, then ϕ cannot be deformed regularly to an immersion with 1-type Gauss map.*

Remark 12.1. The well-known *Whitney immersion* can be defined as follows: Let $\phi : \mathbb{E}^{2n+1} \to \mathbb{E}^{4n}$ be the map of $\mathbb{E}^{2n+1}$ into $\mathbb{E}^{4n}$ given by

$$\phi(x_0, x_1, \ldots, x_{2n}) = (x_1, \ldots, x_{2n}, 2x_0x_1, \ldots, 2x_0x_{2n}).$$

Then ϕ gives rise to a (non-isometric) immersion $w : S^{2n} \to \mathbb{E}^{4n}$, called *Whitney's immersion* with a unique self-intersection point. Although the canonical inclusion of S^{2n} in $\mathbb{E}^{2n+1} \subset \mathbb{E}^{4n}$ has 1-type Gauss map, the Whitney immersion $S^{2n} \subset \mathbb{E}^{4n}$ cannot be deformed regularly into an immersion with 1-type Gauss map according to Corollary 12.2.

13. Submanifolds with 1-type spherical Gauss maps

The geometric behavior of classical Gauss map and spherical Gauss map are different. In fact, the concept of spherical Gauss map is more relevant than the classical one for spherical submanifolds. For instance, the classical Gauss map of every compact Euclidean submanifold is mass-symmetric (see [55]). In contrast, the spherical Gauss map of a spherical closed submanifold is not mass-symmetric in general. Based on these observations, the first author and H.-S. Lue initiated the study spherical submanifolds with finite type spherical Gauss map in [49].

For submanifolds of $S^{m-1} \subset \mathbb{E}^m$ with 1-type spherical Gauss map, they proved the following two results.

Theorem 13.1. [49] *A submanifold of S^{m-1} has mass-symmetric 1-type spherical Gauss map if and only if it is a minimal submanifold of S^{m-1} with constant scalar curvature and with flat normal connection.*

Theorem 13.2. [49] *A non-totally geodesic surface in S^{m-1} has mass-symmetric 1-type spherical Gauss map if and only if it is an open portion of the Clifford minimal torus lying fully in a totally geodesic 3-sphere $S^3 \subset S^{m-1}$.*

For non-mass-symmetric 1-type spherical Gauss map, B. Bektaş and U. Dursun proved the following.

Theorem 13.3. [14] *An n-dimensional submanifold M of S^{m-1} has non-mass-symmetric 1-type spherical Gauss map if and only if M is an open part of a small n-sphere of a totally geodesic $(n+1)$-sphere $S^{n+1} \subset S^{m-1}$.*

14. Submanifolds with 1-type pseudo-spherical Gauss map

B. Bektaş, E. Ö. Canfes, and U. Dursun proved the following theorems.

Theorem 14.1. [12] *A pseudo-Riemannian submanifold M_t with index t of $S_s^{m-1}(1) \subset \mathbb{E}_s^m$ has 1-type pseudo-spherical Gauss map if and only if M_t has zero mean curvature in $S_s^{m-1}(1) \subset \mathbb{E}_s^m$, constant scalar curvature and flat normal connection.*

The standard imbedding of $S^1(2) \times S^1(2) \subset S^3(1)$ is called the *Clifford torus*, which is both flat and minimal in $S^3(1)$.

Theorem 14.2. [12] *A non-totally geodesic space-like surface M in $S_1^4(1) \subset \mathbb{E}_1^5$ has 1-type pseudo-spherical Gauss map if and only if it is an open portion of the Clifford torus lying fully in a totally geodesic 3-sphere $S^3(1) \subset S_1^4(1)$.*

Theorem 14.3. [12] *A non–totally geodesic Lorentzian surface M_1 in $\mathbb{S}_1^4(1) \subset \mathbb{E}_1^5$ has 1–type pseudo-spherical Gauss map if and only if it is an open portion of the pseudo-Riemannian Clifford torus lying fully in a totally geodesic pseudo-sphere $S_1^3(1) \subset S_1^4(1)$.*

Theorem 14.4. [12] *An n-dimensional pseudo-Riemannian submanifold M_t with index t and non-null mean curvature vector of a pseudo-sphere $S_s^{m-1}(1) \subset \mathbb{E}_s^m$ has 1-type pseudo-spherical Gauss map with a non-zero constant component in its spectral decomposition if and only if M_t is an open part of a non-flat, non-totally geodesic and totally umbilical pseudo-Riemannian hypersurface of a totally geodesic pseudo-sphere $S_{s^*}^{n+1}(1) \subset S_s^{m-1}(1)$, ($s^* = t \leq s$ or $s^* = t+1 \leq s$), that is, it is an open portion of $S_t^n(c) \subset S_t^{n+1}(1)$ of curvature c for $c > 1$ or $S_t^n(c) \subset S_{t+1}^{n+1}(1)$ of curvature c for $0 < c < 1$ or $H_t^n(-c) \subset S_{t+1}^{n+1}(1)$ of curvature $-c$ for $c > 0$.*

A surface of a pseudo-sphere is called *marginally trapped* if its mean curvature vector is light-like at each point of the surface.

Theorem 14.5. [12] *A marginally trapped space-like surface M in the de Sitter space $S_1^4(1) \subset \mathbb{E}_1^5$ has 1-type pseudo-spherical Gauss map with non-zero constant component in its spectral decomposition if and only if M is congruent to an open part of the surface of curvature one given by*

$$\phi(u, v) = (1, \sin u, \cos u \cos v, \cos u \sin v, 1). \tag{14.1}$$

The next results are due to B. Bektaş, J. Van der Veken, and L. Vrancken.

Theorem 14.6. [15] *Let M be a non-harmonic space-like surface in $S_s^m(1)$. Then M has 1-type pseudo-spherical Gauss map if and only if it is an open part of the Clifford torus lying fully in a totally geodesic 3-sphere $S^3(1) \subset S_s^m(1)$.*

Theorem 14.7. [15] *Let M be a non-harmonic Lorentzian surface in $S_s^m(1) \subset \mathbb{E}_s^{m+1}$. Then M has 1-type pseudo-spherical Gauss map if and only if it is congruent to an open part of the image of*

$$\phi(u, v) = \frac{1}{\sqrt{2}}(\cos u, \sin u, \cosh v, \sinh v)$$

in a totally geodesic $S_1^3(1) \subset S_s^m(1) \subset \mathbb{E}_s^{m+1}$, or of

$$\phi(u, v) = (\cosh u \cos v, \cosh u \sin v, \sinh u \cos v, \sinh u \sin v)$$

in a totally geodesic $S_2^3(1) \subset S_s^m(1) \subset \mathbb{E}_s^{m+1}$.

The authors of [15] also classified harmonic space-like and harmonic Lorenztian surfaces in $S_s^m(1) \subset \mathbb{E}_s^{m+1}$ with pseudo-spherical Gauss map.

15. Submanifolds with 1-type pseudo-hyperbolic Gauss map

R. Yeğin and U. Dursun investigated pseudo-Riemannian submanifold in a pseudo-hyperbolic space with 1-type pseudo-hyperbolic Gauss map in [154]. They proved the following results.

Theorem 15.1. [154] *An oriented pseudo-Riemannian submanifold M_t^n with index t of a pseudo-hyperbolic space $H_s^{m+1}(-1) \subset \mathbb{E}_{s+1}^m$ has a 1-type pseudo-hyperbolic Gauss map if and only if M_t^n has vanishing mean curvature in $H_s^{m+1}(-1)$, constant scalar curvature, and flat normal connection.*

Then they proved the following results.

Theorem 15.2. [154] *Let M be an oriented space-like surface in a pseudo-hyperbolic space $H^{m-1}_s(-1) \subset \mathbb{E}^m_{s+1}$. Then M has a 1-type pseudo-hyperbolic Gauss map if and only if M is congruent to an open part of maximal surface $H(-2) \times H(-2)$ lying in $H^3_1(-1) \subset H^{m-1}_s(-1) \subset \mathbb{E}^m_{s+1}$ or the totally geodesic space $H^2(-1) \subset H^{m-1}_s(-1) \subset H^{m-1}_s \subset \mathbb{E}^m_{s+1}$.*

Theorem 15.3. [154] *Let M be an oriented space-like surface in a pseudo-hyperbolic $H^4_1(-1) \subset \mathbb{E}^5_2$ with zero mean curvature vector in an anti-de Sitter space $H^4_1(-1)$. Then M has a 1-type pseudo-hyperbolic Gauss map with non-zero constant component in its spectral decomposition if and only if M is an open part of the surface defined by*

$$\phi(u, v) = (1, \cosh u \sinh v, \sinh u, \cosh u \cosh v, 1) \tag{15.1}$$

which is of curvature -1 and totally umbilical with zero mean curvature vector.

Theorem 15.4. [154] *Let M^n_t be an n-dimensional oriented pseudo-Riemannian submanifold with index t and non-zero mean curvature vector in a pseudo-hyperbolic space $H^{m-1}_s(-1) \subset \mathbb{E}^m_{s+1}$. Then M^n_t has a 1-type pseudo-hyperbolic Gauss map with non-zero constant component in its spectral decomposition if and only if M^n_t is an open part of a non-flat, non-totally geodesic and totally umbilical pseudo-Riemannian hypersurface of the totally geodesic pseudo-hyperbolic space $H^{n+1}_{s^*}(-1) \subset H^{m-1}_{s^*}(-1) \subset \mathbb{E}^m_{s+1}$ for $s^* = t \le s$ or $s^* = t+1 \le s$, i.e., M^n_t is an open part of $H^n_t(-c) \subset H^{n+1}_{t+1}(-1)$ of curvature $-c$ for $c > 1$ or $H^n_t(-c) \subset H^{n+1}_t(-1)$ of constant curvature $-c$ for $0 < c < 1$, or $S^n_t(c) \subset H^{n+1}(-1)$ of curvature $c > 0$.*

As easy consequences of Theorem 15.4, they obtained the following.

Corollary 15.1. *A hyperbolic space $H^n(-c)$ of curvature $-c$ for $c > 1$ in the anti-de Sitter space $H^{n+1}_1(-1) \subset \mathbb{E}^{n+2}_2$ is the only space-like hypersurface with 1-type pseudo-hyperbolic Gauss map which has a non-zeroconstant component in its spectral decomposition.*

Corollary 15.2. *An anti-de Sitter space $H^n(-c)$ of curvature $-c$ for $c > 1$ in the pseudo-hyperbolic space $H^{n+1}_2(-1) \subset \mathbb{E}^{n+2}_3$ is the only Lorentzian hypersurface with 1-type pseudo-hyperbolic Gauss map which has a non-zeroconstant component in its spectral decomposition.*

A surface in a pseudo-Riemannian is called *quasi-minimal* if its mean curvature vector light-like. In [150], N. C. Turgay classified quasi-minimal surfaces in $S^4_1(1) \subset \mathbb{E}^5_1$ which have 1-type Gauss map as follows.

Theorem 15.5. [150] *If M is a quasi-minimal surface lying in $S^4_1(1)$, then M has 1-type Gauss map if and only if it is congruent to a surface congruent to one of the surfaces given by*

(1) *A surface defined by $\phi(u, v) = (1, \sin u, \cos u \cos v, \cos u \sin v, 1)$,*
(2) *A surface defined by $\phi(u, v) = \frac{1}{2}\left(2u^2 - 1, 2u^2 - 2, 2u, \sin 2u, \cos 2v\right)$,*
(3) *A surface defined by $\phi(u, v) = \frac{1}{cd}(b, d\cos cu, d\sin cu, c\cos dv, c\sin dv)$, where $c = \sqrt{2-b}$ and $d = \sqrt{2+b}$ with $|b| < 2$,*
(4) *A surface defined by $\phi(u, v) = \frac{1}{cd}(d\cosh cu, d\sinh cu, c\cos dv, c\sin dv, d)$, where $c = \sqrt{b-2}$ and $d = \sqrt{b+2}$ with $|b| > 2$*

or one of the following two type of surfaces:

(a) *A surface of curvature one with constant light-like mean curvature vector which lies in $K_{a=}\{(t, x_2, x_3, x_4, t + a)\} \subset \mathbb{E}^5_1$ for a constant a,*
(b) *A surface of curvature one lying in the light cone $\{(\mathbf{y}, 1) \in \mathbb{E}^5_1 : \langle \mathbf{y}, \mathbf{y} \rangle = 0\}$.*

16. Euclidean submanifolds with pointwise 1-type Gauss maps

Recall that a submanifold of a Euclidean space is said to have 1-type Gauss map ν if its Gauss map satisfies $\Delta\nu = \lambda(\nu + C)$ for some constant λ and a constant vector C. On the other hand, it was known that the Laplacian of the Gauss map of several important surfaces such as helicoid, catenoid and right cones take the following form:

$$\Delta\nu = f(\nu + C) \tag{16.1}$$

for some non-constant function f and constant vector C. For this reason, a submanifold M of a Euclidean space is said to have *pointwise 1-type Gauss map* if its Gauss map satisfies (16.1) for some function f and vector C (see

[119]). In particular, if the function f in (16.1) is non-constant, then M is said to have *proper pointwise Gauss map.* Further, a submanifold with pointwise 1-type Gauss map is said to be of *the first kind* if the vector C in (16.1) is the zero vector. Otherwise, the pointwise 1-type Gauss map is said to be of *the second kind* (see [41]).

In [105], S. M. Jung, D.-S. Kim, and Y. H. Kim proved the following two results.

Theorem 16.1. *If M is a closed surface in $S^3 \subset \mathbb{E}^4$ with non-negative Gaussian curvature, then M has a pointwise 1-type spherical Gauss map of first kind if and only if M is a great sphere S^2 or the Clifford torus $S^1(1/\sqrt{2}) \times S^1(1/\sqrt{2}) \subset S^3$.*

Theorem 16.2. *Let M be a closed hypersurface of S^{n+1} without geodesic points, and suppose that M has pointwise 1-type spherical Gauss map of the second kind. Then we have:*

(1) *M is a small sphere if the mean curvature is constant.*
(2) *The function f in $\Delta\nu = f(\nu + C)$ is a non-vanishing function satisfying $f > n$ and C is a constant vector.*

A *Salkowski curve* in $\mathbb{E}^3$ is a curve which has constant curvature but non-constant torsion. İ. Kişi and G. Öztürk proved the following theorems.

Theorem 16.3. [121] *Let M be a tubular surface given with the parametrization*

$$\phi(u, v) = \gamma(u) + r(M_2(u)\cos v + M_3(u)\sin v) \tag{16.2}$$

in $\mathbb{E}^4$ such that $\phi_1[a_2 + a_3] = 0$ $(a_3 \neq -a_2)$ and $a_1 = 0$. Then M has pointwise 1-type Gauss map of the first kind if and only if the spine curve γ is a Salkowski curve, and the Gauss map is non-proper with the constant function $\lambda = a_2^2 + a_3^2 + \frac{1}{r^2}$.

Theorem 16.4. [121] *There exists no tubular surface M given with the parametrization (16.2) having pointwise 1-type Gauss map of the second kind which satisfies $\phi_1[a_2 + a_3] = 0$ $(a_3 \neq -a_2)$ and $a_1 = 0$ in $\mathbb{E}^4$.*

O. Kaya and M. Önder worked the following special developable ruled surfaces with pointwise 1-type Gauss map.

Theorem 16.5. [110] *If one of the curvature functions κ, τ of base curve α is a non-zero constant, then the developable general rectifying surface M has the pointwise 1-type Gauss map of the first kind if and only if α is a circular helix.*

Theorem 16.6. [110] *A developable generalized normal surface M has pointwise 1-type Gauss map of the first kind, that is $\Delta\nu = f\nu$ where $f = \kappa^2$ if and only if $\kappa = constant$ and $b_1 = 0$.*

Theorem 16.7. [110] *A developable osculating type ruled surface M has pointwise 1-type Gauss map of first kind if and only if M is a plane.*

F. K. Aksoyak and Y. Yaylı proved in [108] the following two theorems.

Theorem 16.8. *Let M be the flat rotational surface given by the parameterization*

$$\phi(s, t) = (x(s)\cos t, x(s)\sin t, y(s)\cos t, y(s)\sin t). \tag{16.3}$$

Then M has pointwise 1-type Gauss map if and only if M is either totally geodesic or it is parameterized by

$$\phi(s, t) = \begin{pmatrix} \lambda\cos(b_0 s + d)\cos t \\ \lambda\cos(b_0 s + d)\sin t \\ \lambda\sin(b_0 s + d)\cos t \\ \lambda\sin(b_0 s + d)\sin t \end{pmatrix}, \quad b_0^2\lambda^2 = 1,$$

where b_0, λ and d are real constants.

Theorem 16.9. *Let M be a non totally geodesic flat rotation surface with pointwise 1-type Gauss map given by the parameterization (16.3) with $d = 2k\pi$. Then M is a Lie group with bicomplex number product if and only if it is a Clifford torus.*

F. Kahraman Aksoyak and Y. Yaylı gave the following theorem in [109] .

Theorem 16.10. *Let M be the flat rotational surface given by the parameterization*

$$\phi(t,s)=\left(x(s)\cos t,\frac{1}{\sqrt{\alpha}}x(s)\sin t,y(s)\cos t,\frac{1}{\sqrt{\alpha}}y(s)\sin t\right),\ \alpha\in\mathbb{R}^{+}.$$

Then, M has pointwise 1-type Gauss map if and only if M is either totally geodesic or parameterized by one of the following

$$\phi(t,s)=\begin{pmatrix}\mu_0\cos\theta(s)\cos t\\ \frac{1}{\sqrt{\alpha}}\mu_0\cos\theta(s)\sin t\\ \frac{1}{\sqrt{\beta}}\mu_0\sin\theta(s)\cos t\\ \frac{1}{\sqrt{\alpha}}\frac{1}{\sqrt{\beta}}\mu_0\sin\theta(s)\sin t\end{pmatrix},$$

where $\theta(s)=\sqrt{\beta}b_0s+\delta$ and $\beta b_0^2\mu_0^2=1$, or

$$\phi(t,s)=\begin{pmatrix}\frac{1}{2}\xi\left(\mu_2e^{\theta(s)}+\mu_1e^{-\theta(s)}\right)\cos t\\ \frac{1}{\sqrt{\alpha}}\frac{1}{2}\xi\left(\mu_2e^{\theta(s)}+\mu_1e^{-\theta(s)}\right)\sin t\\ \frac{1}{\sqrt{\beta}}\frac{1}{2}\xi\left(\frac{1}{\sqrt{-\beta}}\mu_2e^{\theta(s)}-\frac{1}{\sqrt{-\beta}}\mu_1e^{-\theta(s)}\right)\cos t\\ \frac{1}{\sqrt{\alpha}}\frac{1}{2}\xi\left(\frac{1}{\sqrt{-\beta}}\mu_2e^{\theta(s)}-\frac{1}{\sqrt{-\beta}}\mu_1e^{-\theta(s)}\right)\sin t\end{pmatrix},$$

where $\theta(s)=-\sqrt{-\beta}b_0s+\delta$ and $\beta b_0^2\mu_1\mu_2=1$.

İ. Kişi and G. Öztürk proved the following theorem.

Theorem 16.11. [123] *Let M be a tubular surface given with the parametrization*

$$\phi(u,v)=\gamma(u)+r\left(b_1(u)\cos v+b_2(u)\sin v\right)$$

in $\mathbb{E}^4$. If M has pointwise 1-type Gauss map of the first kind, then M is one of the following surfaces in $\mathbb{E}^4$:

(1) circular cylinder,
(2) torus with the parametrization $\phi(u,v)=(\cos u,\sin u,r\cos v,r\sin v)$.

D.-S. Kim and Y. H. Kim considered the relationship of the shape operator of a surface of $\mathbb{E}^3$ with its Gauss map of pointwise 1-type. Surfaces with constant mean curvature and right circular cones with respect to some properties of the shape operator are characterized when their Gauss map is of pointwise 1-type and gave the results in [113]. With the helps of [22], they gave the following.

Theorem 16.12. [113] *Let M be an n-dimensional submanifold of m-dimensional Euclidean space $\mathbb{E}^m$. Then, the following are equivalent:*

(1) The immersion ϕ of M into $\mathbb{E}^m$ satisfies $\Delta\phi=f(\phi+C)$.
(2) M is a minimal submanifold of $\mathbb{E}^m$ or a minimal submanifold of a hypersphere in $\mathbb{E}^m$.

U. Dursun studied a class of space-like rotational surfaces in Minkowski 4-space $\mathbb{E}^4_1$ with meridian curves lying in 2-dimensional space-like planes and having pointwise 1-type Gauss map. He found all such surfaces with pointwise 1-type Gauss map of the first kind. Then he proved the space-like rotational surface with flat normal bundle and pointwise 1-type Gauss map of the second kind is an open part of a space-like 2-plane in $\mathbb{E}^4_1$. He gave the following theorems.

Theorem 16.13. [78] *Let M be a space-like rotational surface in $\mathbb{E}^4_1$ defined by*

$$\phi(s,t)=\left(x(s)\cos(at),x(s)\sin at,z(s)\cosh bt,z(s)\sinh bt\right).\tag{16.4}$$

Then M has pointwise 1-type Gauss map of the first kind if and only if M is an open part of a space-like plane or the surface given by

$$\phi(s,t)=\left(r_0\cos(\frac{s}{r_0})\cos at,r_0\cos(\frac{s}{r_0})\sin at,r_0\sin(\frac{s}{r_0})\cosh bt,r_0\sin(\frac{s}{r_0})\sinh bt\right).\tag{16.5}$$

Moreover, the Gauss map $\nu = e_3 \wedge e_4$ of the rotational surface (16.5) satisfies $\Delta\nu = f(\nu + C)$ for $C = 0$ and the function

$$f = \|h\|^2 = \frac{2}{r_0^2}\left(1 + \frac{a^2b^2}{\left(a^2\cos^2(\frac{s}{r_0}) - b^2\sin^2(\frac{s}{r_0})\right)^2}\right), \quad \tan^2(\frac{s}{r_0}) < (\frac{a}{b})^2.$$

Theorem 16.14. [78] *A space-like rotational surface M in $\mathbb{E}^4_1$ defined by (16.4) with flat normal bundle has pointwise 1-type Gauss map of the second kind if and only if M is an open part of a space-like plane in $\mathbb{E}^4_1$.*

D.-S. Kim studied generalized slant cylindrical surfaces ($GSCS$'s) with pointwise 1-type Gauss map of the first and second kinds, and he gave the following results.

Proposition 16.1. [112] *Let M be a $GSCS$ given by*

$$\phi(s,t) = X(s) + Y_s(t).$$

Then the following are equivalent.

(1) M has pointwise 1-type Gauss map ν of the first kind.
(2) M has constant mean curvature.
(3) M is a surface of revolution with constant mean curvature.

Corollary 16.1. [112] *Suppose that a $GSCS$ M has pointwise 1-type Gauss map ν of the second kind. Then the following are equivalent.*

(1) M is of rational kind.
(2) M is of polynomial kind.
(3) M is a right circular cone.

U. Dursun and N. C. Turgay studied general rotational surfaces in $\mathbb{E}^4$ with pointwise 1-type Gauss map and proved the following result.

Theorem 16.15. [82] *Let M be a general rotational surface in $\mathbb{E}^4$ defined by*

$$\phi(s,t) = (x(s)\cos at, x(s)\sin at, z(s)\cos bt, z(s)\sin bt)\,.$$

Then M has pointwise 1-type Gauss map of the first kind if and only if M is an open part of a plane or a surface given by

$$\phi(s,t) = \left(r_0\cos(\frac{s}{r_0})\cos at, r_0\cos(\frac{s}{r_0})\sin at, r_0\sin(\frac{s}{r_0})\cos bt, r_0\sin(\frac{s}{r_0})\sin bt\right). \tag{16.6}$$

Moreover, the Gauss map $\nu = e_3 \wedge e_4$ of the rotational surface (16.6) satisfies $\Delta\nu = f(\nu + C)$ for the function

$$f = \frac{2}{r_0^2}\left(1 + \frac{a^2b^2}{\left(a^2\cos^2\frac{s}{r_0} + b^2\sin^2\frac{s}{r_0}\right)^2}\right).$$

K. Arslan et. al. studied the tensor product surfaces of two Euclidean plane curves, and they proved the following.

Theorem 16.16. [1] *Let M be a tensor product surface of a plane circle γ_1 centered at the origin with a Euclidean planar curve $\gamma_2(s) = (\alpha(s), \beta(s))$. Then M has pointwise 1-type Gauss map if and only if M is either totally geodesic or parameterized by*

$$\alpha(s) = \int \cos(\lambda\ln|s+\mu|)ds, \quad \beta(s) = \int \sin(\lambda\ln|s+\mu|)ds.$$

In [2], K. Arslan et. al. studied Vranceanu rotation surfaces with pointwise 1-type Gauss map in $\mathbb{E}^4$. They proved the following two results.

Theorem 16.17. *Let M be a flat Vranceanu surface given with the parametrization*

$$\phi(s,t) = (r(s)\cos s\cos t, r(s)\cos s\sin t, r(s)\sin s\cos t, r(s)\sin s\sin t). \tag{16.7}$$

If M has pointwise 1-type Gauss map ν of the second kind so that $\Delta\nu = f(\nu + C)$, then $f = 4e^{-2bs}$ for a constant b.

Theorem 16.18. *Let M be a non-flat Vranceanu surface given with the parametrization (16.7). If M has pointwise 1-type Gauss map of the second kind, then up to homothety M is given by*

$$\phi(s,t)=\left(\frac{1}{2}\sin s\cos t,\frac{1}{2}\sin s\sin t,\sin^2 s\cos t,\sin^2 s\sin t\right).$$

K. Arslan et. al. gave necessary and sufficient conditions for the flat Ganchev-Milousheva rotational surface to have pointwise 1-type Gauss map, and they proved the following.

Theorem 16.19. [3] *Let M be a flat rotational embedded surface in Euclidean 4-space $\mathbb{E}^4$. Then M has pointwise 1-type Gauss map if and only if*

$$f_1(s)=\int\mu\cos\left(\frac{\lambda}{a\mu}\ln|as+b|\right)ds,\ f_2(s)=\int\mu\sin\left(\frac{\lambda}{a\mu}\ln|as+b|\right)ds,$$
$$f_3(s)=as+b,$$

for some constants $\lambda\neq 0$, $\mu>0$, $a\neq 0$ and b.

U. Dursun proved the following.

Theorem 16.20. [76] *Let M be an oriented flat regular surface in Euclidean space $\mathbb{E}^3$. Then M has pointwise 1-type Gauss map of the second kind if and only if M is an open part of the following surfaces:*

(1) A right circular cone in $\mathbb{E}^3$,
(2) a plane in $\mathbb{E}^3$,
(3) a cylinder given, up to a rigid motion, by

$$\phi(s,t)=\left(\pm\frac{q_0^2}{d_0}\mu(s)-\frac{s}{d_0}+d_1,-\frac{q_0}{2d_0\kappa^2(s)}+d_2,t\right),$$

where $d_0\neq 0$, $q_0\neq 0$, d_1, and d_2 are arbitrary constants, while the function $\mu(s)$ and the curvature function $\kappa(s)$ of the base curve are related by

$$\mu(s)=\int\frac{d\kappa}{\kappa^3\sqrt{(d_0^2-1)\kappa^2+2q_0\kappa-q_0^2}},$$

and $\kappa(s)$ satisfies the differential equation

$$q_0^2\kappa'^2=\kappa^4\left[\left(d_0^2-1\right)\kappa^2+2q_0\kappa-q_0^2\right].$$

M. Choi et. al. completely classified ruled surfaces in a 3-dimensional Euclidean space with pointwise 1-type Gauss map of the first kind and the second kind, and gave the following.

Theorem 16.21. [62] *Let M be a ruled surface in $\mathbb{E}^3$. Then, M has pointwise 1-type Gauss map if and only if it is a part of a plane, a circular cylinder, a helicoid, a cylinder of an infinite type satisfying*

$$\sin^{-1}\left(\frac{c^2f^{-\frac{1}{3}}-1}{\sqrt{c_1^2+c_2^2}}\right)-\sqrt{c_1^2+c_2^2-\left(c^2f^{-\frac{1}{3}}-1\right)^2}=\pm c^3(s+k),$$

where k is the constant of integration, or a rotational ruled surface of the first kind or the second kind.

M. Choi, D.-S. Kim, and Y. H. Kim worked helicoidal surfaces with pointwise 1-type Gauss map ν, and they proved the following.

Lemma 16.1. [57] *Let M be a helicoidal surface in $\mathbb{E}^3$. If the Gauss map ν of M satisfies the equation*

$$\Delta\nu=f(\nu+C)\tag{16.8}$$

for some smooth function f and a constant vector C, then either the Gauss map is harmonic, that is, $\Delta\nu=0$ or the function f defined by (16.8) depends only on t and the vector C in (16.8) is parallel to the axis of the helicoidal surface.

Theorem 16.22. [57] *A rotational helicoidal surface M with pointwise 1-type Gauss map if and only if M is a part of a circular cylinder, a right cone or an ordinary helicoid.*

Theorem 16.23. [57] *Let M be a rotational helicoidal surface in $\mathbb{E}^3$. Then, the Gauss map ν is harmonic or of pointwise 1-type if and only if M is a part of a plane, a circular cylinder, a helicoid and a right cone.*

U. Dursun proved the following.

Theorem 16.24. [74] *A rational hypersurface of revolution of Euclidean space $\mathbb{E}^{n+1}$ has pointwise 1-type Gauss map if and only if it is an open portion of a hyperplane, a generalized cylinder, or a right n-cone.*

B.-Y. Chen, M. Choi, and Y. H. Kim studied surfaces of revolution with pointwise 1-type Gauss map, and proved the following.

Theorem 16.25. [41] *A surface of revolution in $\mathbb{E}^3$ has constant mean curvature if and only if it has pointwise 1-type Gauss map of the first kind.*

Theorem 16.26. [41] *A surface of revolution of polynomial kind has pointwise 1-type Gauss map of the second kind if and only if it is a right cone.*

Theorem 16.27. [41] *A rational surface of revolution has pointwise 1-type Gauss map if and only if it is it is an open part of a plane, a circular cylinder, or a right cone.*

A. Niang worked rotation surfaces with 1-type Gauss map, and proved the following.

Theorem 16.28. [134] *A rotation surface M in $\mathbb{E}^3$ is pointwise 1-type Gauss map if and only if its mean curvature is a constant.*

M. Choi and Y. H. Kim characterized the helicoid as ruled surfaces with pointwise 1-type Gauss map, and they proved the following.

Theorem 16.29. [58] *A ruled surfaces in $\mathbb{E}^3$ with pointwise 1-type Gauss map are the open portions of the plane, the circular cylinder and the minimal helicoid.*

Y. H. Kim and D. W. Yoon studied ruled surfaces in a three-dimensional Minkowski space with pointwise 1-type Gauss map and obtained the complete classification theorems in [119]. They gave the following characterizations.

Theorem 16.30. *Let M be a space-like ruled surface in a three-dimensional Minkowski space. Then, the Gauss map is of pointwise 1-type if and only if M is an open part of one of the following surfaces: (1) a Euclidean plane, (2) the hyperbolic cylinder, (3) the helicoid of the 1st kind, (4) the helicoid of the 2nd kind, or (5) the conjugate of Enneper's surface of the 2nd kind.*

Theorem 16.31. [119] *Let M be a space-like ruled surface in a three-dimensional Minkowski space. Then, the Gauss map is of pointwise 1-type if and only if M is an open part of one of the following surfaces: (1) a Minkowski plane, (2) the Lorentz circular cylinder, (3) the circular cylinder of index 1, (4) the helicoid of the 1st kind, (5) the helicoid of the 2nd kind, (6) the helicoid of the 3rd kind, (7) the conjugate of Enneper's surfaces of the 2nd kind, (8) a flat B-scroll if $B'$is light-like, or (9) a non-flat B-scroll if B' is non-null.*

J. Qian, M. Su, and Y. H. Kim proved the following theorem.

Theorem 16.32. [141] *An oriented canal surface M in $\mathbb{E}^3$ has generalized 1-type Gauss map if and only if it is one of the following surfaces:*

(1) a surface of revolution such as

$$\phi(s,\theta) = (r\left(s\right)\cos\varphi\left(s\right) + s, r\left(s\right)\sin\varphi\left(s\right)\cos\theta, r\left(s\right)\sin\varphi\left(s\right)\sin\theta),$$

where $r\left(s\right) = -\frac{3c^2}{\sqrt[3]{4B}} - \frac{B}{6c\sqrt[3]{2}} - \frac{c}{2}$, where A and B are given by

$$A = -972c^4s^2 - 648c^3c_0s - 108c^2c_0^2 - 54c^6, \quad B = \left(A + \sqrt{-2916c^{16} + A^2}\right)^{\frac{1}{3}},$$

(2) a torus.

Let $\{e_1, e_2, e_3, e_4\}$ be the standard orthonormal frame in $\mathbb{E}^4$, and let $S^2(1)$ be the 2-sphere in $\mathbb{E}^3$ spanned by $\{e_1, e_2, e_3\}$ centered at the origin o. Consider a unit speed curve $\gamma = \gamma(v)$ lying in $S^2(1) \subset \mathbb{E}^3 \subset \mathbb{E}^4$ defined on an open interval I. Put $\mathbf{t}(v) = r'(v)$ and consider the moving frame $\{\mathbf{t}, \mathbf{n}, \gamma\}$ of the curve γ on $S^2(1)$. Then we have the following formulas:

$$\gamma' = \mathbf{t}, \quad \mathbf{t}' = \kappa\mathbf{n} - \gamma, \quad \mathbf{n}' = -\kappa\mathbf{t}, \tag{16.9}$$

where $\kappa(v) = \langle \mathbf{t}'(v), \mathbf{n}(v)\rangle$ is the spherical curvature of γ. Let $f = f(u)$ and $g = g(u)$ be two non-zero functions defined on an interval $J \subset \mathbb{R}$ such that $f'^2 + g'^2 = 1$. Let us consider the surface M^2 in $\mathbb{E}^4$ given by (see [86])

$$\phi(u, v) = f(u)\gamma(v) + g(u)\, e_4, \quad u \in J,\, v \in I. \tag{16.10}$$

Then this surface lies on the rotational hypersurface $M^3 \subset \mathbb{E}^4$ obtained by the rotation of the meridian curve $\alpha : u \to (f(u), g(u))$ about the e_4-axis in $\mathbb{E}^4$. This surface M^2 is called a *meridian surface* on M^3 since it is a one-parameter system of meridians of M^3.

K. Arslan, B. Bulca, and V. Milousheva studied meridian surfaces in $\mathbb{E}^4$ with pointwise 1-type Gauss map. They obtained the following three theorems.

Theorem 16.33. [4] *Let M be a meridian surface given with parametrization (16.10), and $g' \neq 0$. Then M has pointwise 1-type Gauss map of the second kind if and only if the curve c is a circle with non-zero constant spherical curvature and the meridian curve α is determined by $f(u) = \pm u + a$; $g(u) = b$, where $a = const.$, $b = const.$ In this case M is a developable ruled surface lying in 3-dimensional space.*

Theorem 16.34. [4] *Let M be a meridian surface given with parametrization (16.10) and $g' \neq 0$. Then M has pointwise 1-type Gauss map of the first kind if and only if one of the following two cases occurs:*

(1) the curve c is a great circle on $S^2(1)$ and the meridian curve α is determined by the solutions of the following differential equation

$$f'\sqrt{1 - f'^2} + f\left(\frac{ff''}{\sqrt{1-f'^2}}\right)' = 0,$$

(2) the curve c is a circle on $S^2(1)$ with non-zero constant spherical curvature and the meridian curve α is determined by $f(u) = a$; $g(u) = \pm u + b$, where $a = const.$, $b = const.$ In this case M is a developable ruled surface in a 3-dimensional space. Moreover, M is non-proper.

Theorem 16.35. [4] *Let M be a meridian surface given with parametrization (16.10) and $g' \neq 0$. Then M has pointwise 1-type Gauss map of the second kind if and only if one of the following holds:*

(1) the curve c is a circle on $S^2(1)$ and the meridian curve α is determined by $f(u) = \pm au + a_1$; $g(u) = bu + b_1$, where a, a_1, b, b_1 are constants. In this case M is a developable ruled surface lying in a 3-dimensional space;

(2) the curve c is a great circle on $S^2(1)$ and the meridian curve α is determined by the solutions of the following differential equation

$$\left(\ln \frac{-\sqrt{1-f'^2}\left[f\left(1-f'^2\right)\left(ff''\right)'^2 f' f''^2 + f'\left(1-f'^2\right)^2\right]}{ff'(ff'')'(1-f'^2)+f^2 f''^2+(1-f'^2)^2}\right)' = -\frac{ff''}{1 - f'^2}.$$

U. Dursun and G. G. Arsan proved the following theorems.

Theorem 16.36. [79] *An oriented non-minimal surface M in Euclidean space $\mathbb{E}^4$ has a pointwise 1-type Gauss map of the first kind if and only if M has parallel mean curvature vector in $\mathbb{E}^4$.*

Corollary 16.2. [79] *An oriented non-minimal surface M in Euclidean space $\mathbb{E}^4$ has pointwise 1-type Gauss map of the first kind if and only if M is a surface in a 3-sphere $S^3(a)$ of $\mathbb{E}^4$ with constant mean curvature.*

Theorem 16.37. [79] *An oriented minimal surface M in Euclidean space $\mathbb{E}^4$ has pointwise 1-type Gauss map of the first kind if and only if M has a flat normal bundle.*

Theorem 16.38. [79] *A non-planar minimal oriented surface M in Euclidean space $\mathbb{E}^4$ has pointwise 1-type Gauss map of the second kind if and only if, with respect to some suitable local orthonormal frame $\{e_1, e_2, e_3, e_4\}$ on M, the shape operators of M are given by A_3 =diag$(\rho, -\rho)$ and A_4 =adiag$(\pm\rho, \pm\rho)$, where ρ is a smooth non-zero function on M and adiag(a, b) means a 2×2 anti-diagonal matrix.*

Theorem 16.39. [79] *Let M be an oriented surface in Euclidean space $\mathbb{E}^4$ with non-parallel mean curvature direction, non-zero CMC, $\dim(N_1(M)) = 1$, where $N_1(M)$ denotes the first normal space of M. Then, M has pointwise 1-type Gauss map of the second kind if and only if M is an open portion of a helical cylinder in $\mathbb{E}^4$.*

Remark 16.1. In a very recent article [128], Y. Li et. al. studied developable surfaces with pointwise 1-type Gauss map of Frenet type framed base curves in $\mathbb{E}^3$.

17. Submanifolds of $\mathbb{E}^m_s (s \geq 1)$ with pointwise 1-type Gauss maps

In [150], N. C. Turgay classified quasi-minimal surfaces in $S^4_1(1) \subset \mathbb{E}^5_1$ which have pointwise 1-type Gauss map. He obtained the following.

Theorem 17.1. *[150] If M be a quasi-minimal surface lying in $S^4_1(1)$, then M, then M has proper pointwise 1-type Gauss map if and only if it is congruent to a surface congruent to one of the the following two type of surfaces:*

(1) A surface lying in $S^4_1(1) \cap S^4(c_0, r^2)$ with $c_0 \neq 0$ and $r > 0$;
(2) A surface lying in $S^4_1(1) \cap H^4(c_0, -r^2)$ with $c_0 \neq 0$ and $r > 0$.

D. W. Yoon et. al. classified flat surfaces with generalized 1-type Gauss map in Minkowski 3-space $\mathbb{L}^3$ in [159]. They proved the following.

Theorem 17.2. [159] *All cylindrical surfaces in $\mathbb{L}^3$ have generalized 1-type Gauss map.*

Theorem 17.3. [159] *Let M be a conical surface in Minkowski 3-space $\mathbb{L}^3$. Then, M is of generalized 1-type Gauss map if and only if it is an open part of one of the following surfaces:*

(1) a Euclidean plane,
(2) a Minkowski plane,
(3) a hyperbolic conical surface of the first kind,
(4) a hyperbolic conical surface of the second kind,
(5) an elliptic conical surface,
(6) a conical surface parameterized by

$$\phi(s,t) = \alpha_0 + t\beta(s),$$

where α_0 is a constant vector and $\beta(s)$ is a unit speed pseudo-spherical curve in $\mathbb{Q}^2(\varepsilon)$ with the non-constant geodesic curvature κ_g which is, for some indefinite integral F(v) of the function

$$\psi(v) = \left(a_1 \ln v + a^2 + \epsilon_2 v^2 + \epsilon_G (\ln v)^2\right)^{-1/2}$$

with a_1, $a_2 \in \mathbb{R}$, given by $\kappa_g(s) = 1/F^{-1}(\pm s + a_3)$, where a_3 is constant.

K. Arslan and V. Milousheva studied meridian surfaces of elliptic or hyperbolic type with pointwise 1-type Gauss map in Minkowski 4-space, and gave the following.

Theorem 17.4. [5] *Let M'_m be a meridian surface of elliptic type, defined by*

$$z(u,v) = f(u)l(v) + g(u)e_4,\ u \in I,\ v \in J. \tag{17.1}$$

Then M'_m has pointwise 1-type Gauss map of first kind if and only if the curve c has zero spherical curvature and the meridian curve m is determined by a solution $f(u)$ of the following differential equation

$$f\left(\frac{ff''}{\sqrt{f'^2-1}}\right) - f'\sqrt{f'^2 - 1} = 0,$$

$g(u)$ is defined by $g'(u) = \sqrt{f'^2 - 1}$.

Theorem 17.5. [5] *Let M'_m be a meridian surface of elliptic type, defined by (17.1). Then M'_m has pointwise 1-type Gauss map of second kind if and only if one of the following cases holds:*

(1) the curve c has non-zero constant spherical curvature κ and the meridian curve m is determined by $f(u) = \pm u + a$; $g(u) = b$, where $a = const.$, $b = const.$ In this case M'_m is a developable ruled surface lying in a constant hyperplane $\mathbb{E}^3$ of $\mathbb{E}^4_1$.
(2) the curve c has constant spherical curvature κ and the meridian curve m is determined by $f(u) = au + a_1$; $g(u) = bu + b_1$, where a, a_1, b and b_1 are constants, $a^2 \geq 1$, $a^2 - b^2 = 1$. In this case M'_m is either a marginally trapped developable ruled surface (if $\kappa^2 = b^2$) or a developable ruled surface lying in a constant hyperplane $\mathbb{E}^3$ (if $\kappa^2 - b^2 > 0$) or $\mathbb{E}^3_1$ (if $\kappa^2 - b^2 < 0$) of $\mathbb{E}^4_1$.

(3) *the curve c has zero spherical curvature and the meridian curve m is determined by the solutions of the following differential equation*

$$\left(\ln \frac{\sqrt{f'^2-1}\left[f(f'^2-1)(ff'')'-f^2f'f''^2-f'(f'^2-1)^2\right]}{(f'^2-1)^2+f^2f''^2+ff'(f'^2-1)(ff'')'}\right)' = \frac{ff''}{f'^2-1}.$$

$g(u)$ is defined by $g'(u)=\sqrt{f'^2-1}$.

They also classified the meridian surfaces of hyperbolic type with pointwise 1-type Gauss map of second kind in [5].

B. Bektaş and U. Dursun studied time-like rotational surfaces of elliptic type with pointwise 1-type Gauss map in Minkowski 4-space, and they proved the following.

Theorem 17.6. [13] *Let M_1 be a flat time-like rotational surface of elliptic type in $\mathbb{E}^4_1$ defined by*

$$\phi(s,t)=(x(s)\cos t, x(s)\sin t, z(s), w(s)), \quad s\in I,\, t\in[0,2\pi).$$

If the profile curve $\alpha(s)=(x(s),0,z(s),w(s))$ has the non-null principal curvature vector $\alpha''(s)$, then

(1) *M_1 has global 1-type Gauss map of the first kind if and only if $\alpha(s)$ is given by*

$$x(s)=x_1,\ z(s)=\frac{1}{q_0}\cosh(\kappa_1+q_0s)+z_0,\ w(s)=\frac{1}{q_0}\sinh(\kappa_1+q_0s)+w_0,$$

where $q_0=\pm\kappa_0$ and x_1,z_0,w_0,κ_0,$\kappa_1\in\mathbb{R}$ with x_1,$\kappa_0>0$. Moreover, $\Delta v=\left(\frac{1}{x_1^2}+\kappa_0^2\right)v$.

(2) *M_1 has pointwise 1-type Gauss map of the second kind if and only if $\alpha(s)$ is given by $\alpha(s)=(x(s),0,z(s),w(s))$ with*

$$x(s)=x_0s+x_1,$$
$$z(s)=\sqrt{x_0^2+1}\sinh(q_0\ln(x_0s+x_1)+\psi_0)ds+z_0,$$
$$w(s)=\sqrt{x_0^2+1}\cosh(q_0\ln(x_0s+x_1)+\psi_0)ds+w_0,$$

and the Gauss map $v=e_3\wedge e_4$ satisfies $\Delta v=f(v+C)$ for the function

$$f(s,t)=\frac{\kappa_0^2+x_0^2+1}{(x_0^2+1)(x_0s+1)^2}$$

and for the constant vector

$$C=x_0^2e_3\wedge e_4-\frac{q_0x_0^2\left(x_0^2+1\right)}{\kappa_0}e_1\wedge e_3,$$

where $s>-\frac{x_1}{x_0}$, $q_0=\pm\frac{\kappa_0}{x_0\sqrt{x_0^2+1}}$ and κ_0,x_0,x_1,z_0, w_0,κ_1,$\psi_0\in\mathbb{R}$ with $\kappa_0>0$. The integrals given above can be evaluated according to $q_0\neq\pm1$, $q_0=1$ or $q_0=-1$. Moreover, the profile curve α is a helix.

They also obtained the similar results for hyperbolic and parabolic types with pointwise 1-type Gauss map in Minkowski 4-space in [13].

F. Kahraman Aksoyak and Y. Yaylı proved in [106] the following theorem.

Theorem 17.7. *Let M be the marginally trapped surface given by*

$$\phi(t,s)=(\alpha_1(s)\cosh t, \alpha_1(s)\sinh t, \alpha_3(s), \alpha_4(s))$$

in Minkowski 4-space. Then M has pointwise 1-type Gauss map if and only if the profile curve is given by

$$\alpha_1(s)=(\lambda_1-1)^{\frac{1}{2}}\left(u^2(s)+\lambda\right)^{\frac{1}{2}},$$
$$\alpha_3(s)=\int\left(\frac{\lambda_1u^2+\lambda}{u^2+\lambda}\right)^{\frac{1}{2}}\cos\theta(s)\,ds,$$
$$\alpha_4(s)=\int\left(\frac{\lambda_1u^2+\lambda}{u^2+\lambda}\right)^{\frac{1}{2}}\sin\theta(s)\,ds,$$

and

$$\theta(s)=-\epsilon\frac{\lambda_1}{(\lambda_1-1)^{\frac{1}{2}}}\int\frac{(u^2+\lambda)^{\frac{1}{2}}}{\lambda_1u^2+\lambda}ds,$$

where $u(s)=\delta s+\lambda_3$, $\lambda=\frac{\lambda_2}{\lambda_1-1}$, λ_1,λ_2,λ_3,a_1 and a_2 are real constants.

U. Dursun and B. Bektaş proved the following for space-like rotational surfaces of elliptic type with pointwise 1-type Gauss map in Minkowski space $\mathbb{E}_1^4$.

Theorem 17.8. [80] *Let M_1 be a flat space-like rotational surface of elliptic type in $\mathbb{E}_1^4$ defined by*

$$\phi(s,t) = (x(s)\cos t, x(s)\sin t, z(s), w(s)),\ s \in I,\ t \in [0, 2\pi).$$

If the profile curve $\alpha(s) = (x(s), 0, z(s), w(s))$ has the non-null principal curvature vector $\alpha''(s)$, then

(1) M_1 has global 1-type Gauss map ν of the first kind if and only if $\alpha(s)$ is given by

$$x(s) = x_1,$$
$$z(s) = \mp\frac{1}{k_0}\sinh(k_1 \mp k_0 s) + z_0,$$
$$w(s) = \mp\frac{1}{k_0}\cosh(k_1 \mp k_0 s) + w_0,$$

where $x_1,z_0,w_0,k_0,k_1 \in \mathbb{R}$ with $x_1,k_0 > 0$. Moreover, $\Delta\nu = \left(\frac{1}{x_1^2} - k_0^2\right)\nu$, and ν is harmonic if and only if $k_0 = \frac{1}{x_1}$. Also, M_1 is a marginally trapped surface if $k_0 = \frac{1}{x_1}$.

(2) M_1 has pointwise 1-type Gauss map of the second kind if and only if $\alpha(s)$ is given by

$$x(s) = x_0 s + x_1,$$
$$z(s) = \sqrt{x_0^2 - 1}\sinh(q_0 \ln(x_0 s + x_1) + \psi_0)ds + z_0,$$
$$w(s) = \sqrt{x_0^2 - 1}\cosh(q_0 \ln(x_0 s + x_1) + \psi_0)ds + w_0,$$

where $s > -\frac{x_1}{x_0}$, $k_0^2 \neq x_0^2 - 1$, $q_0 = \pm\frac{k_0}{x_0\sqrt{x_0^2-1}}$ and $k_0,x_0,x_1,z_0,w_0,\psi_0 \in \mathbb{R}$ with $k_0 > 0$. The integrals given above can be evaluated according to $q_0 \neq \pm 1$, $q_0 = 1$ or $q_0 = -1$. Moreover, the profile curve α is a helix, and the Gauss map $\nu = e_3 \wedge e_4$ satisfies $\Delta\nu = f(\nu + C)$ for the function

$$f(s,t) = \frac{x_0^2 - 1 - k_0^2}{(x_0^2 - 1)(x_0 s + 1)^2}$$

and for the constant vector $C = x_0^2 e_3 \wedge e_4 - x_0\sqrt{x_0^2 - 1}e_1 \wedge e_3$.

They also used similar techniques, and proved two theorems for space-like rotational surfaces of hyperbolic and parabolic types with pointwise 1-type Gauss map in $\mathbb{E}_1^4$ in [80].

F. Kahraman Aksoyak and Y. Yaylı worked general rotational surfaces with pointwise 1-type Gauss map in pseudo-Euclidean space $\mathbb{E}_2^4$, and they proved the following theorems in [107].

Theorem 17.9. [107] *Let M_1 be a flat rotation surface in $\mathbb{E}_2^4$ given by the parametrization*

$$\phi(t,s) = (y(s)\sinh t, x(s)\cosh t, x(s)\sinh t, y(s)\cosh t).$$

Then M_1 has pointwise 1-type Gauss map if and only if M_1 is either totally geodesic or parametrized by

$$\phi(t,s) = \frac{\varepsilon}{2}\begin{pmatrix} (\mu_2 e^{-b_0 s + d} - \mu_1 e^{b_0 s - d})\sinh t \\ (\mu_2 e^{-b_0 s + d} + \mu_1 e^{b_0 s - d})\cosh t \\ (\mu_2 e^{-b_0 s + d} + \mu_1 e^{b_0 s - d})\sinh t \\ (\mu_2 e^{-b_0 s + d} - \mu_1 e^{b_0 s - d})\cosh t \end{pmatrix},\ \mu_1\mu_2 = -\frac{1}{b_0^2},$$

where μ_1,μ_2 and d are real constants.

Theorem 17.10. [107] *Let M_2 be a flat rotation surface in $\mathbb{E}_2^4$ given by the parametrization*

$$\phi(t,s) = (x(s)\cos t, x(s)\sin t, y(s)\cos t, y(s)\sin t),\ s \in I,\ t \in [0, 2\pi).$$

Then M_2 has pointwise 1-type Gauss map if and only if M_2 is either totally geodesic or parametrized by

$$\phi(t,s) = \frac{\varepsilon}{2}\begin{pmatrix} (\mu_2 e^{-b_0 s + d} + \mu_1 e^{b_0 s - d})\cos t \\ (\mu_2 e^{-b_0 s + d} + \mu_1 e^{b_0 s - d})\sin t \\ (\mu_2 e^{-b_0 s + d} - \mu_1 e^{b_0 s - d})\cos t \\ (\mu_2 e^{-b_0 s + d} - \mu_1 e^{b_0 s - d})\sin t \end{pmatrix},\ \mu_1\mu_2 = -\frac{1}{b_0^2},$$

where μ_1,μ_2 and d are real constants.

M. H. Jin and D. H. Pei studied time-like axis surface of revolution with pointwise 1-type Gauss map in Minkowski 3-space in [104], and they proved the following.

Theorem 17.11. *We have:*

(1) *Time-like axis surfaces of revolution with pointwise 1-type Gauss map of first kind coincide with surfaces of revolution with constant mean curvature.*
(2) *Non light-like Lorentzian right cones are the only rational time-like axis surfaces of revolution with pointwise 1-type Gauss map of the second kind.*

In [81], U. Dursun and E. Coşkun obtained the following.

Theorem 17.12. [81] *Let M be a flat ruled surface in Minkowski space $\mathbb{E}^3_1$. Then, M has pointwise 1-type Gauss map of the second kind if and only if it is a part of a plane, a right circular cone, a hyperbolic cone or an open part of the following cylinders given by*

(1) *the time-like cylinder parametrized by*

$$\phi(k,t)=\left(t,\ \pm\left(\tfrac{(k+k_0)\sqrt{R(k)}}{2c_0k_0k^2}+\tfrac{c_0}{2k_0}\arctan\left(\tfrac{(k-k_0)}{\sqrt{R(k)}}\right)\right),\ -\tfrac{k_0}{2c_0k^2}\right),$$

where $R(k)=c_0^2k^2-(k-k_0)^2>0$,
(2) *the space-like cylinder parametrized by*

$$\phi(k,t)=\left(\pm\varphi\left(k\right),\ \tfrac{k_0}{2c_0k^2},\ t\right),$$

(3) *the space-like cylinder parametrized by*

$$\phi(k,t)=\left(\tfrac{k_0}{2c_0k^2},\ \pm\psi\left(k\right),\ t\right),$$

(4) *the space-like cylinder parametrized by*

$$\phi(k,t)=\left(\pm\left(\tfrac{k_0}{4k^2}-\theta\left(k\right)\right),\ \tfrac{k_0}{4k^2}+\theta\left(k\right),t\right),$$

(5) *the time-like cylinder parametrized by*

$$\phi(k,t)=\left(\pm\psi\left(k\right),\ \tfrac{k_0}{2c_0k^2},\ t\right),$$

(6) *the time-like cylinder parametrized by*

$$\phi(k,t)=\left(\tfrac{k_0}{2c_0k^2},\ \pm\varphi\left(k\right),\ t\right),$$

(7) *the time-like cylinder parametrized by*

$$\phi(k,t)=\left(\tfrac{k_0}{4k^2}+\theta\left(k\right),\ \pm\left(\tfrac{k_0}{4k^2}-\theta\left(k\right)\right),\ t\right),$$

where

$$\varphi\left(k\right)=\frac{(k+k_0)}{2c_0k_0k^2}\sqrt{c_0^2k^2+(k-k_0)^2}-\frac{c_0}{2k_0}\ln\left|\tfrac{k_0-k+\sqrt{c_0^2k^2+(k-k_0)^2}}{k}\right|,$$
$$\psi\left(k\right)=\frac{(k+k_0)}{2c_0k_0k^2}\sqrt{(k-k_0)^2-c_0^2k^2}+\frac{c_0}{2k_0}\ln\left|\tfrac{k_0-k+\sqrt{(k-k_0)^2-c_0^2k^2}}{k}\right|,$$

with $(k-k_0)^2-c_0^2k^2>0$,

$$\theta\left(k\right)=\frac{1}{2\left(k-k_0\right)}-\frac{1}{2k_0}\ln\left|\frac{k}{k-k_0}\right|,$$

and, p_0, c_0 and k_0 are nonzero constants.

M. Yıldırım studied the tensor product surfaces of two Lorentzian planar, non-null curves to have pointwise 1-type Gauss map of first kind in pseudo-Euclidean 4-space $\mathbb{E}^4_2$, and he proved the following.

Theorem 17.13. [155] *Let $M \subset \mathbb{E}^4_2$ defined by*

$$\phi(t,s) = (\alpha(s)\sinh t, \beta(s)\cosh t, \alpha(s)\sinh t, \beta(s)\cosh t)$$

be a tensor product surface of a Lorentzian plane circle $\gamma_1(t) = (\sinh t, \cosh t)$ centered at the origin space-like or time-like with unit speed curve $\gamma_2(s) = (\alpha(s), \beta(s))$ in $\mathbb{E}^2_1$. Then we have the followings:

(1) If $\varepsilon_1 = \varepsilon_2$, M doesn't have pointwise 1-type Gauss map of first kind,
(2) If $\varepsilon_1 = -\varepsilon_2 = 1$, M has pointwise 1-type Gauss map of first kind if and only if $b = c = \lambda(\beta^2 - \alpha^2)^{-3/2}$ and $' - a^2 = 2b^2$,
(3) If $\varepsilon_1 = -\varepsilon_2 = -1$, M has pointwise 1-type Gauss map of first kind if and only if $b = c = \lambda(\alpha^2 - \beta^2)^{3/2}$ and $a' + a^2 = -2b^2$.

U. Dursun and N. C. Turgay proved the following.

Theorem 17.14. [83] *Let M be an oriented space-like surface in Minkowski space $\mathbb{E}^4_1$ with flat normal bundle and non-zero constant mean curvature. Then, M has pointwise 1-type Gauss map of the second kind if and only if it is congruent to one of the helical cylinders given by*

$$\begin{aligned}\phi_1(s,t) &= (a_1 s, b_1 \cos s, b_1 \sin s, t)\,,\\ \phi_2(s,t) &= (b_2 \cosh s, b_2 \sinh s, a_2 s, t)\,,\\ \phi_3(s,t) &= (b_3 \sinh s, a_1 s, b_3 \cosh s, a_3 s, t)\,.\end{aligned}$$

U. Dursun classified hypersurfaces of a Lorentz–Minkowski space $\mathbb{L}^{n+1}$ with pointwise 1-type Gauss map, and he proved the following.

Theorem 17.15. [75] *(Classification)*

(1) A rational rotation hypersurface $M_{q,T}$ of $\mathbb{L}^{n+1}$ parametrized by

$$\begin{aligned}\phi_T(u_1,\ldots,u_{n-1},t) =\, &\varphi(t)\sin u_{n-1}\Theta(u_1,\ldots,u_{n-2})\\ &+ \varphi(t)\cos u_{n-1}\eta_n + \psi(t)\eta_{n+1},\end{aligned} \tag{17.2}$$

has pointwise 1-type Gauss map of the first kind if and only if it is an open portion of a space-like hyperplane or a Lorentzian cylinder $S^{n-1} \times \mathbb{L}^1$ of $\mathbb{L}^{n+1}$.
(2) A rational rotation hypersurface M_{q,S_1} of $\mathbb{L}^{n+1}$ parametrized by

$$\begin{aligned}\phi_{S_1}(u_1,\ldots,u_{n-1},t) =\, &\varphi(t)\sinh u_{n-1}\Theta(u_1,\ldots,u_{n-2})\\ &+ \psi(t)\eta_n + \varphi(t)\cosh u_{n-1}\eta_{n+1},\end{aligned} \tag{17.3}$$

has pointwise 1-type Gauss map of the first kind if and only if it is an open portion of a time-like hyperplane or a hyperbolic cylinder $H^{n-1} \times \mathbb{R}$ of $\mathbb{L}^{n+1}$.
(3) A rational rotation hypersurface M_{q,S_2} of $\mathbb{L}^{n+1}$ parametrized by

$$\begin{aligned}\phi_{S_2}(u_1,\ldots,u_{n-1},t) =\, &\varphi(t)\cosh u_{n-1}\Theta(u_1,\ldots,u_{n-2})\\ &+ \psi(t)\eta_n + \varphi(t)\sinh u_{n-1}\eta_{n+1},\end{aligned} \tag{17.4}$$

has pointwise 1-type Gauss map of the first kind if and only if it is an open portion of a time-like hyperplane or a pseudo-spherical cylinder $S^{n-1}_1 \times \mathbb{R}$ of $\mathbb{L}^{n+1}$.
(4) A rational rotation hypersurface $M_{q,L}$ of $\mathbb{L}^{n+1}$ parametrized by

$$\begin{aligned}\phi_L(u_1,\ldots,u_{n-1},t) =\, &2\varphi(t)u_{n-1}\Theta(u_1,\ldots,u_{n-2}) + \sqrt{2}\varphi(t)\widehat{\eta}_n\\ &+ \sqrt{2}\left(\psi(t) - \varphi(t)u^2_{n-1}\right)\widehat{\eta}_{n+1},\end{aligned}$$

$u_{n-1} \neq 0$, has pointwise 1-type Gauss map of the first kind if and only if it is an open portion of hyperbolic n-space H^n, de Sitter n-space S^n_1 or Enneper's hypersurface of the second kind or the third kind. Moreover, the Enneper's hypersurfaces of the second kind and the third kind of $\mathbb{L}^{n+1}$ are the only polynomial rotation hypersurfaces of $\mathbb{L}^{n+1}$ with proper pointwise 1-type Gauss map of the first kind.

Theorem 17.16. [75] *Let M_q be one of the rotation hypersurfaces in $\mathbb{L}^{n+1}$ given by* (17.2), (17.3), *and* (17.4), *respectively. If M_q is a polynomial kind rotation hypersurface, then it has proper pointwise 1-type Gauss map of the second kind if and only if it is an open portion of a spherical n-cone, hyperbolic n-cone, or pseudo-spherical n-cone.*

U-H. Ki et. al. gave a complete classification of rational surfaces of revolution with pointwise 1-type Gauss map in Minkowski 3-space $\mathbb{E}^3_1$ in [111], and they proved the following.

Theorem 17.17. [111] *A rational surface of revolution of type I (i.e., the axis of revolution is a space-like line) has pointwise 1-type Gauss map of the first kind if and only if it is an open part of a plane or a hyperbolic cylinder. A rational surface of revolution of type II (i.e., the axis of revolution is a time-like line) has pointwise 1-type Gauss map of the first kind if and only if it is an open part of a plane or a circular cylinder. A rational surface of revolution of type III (i.e., the axis of revolution is a light-like line) has pointwise 1-type Gauss map of the first kind if and only if it is an open part of an Enneper's surface of second kind, a de Sitter space or an anti-de Sitter space up to rigid motion.*

Theorem 17.18. [111] *Let M be a rational surface of revolution. Then, M has pointwise 1-type Gauss map of the second kind in $\mathbb{E}^3_1$ if and only if M is part of either a right cone or a hyperbolic cone.*

A. Niang studied rotation surfaces with pointwise 1-type Gauss map in three dimensional Minkowski space, and proved the following in [133].

Theorem 17.19. [133] *Let M be a connected pseudo-Riemannian surface of revolution in a 3-dimensional Minkowski space $\mathbb{E}^3_1$ whose axis of rotation is L. Let M' be any component of the subset of the subset $M - L$. Then we have M' is pointwise 1-type Gauss map if and only if M' is a constant mean curvature.*

E. Ö. Canfes and N. C. Turgay studied Gauss map of minimal Lorentzian surfaces in 4-dimensional pseudo-Riemannian space forms They proved the following.

Theorem 17.20. [18] *Let M be a minimal Lorentzian surface properly contained in semi-Euclidean space $\mathbb{E}^4_2$. Then, the following statements are equivalent:*

(1) M has pointwise 1-type Gauss map of the first kind,
(2) M has harmonic Gauss map,
(3) M is congruent to either the surface given by

$$\phi(s,t) = s\eta_0 + \beta(t), \quad \langle \eta_0, \beta(t)\rangle \neq 0,$$

where η_0 is a constant light-like vector and β is a null curve in $\mathbb{E}^m_s$ which contains no open part of a line or the surface given by

$$\phi(s,t) = \left(\varphi_1(s) + \varphi_2(t), \frac{\sqrt{2}}{2}(s+t), \frac{\sqrt{2}}{2}(s-t), \varphi_1(s) + \varphi_2(t)\right),$$

where $\varphi_i : I_i \to \mathbb{R}$ are some smooth, non-vanishing functions, and I_i are some open intervals for $i = 1, 2$.

Theorem 17.21. [18] *Let M be a minimal Lorentzian surface properly contained in semi-Euclidean space $\mathbb{E}^4_2$ with non-harmonic Gauss map. Then M has pointwise 1-type Gauss map of the second kind if and only if it is locally congruent to the surface given by*

$$\phi(s,t) = (\phi_1(s) + \phi_2(t), s+t, s + (\cos c)t + (\sin c)\phi_2(t), \phi_1(s) - (\sin c)t + (\cos c)\phi_2(t))$$

for some non-linear functions ϕ_1,ϕ_2 and a constant $c \in (0, 2\pi)$, where $\varepsilon = \pm 1$. In this case, $\Delta\nu = f(\nu + C)$ is satisfied for $f = 4K$.

Theorem 17.22. [18] *Let M be a connected minimal Lorentzian surface in $\mathbb{S}^4_2(1)$. Then, M has a 2-type Gauss map if and only if it has constant Gaussian curvature and non-zero constant normal curvature.*

N. C. Turgay gave complete classification of minimal surfaces with pointwise 1-type Gauss map in Minkowski 4-space $\mathbb{E}^4_1$, and he presented following two classification theorems in [151].

Theorem 17.23. [151] *Let M be a Lorentzian minimal surface in $\mathbb{E}^4_1$. Also, suppose that no open part of M is contained in a hyperplane of $\mathbb{E}^4_1$. Then, the following conditions are equivalent:*

(1) M has pointwise 1-type Gauss map,

(2) *M has pointwise 1-type Gauss map of the first kind,*
(3) *M has degenerate relative null bundle,*
(4) *M is congruent to the surface given by $x(s,t) = s\eta_0 + \beta(t)$ for a constant light-like vector $\eta_0 \in \mathbb{E}^4_1$ and a null curve β in $\mathbb{E}^4_1$ satisfying $\langle \eta_0, \beta(t)\rangle \neq 0$.*

The other theorem is for Lorentzian surfaces M with non-zero constant mean curvature and finite type Gauss map in 4-dimensional Minkowski space, saying that M is congruent to one of five types of parametrized surface.

Theorem 17.24. [151] *Let M be a non-minimal Lorentzian surface in $\mathbb{E}^4_1$ with normal flat bundle and constant mean curvature. Then M has pointwise 1-type Gauss map of the second kind if and only if it is congruent to one of the following surfaces:*

(1) *a surface given by*
$$\phi(s,t) = \left(s, \frac{a}{\lambda}\cos\lambda t, \frac{a}{\lambda}\sin\lambda t, \sqrt{1-a^2}t\right),\ 0 < a < 1,$$
(2) *a surface given by*
$$\phi(s,t) = \left(\frac{a^2}{3}t^3, \sqrt{2}at, \frac{a^2}{3}t^3, s\right),$$
(3) *a surface given by*
$$\phi(s,t) = \left(\frac{\sqrt{a^2-1}}{\lambda}\cosh\lambda t, \frac{\sqrt{a^2-1}}{\lambda}\sinh\lambda t, at, s\right),\ a > 1,$$
(4) *a surface given by*
$$\phi(s,t) = \left(\frac{\sqrt{a^2+1}}{\lambda}\cosh\lambda t, \frac{\sqrt{a^2+1}}{\lambda}\sinh\lambda t, at, s\right),$$
(5) *a surface given by*
$$\phi(s,t) = \left(\sqrt{1+a^2}t, \frac{a}{\lambda}\cos\lambda t, \frac{a}{\lambda}\sin\lambda t, s\right)$$

for a nonzero constant λ.

M. Choi, Y. H. Kim, and D. W. Yoon studied ruled surfaces with pointwise 1-type Gauss map in Minkowski 3-space $\mathbb{E}^3_1$, and they proved the following in [59].

Theorem 17.25. [59] *(Classification) Let M be a ruled surface in Minkowski 3-space $\mathbb{E}^3_1$ with pointwise 1-type Gauss map. Then, M is an open part of a Euclidean plane, a Minkowski plane, a hyperbolic cylinder, a Lorentz circular cylinder, a circular cylinder of index 1, a cylinder of an infinite type, a helicoid of the first kind, a helicoid of the second kind, a helicoid of the third kind, the conjugate of Enneper's surface of the second kind, a rotational ruled surface of type I or type II, a transcendental ruled surface, or a B-scroll.*

Theorem 17.26. [59] *Let M be a non-cylindrical ruled surface of type M^1_+ , M^1_- or M^3_+ in Minkowski 3-space $\mathbb{E}^3_1$. Suppose that M has pointwise 1-type Gauss map of the second kind. Then, M is an open part of a rotational ruled surfaces of type I or type II.*

Theorem 17.27. [59] *(Classification). Let M be a non-cylindrical ruled surface over a non-null base curve in Minkowski 3-space $\mathbb{E}^3_1$. Then, M has pointwise 1-type Gauss map if and only if M is an open part of a helicoid of the first kind, a helicoid of the second kind, a helicoid of the third kind, the conjugate of Enneper's surface of the second kind, a rotational ruled surface of type I or II, or a transcendental ruled surface.*

Theorem 17.28. [59] *(Classification). Let M be a null scroll with pointwise 1-type Gauss map in Minkowski 3-space $\mathbb{E}^3_1$. Then, M is an open part of a Minkowski plane or a B-scroll.*

In [152], N. C. Turgay studied the Gauss map for marginally trapped (or quasi-minimal) surfaces in all three cases of spacetimes of constant curvature, namely, Minkowski, de Sitter and anti-de Sitter spacetimes. The author gave complete classifications of all such surfaces, and he proved the following.

Theorem 17.29. [152] *Let M be a marginally trapped surface in Minkowski space-time $\mathbb{E}^4_1$. Then, M has proper pointwise 1-type Gauss map of the first kind if and only if it is a non-flat surface lying in $S^3_1(r^2)$ or $H^3_1(-r^2)$.*

Theorem 17.30. [152] *Let M be a marginally trapped surface in Minkowski space-time $\mathbb{E}_1^4$. Then, M has non-harmonic 1-type Gauss map, if and only if it is congruent to the surface given by*

$$\phi(u,v) = (\varphi(u,v), u, v, \varphi(u,v))$$

for a function $\varphi : \Omega \subset \mathbb{R}^2 \to \mathbb{R}$ satisfying Helmholtz equation: $\Delta\varphi + \lambda\varphi = c_1 u + c_2 v$ for some constants $\lambda \neq 0$, c_1, c_2.

In [84], U. Dursun and N. C. Turgay classified space-like surfaces in Minkowski space $\mathbb{E}_1^4$, de Sitter space $\mathbb{S}_1^3$, and hyperbolic space $\mathbb{H}^3$ with harmonic Gauss map, and classified the following space-like surfaces with pointwise 1-type Gauss map of the first kind.

Theorem 17.31. [84] *Let M be an oriented maximal surface in Minkowski space $\mathbb{E}_1^4$. Then M has pointwise 1-type Gauss map of the first kind if and only if M has flat normal bundle. Moreover, the Gauss map ν satisfies $\Delta\nu = f(\nu + C)$ for $f = \|h\|^2$ and $C = 0$.*

Theorem 17.32. [84] *Let M be an oriented non-maximal space-like surface in $\mathbb{E}_1^4$. Then M has pointwise 1-type Gauss map of the first kind if and only if M has parallel mean curvature vector.*

Theorem 17.33. [84] *Let M be an oriented non-maximal space-like surface in $\mathbb{E}_1^4$ with space-like or time-like mean curvature vector. Then M has pointwise 1-type Gauss map of the first kind if and only if M is a CMC surface lying in the light cone $LC \subset \mathbb{E}_1^4$, a Euclidean hyperplane $\mathbb{E}^3 \subset \mathbb{E}_1^4$, a Lorentzian hyperplane $\mathbb{E}_1^3 \subset \mathbb{E}_1^4$, the de Sitter space-time $S_1^3\left(c^2\right) \subset \mathbb{E}_1^4$, or the hyperbolic space $H^3\left(-c^2\right) \subset \mathbb{E}_1^4$.*

Theorem 17.34. [84] *Let M be an oriented non-maximal surface in Minkowski space $\mathbb{E}_1^4$. Then M has (global) 1-type Gauss map of the first kind if and only if M has parallel mean curvature vector and constant Gaussian curvature.*

V. Milousheva and N. C. Turgay considered quasi-minimal Lorentz surfaces with pointwise 1-type Gauss map in pseudo-Euclidean 4-space $\mathbb{E}_2^4$, and they proved the following in [129].

Theorem 17.35. [129] *Let M_1^2 be a flat quasi-minimal surface in pseudo-Euclidean space $\mathbb{E}_2^4$. Then, M_1^2 has pointwise 1-type Gauss map ν if and only if it is congruent to the surface given by*

$$\phi(u,v) = \left(\theta(u,v), \frac{u-v}{\sqrt{2}}, \frac{u+v}{\sqrt{2}}, \theta(u,v)\right),$$

for a smooth function θ satisfying

$$\frac{\partial^2\theta(u,v)}{\partial u \partial v} = (F \circ \psi)(u,v), \quad \psi(u,v) = \theta(u,v) + c_1 u + c_2 v,$$

where F is a non-constant function, c_1 and c_2 are constants. In this case,

$$\Delta\nu = \phi(\nu + C)$$

is satisfied for the smooth function $\phi = F' \circ \psi$ and the non-zero constant vector $C = c_1\eta_0 \wedge \eta_2 - c_2\eta_0 \wedge \eta_1 - \eta_1 \wedge \eta_2$.

B. Bektaş, E. Ö. Canfes, and U. Dursun studied rotational surfaces with pointwise 1-type Gauss map in pseudo-Euclidean spaces $\mathbb{E}_1^4$ and $\mathbb{E}_2^4$, and they proved the following in [11].

Theorem 17.36. [11] *Let M be an oriented time-like surface with zero mean curvature in $\mathbb{E}_1^4$. Then M has pointwise 1-type Gauss map of the first kind if and only if M has flat normal bundle. Hence, the Gauss map ν satisfies $\Delta\nu = f(\nu + C)$ for $f = \|h\|^2$ and $C = 0$.*

Theorem 17.37. [11] *Let M be a time-like rotational surface in $\mathbb{E}_1^4$ defined by*

$$\phi(s,t) = (x(s)\cos(at), x(s)\sin at, w(s)\sinh bt, w(s)\cosh bt). \tag{17.5}$$

Then M has zero mean curvature vector, and its normal bundle is flat if and only if M is an open part of a time-like plane in $\mathbb{E}_1^4$.

Theorem 17.38. [11] *A time-like rotational surface in $\mathbb{E}_1^4$ defined by (17.5) has parallel nonzero mean curvature if and only if it is an open part of the time-like surface defined by*

$$\phi(s,t) = \left(r_0\cosh(\frac{s}{r_0})\cos at, r_0\cosh(\frac{s}{r_0})\sin at, r_0\sinh(\frac{s}{r_0})\sinh bt, r_0\sinh(\frac{s}{r_0})\cosh bt\right) \tag{17.6}$$

which has zero mean curvature in de Sitter space $S_1^3\left(r_0^{-2}\right) \subset \mathbb{E}_1^4$.

Theorem 17.39. [11] *Let M be a time-like rotational surface in $\mathbb{E}^4_1$ defined by (17.5). Then M has pointwise 1-type Gauss map of the first kind if and only if M is an open part of a time-like plane or the surface given by (17.6). Moreover, the Gauss map $\nu = e_3 \wedge e_4$ of the surface (17.6) satisfies $\Delta\nu = f(\nu + C)$ for $C = 0$ and the function*

$$f = \|h\|^2 = \frac{2}{r_0^2}\left(1 - a^2b^2\left(a^2\cosh^2(\frac{s}{r_0}) + b^2\sinh^2(\frac{s}{r_0})\right)^{-2}\right).$$

18. Other submanifolds with pointwise 1-type Gauss maps

18.1. L_k operators

Let M be a hypersurface of $\mathbb{E}^{n+1}$ with $\kappa_1(x), \ldots, \kappa_n(x)$ as its principal curvatures. Associated with the principal curvatures, there are n algebraic invariants given by

$$s_k(x) = \sigma_k(\kappa_1(x), \ldots, \kappa_n(x)), \quad 1 \leq k \leq n,$$

where $\sigma_k : \mathbb{R}^n \to \mathbb{R}$ is the elementary symmetric function defined by

$$\sigma_k(x_1, \ldots, x_n) = \sum_{i_1 < \cdots < i_r} x_{i_1} \ldots x_{i_k}.$$

Then the characteristic polynomial of the shape operator A of M can be expressed in terms of the s_k's as

$$Q_A(t) = \det(tI - A) = \sum_{k=0}^{n} (-1)^k s_k t^{n-k}, \quad s_0 = 1. \tag{18.1}$$

The *k-th mean curvature* H_k of M is $s_k = \binom{n}{k} H_k$, $0 \leq k \leq n$. In fact, H_1 is the mean curvature, H_k is intrinsic for even k, and H_k is extrinsic for odd k.

The *Newton transformations* $P_k : \mathfrak{X}(M) \to \mathfrak{X}(M)$ $(k = 1, \ldots, n)$ are defined inductively from the shape operator A by

$$P_0 = I, \ldots, P_k = s_k I - A \circ P_{k-1} = \binom{n}{k} H_k I - A \circ P_{k-1}, \tag{18.2}$$

where I denotes the identity map on $\mathfrak{X}(M)$. By applying Cayley-Hamilton's theorem, we have $P_n = 0$ from (18.1). If k is even, P_k does not depend on the chosen orientation, but if k is odd there is a change of sign in P_k.

Associated with each Newton transformation P_k, one has the linear differential operator $L_k : \mathcal{F}(M) \to \mathcal{F}(M)$ given by

$$L_k(f) = -\mathrm{Tr}(P_k \circ \nabla^2 f),$$

where $\nabla^2 f : \mathfrak{X}(M) \to \mathfrak{X}(M)$ is the self-adjoint linear operator metrically equivalent to the Hessian of f and given by

$$\left\langle \nabla^2 f(X), Y \right\rangle = \langle \nabla_X(\nabla f), Y \rangle, \quad X, Y \in \mathfrak{X}(M).$$

Note that L_k is the linearized operator of the first variation of the $(k+1)$-th mean curvature arising from the normal variations of M for $k = 2, 3, ..., n-1$. In particular, $L_1 = \square$ is called the Cheng-Yau operator introduced in [56].

18.2. Submanifolds with L_1 pointwise 1-type Gauss map

In recent years, the definition of L_k-finite type hypersurface has been given by changing the Laplace operator Δ in the definition of finite type hypersurfaces with the sequence of operators $L_0, L_1, L_2, ..., L_{n-1}$ such that $L_0 = -\Delta$.

The following definition was given by Y. H. Kim and N. C. Turgay in [115].

Definition 18.1. A surface M of the Euclidean space $\mathbb{E}^3$ is said to have L_1-pointwise 1-type Gauss map if its Gauss map ν satisfies

$$L_1 \nu = f(\nu + C) \tag{18.3}$$

for a function $f \in C^\infty(M)$ and a constant vector $C \in \mathbb{E}^3$. More precisely, a L_1-pointwise 1-type Gauss map is said to be of the first kind if (18.3) is satisfied for $C = 0$; otherwise, it is said to be of the second kind. In addition, if (18.3) is satisfied for a constant function f, then we say that M has L_1-(global) 1-type Gauss map.

Y. H. Kim and N. C. Turgay studied surfaces in $\mathbb{E}^3$ whose Gauss map ν satisfies the equation $L_1\nu = f(\nu + C)$ for a function f and a constant vector C, and gave the following.

Theorem 18.1. [115] *An oriented surface M in $\mathbb{E}^3$ has L_1-pointwise 1-type Gauss map of the first kind if and only if it has constant Gaussian curvature.*

Theorem 18.2. [115] *An oriented minimal surface M in $\mathbb{E}^3$ has L_1-pointwise 1-type Gauss map if and only if it is an open part of a plane.*

Theorem 18.3. [115] *Let M be a surface with constant mean curvature in $\mathbb{E}^3$. Then M has L_1-(global) 1-type Gauss map if and only if it is an open part of a sphere, a right circular cylinder or a plane.*

Theorem 18.4. [115] *Let M be a surface with a constant principal curvature in $\mathbb{E}^3$. Then M has L_1-pointwise 1-type Gauss map if and only if it is either a flat surface or an open part of a sphere.*

Y. H. Kim and N. C. Turgay studied ruled surfaces with L_1-pointwise 1-type Gauss map in 3-dimensional Euclidean and Minkowski spaces, and proved the following two theorems in [118] .

Theorem 18.5. *Let M be a non-cylindrical ruled surface in $\mathbb{E}^3$ whose position vector ϕ satisfies $L_k\phi = div(P_k(\nabla\phi))$. Then, the following three statements are equivalent:*

(1) M has L_1-pointwise 1-type Gauss map ν of the second kind.
(2) The Gauss map ν of M satisfies $L_1\nu = A\nu$ for a matrix $A \in \mathbb{R}^{3\times 3}$.
(3) M is flat.

Theorem 18.6. *Let M be a null scroll in $\mathbb{E}^3_1$. Then the following four conditions are equivalent.*

(1) M has L_1-pointwise 1-type Gauss map ν.
(2) The Gauss map of M satisfies $\Delta\nu = A\nu$ for a constant 3×3-matrix A.
(3) The Gauss map of M satisfies $L_1\nu = A\nu$ for a constant 3×3-matrix A.
(4) M is a B-scroll.

Y. H. Kim and N. C. Turgay studied helicoidal surfaces with L_1-pointwise 1-type Gauss map in $\mathbb{E}^3$, and they proved the following.

Theorem 18.7. *[116] Let M be a rotational surface in $\mathbb{E}^3$. Then, M has L_1-pointwise 1-type Gauss map of the first kind if and only if it is congruent to the surface given by*

$$F(y,t) = \left(\int_{x_0}^{y} \sqrt{\tfrac{-\kappa y^2+1-\lambda}{\kappa y^2+\lambda}}\, dy, y\cos t, y\sin t \right)$$

for some constants x_0, κ and λ.

Theorem 18.8. *[116] Let M be a rotational surface in $\mathbb{E}^3$. Then, M has L_1-pointwise 1-type Gauss map of the second kind if and only if M is an open part of one of the following four types of surfaces: (1) a plane, (2) a right circular cylinder, (3) a right circular cone, or (4) a surface which is locally congruent to the surface given by*

$$F(y,t) = \left(\int_{y_0}^{y} \sqrt{\left(\eta^{-1}\ln\xi\right)^2}\, d\xi, y\cos t, y\sin t \right) \tag{18.4}$$

for some constants y_0, A and $c \neq 0$ where η is the function defined by

$$\eta(\psi) = \int_{\psi_0}^{\psi} \frac{\xi + c}{\xi\left(A\xi^3 - c\xi^2 - \xi - \frac{c}{3}\right)} d\xi. \tag{18.5}$$

Theorem 18.9. *[116] Let M be a genuine helicoidal (i.e., $a \neq 0$) surface in $\mathbb{E}^3$ given by*

$$F(y,t) = (\phi + at, y\cos t, y\sin t),$$

where $\phi = \phi(y)$ is a smooth function defined on an open interval of $\mathbb{R}$ and a is a constant. Then M has L_1-pointwise 1-type Gauss map if and only if its Gaussian curvature is constant.

Theorem 18.10. *[116] Let M be a helicoidal surface in $\mathbb{E}^3$. Then, M has L_1-pointwise 1-type Gauss map of the second kind if and only if M is an open part of one of the following four types of surfaces: (1) a plane, (2) a right circular cylinder, (3) a right circular cone, or (4) a surface which is locally congruent to the rotational surface given by* (18.4) *and* (18.5)*.*

Y. H. Kim and N. C. Turgay studied the following pointwise 1-type Gauss map of surfaces in $\mathbb{E}^3_1$ via the operator L_1 in [117].

Lemma 18.1. *[117] Let M be a surface in $\mathbb{E}^3_1$. Then, its Gauss map ν satisfies*

$$L_1\nu = -\nabla K + 2\varepsilon KH\nu,$$

where H and K are the mean and Gaussian curvatures of M respectively, and

$$\varepsilon = \begin{cases} 1, & \text{if } M \text{ is Riemannian,} \\ -1, & \text{if } M \text{ is Lorentzian.} \end{cases}$$

Theorem 18.11. *[117] If M is a surface in $\mathbb{E}^3_1$, then, it has L_1-pointwise 1-type Gauss map of the first kind if and only if it has constant Gaussian curvature.*

Corollary 18.1. *[117] A surface with diagonalizable shape operator and L_1-pointwise 1-type Gauss map of the first kind in $\mathbb{E}^3_1$ if and only if it is congruent to one of the following five surfaces:*

(1) A cylinder $S^1(r) \times \mathbb{E}^1_1$, where $S^1(r)$ is a circle in $\mathbb{E}^2$ with radius r,
(2) A cylinder $S^1_1(r) \times \mathbb{E}^1$, where $S^1_1(r)$ is the curve in $\mathbb{E}^2_1$ given by $\beta(s) = (r\sinh s, r\cosh s)$,
(3) A cylinder $H^1(r) \times \mathbb{E}^1$, where $H^1(r)$ is the curve in $\mathbb{E}^2_1$ given by $\widetilde{\beta}(s) = (r\sinh s, r\cosh s)$,
(4) The de-Sitter space $S^2_1(r^2)$,
(5) The hyperbolic space $H^2(-r^2)$.

J. Qian and Y. H. Kim studied canal surfaces with L_1-pointwise 1-type Gauss map in [140], and they gave the following.

Theorem 18.12. *[140] An oriented canal surface M given by*

$$\phi(s,\theta) = c(s) + r(s)\left\{\sin\varphi\cos\theta N + \sin\varphi\sin\theta B + \cos\varphi T\right\},$$

where $\theta \in [0, 2\pi)$, $\varphi \in [0, \pi)$, $-r(s) = \cos\varphi$ for some smooth function $\varphi = \varphi(s)$, of $\mathbb{E}^3$ has L_1-pointwise 1-type Gauss map of the second kind if and only if it is a surface of revolution represented by

$$\phi(s,\theta) = (r(s)\cos\varphi + s, r(s)\sin\varphi\cos\theta, r(s)\sin\varphi\sin\theta)$$

satisfying $(\kappa_1\kappa_2)'/(\kappa_1+\kappa_2) = \frac{1}{r}\left(\log|C_1r'-1|\right)'$, where the principal curvatures are given by $\kappa_1 = r''/(rr''+r'^2-1)$ and $\kappa_2 = -r^{-1}$, respectively.

18.3. Submanifolds with L_k pointwise 1-type Gauss map

In [96], E. Güler and N. C. Turgay studied rotational hypersurfaces in the n-dimensional Euclidean space $\mathbb{E}^n$. They considered the Gauss map ν of rotational hypersurface in $\mathbb{E}^n$ with respect to the operator L_k acting on the functions defined on the hypersurfaces with some integers $k = n-3$, $n \geq 3$. They obtained the following classification theorem.

Theorem 18.13. *[96] Let M be a rotational hypersurface in $\mathbb{E}^n$ given by*

$$\phi(r,\theta_1,\theta_2,\ldots,\theta_{n-2}) = \begin{pmatrix} f(r)\prod_{i=1}^{n-2}\cos\theta_i \\ f(r)\sin\theta_1\prod_{i=2}^{n-2}\cos\theta_i \\ f(r)\sin\theta_2\prod_{i=3}^{n-2}\cos\theta_i \\ \vdots \\ f(r)\sin\theta_{n-3}\cos\theta_{n-2} \\ f(r)\sin\theta_{n-2} \\ \varphi(r) \end{pmatrix}$$

where f, φ are the differentiable functions, $r, \theta_i \in \mathbb{R}\setminus\{0\}$. Then the Gauss map ν of M satisfies $L_{n-3}\nu = A\nu$ for some $n \times n$ matrix A with some integers $n \geq 3$ if and only if M is an open part of the followings: (1) a hyperplane, (2) a right circular hypercone, (3) a circular hypercylinder, or (4) a hypersphere.

A. Mohammadpouri proved the following in [130].

Theorem 18.14. *[130] An oriented hypersurface M in $\mathbb{E}^{n+1}$ with at most 2 distinct principal curvatures has L_{n-1}-(global) 1-type Gauss map of the first kind if and only if it is either an n-minimal hypersurface or an open part of a hypersphere, a hyperplane or a generalized cylinder.*

Theorem 18.15. *[130] A rotational hypersurface of rational kind of Euclidean space $\mathbb{E}^{n+1}$ has L_k-pointwise 1-type Gauss map if and only if it is an open portion of a hyperplane, a generalized cylinder, or a right n-cone.*

A. Mohammadpouri gave the following in [131].

Theorem 18.16. *[131] An oriented hypersurface M in $\mathbb{E}^{n+1}$ has an L_k-pointwise 1-type Gauss map of the first kind if and only if it has a constant $(k+1)$-st mean curvature.*

Corollary 18.2. *[131] An oriented minimal hypersurface M in $\mathbb{E}^{n+1}$ with at most 2 distinct principal curvatures has an L_k-pointwise 1-type Gauss map of the first kind if and only if it is an open domain of a hyperplane.*

Proposition 18.1. *[131] Let M be a connected orientable hypersurface in $\mathbb{E}^{n+1}$ with at most 2 distinct principal curvatures. If $nH_1H_{k+1} = (n-k-1)H_{k+2}$, then M has an L_k-pointwise 1-type Gauss map of the second kind if and only if it is an open domain of a hyperplane.*

19. Surfaces with 1-type Gauss map in Galilean $\mathbb{G}_3$, $\mathbb{G}_3^1$ and in Sol_3

19.1. Surfaces in Galilean space $\mathbb{G}_3$

According to F. Klein's Erlangen Program, each geometry is associated with a group of transformations. Hence there are as many geometries as groups of transformations. Associated with group of transformations that in physics guarantees the invariance of many mechanical systems, so the Galilei group gave rise to the Galilean geometry. The Galilean space $\mathbb{G}_3$ is one of the Cayley-Klein spaces endowed with the projective metric of signature $(0, 0, +, +)$. The absolute figure of the Galilean space is the ordered triple $\{w, f, I\}$, where w is an ideal (absolute) plane, f is a line (absolute line) in w, and I is a fixed elliptic involution of points of f.

In non-homogeneous coordinates, the group of motion of $\mathbb{G}_3$ has the form:

$$\begin{aligned}
\overline{x} &= a_1 + x, \\
\overline{y} &= a_2 + a_3 x + y\cos\theta + z\sin\theta, \\
\overline{z} &= a_4 + a_5 x - y\sin\theta + z\cos\theta,
\end{aligned}$$

where a_1, a_2, a_3, a_4, a_5, and θ are real numbers. If the first component of a vector is not zero, then the vector is called as non-isotropic, otherwise it is called isotropic vector (see [138]). The scalar product of two vectors $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ in $\mathbb{G}_3$ is defined by

$$\langle \mathbf{v}, \mathbf{w} \rangle = \begin{cases} v_1 w_1, & \text{if } v_1 \neq 0 \text{ or } w_1 \neq 0, \\ v_2 w_2 + v_3 w_3, & \text{if } v_1 = 0 \text{ and } w_1 = 0\,. \end{cases}$$

The length of the vector $\mathbf{v} = (v_1, v_2, v_3)$ is given by

$$\|v\| = \begin{cases} v_1, & \text{if } v_1 \neq 0, \\ \sqrt{v_2^2 + v_3^2}, & \text{if } v_1 = 0. \end{cases}$$

The group of motion of $\mathbb{G}_3$ leaves invariant the Galilean length of the vector. The distance between $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ in $\mathbb{G}_3$ is defined by

$$d(p, q) = \begin{cases} |q_1 - p_1| & \text{if } p_1 \neq q_1, \\ \sqrt{(q_2 - p_2)^2 + (q_3 - p_3)^2} & \text{if } p_1 = q_1. \end{cases}$$

İ. Kişi and G. Öztürk studied spherical product surface having pointwise 1-type Gauss map in $\mathbb{G}_3$, and they proved the following.

Theorem 19.1. *[122] Let M be a spherical product surface in $\mathbb{G}_3$ given by*

$$\phi(u,v) = (u, p(u)v, p(u)q(v)).$$

Then M has pointwise 1-type Gauss map of the first kind if and only if M has the following parametrization

$$\phi(u,v) = \left(u, p(u)v, \pm p(u)\sqrt{-v^2 - 2c_1 v - c_1^2 + c_2} + c_3\right),$$

where c_1, c_2, c_3 are constants.

In addition, they provided in [122] a theorem for pointwise 1-type Gauss map of the second kind.

Let $\gamma(s)$ be a curve in $\mathbb{G}_3^1$ whose position vector is given by $\gamma(s) = (s, y(s), z(s))$. Denote by $(\mathbf{t}(s), \mathbf{t}(s), \mathbf{t}(s))$ be the Frenet frame of $\gamma(s)$. In [156], D. W. Yoon studied tubular surface in $\mathbb{G}_3$ defined by

$$\phi(s,t) = \gamma(s) + r\cos(t)\mathbf{n}(s) + r\sin(t)\mathbf{b}(s), \tag{19.1}$$

where r is a positive number. He proved the following results in [156].

Theorem 19.2. *There are no harmonic tubular surface in $\mathbb{G}_3$ given by (19.1).*

Theorem 19.3. *Let M be a tubular surface given by (19.1) in $\mathbb{G}_3$. If the immersion ϕ of M satisfies $\Delta\phi = A\phi$ for a 3×3 matrix A, then it is parametrized by*

$$\begin{aligned}\phi(s,t) = \Big(s, &-c_1 r^2 \sin(\tfrac{1}{r}s + b) + \tfrac{r}{\sqrt{c_1^2+d_1^2}}(c_1\cot t - d_1 \sin t) + c_2 s + c_3,\\ &- d_1 r^2 \sin(\tfrac{1}{r}s + b) + \tfrac{r}{\sqrt{c_1^2+d_1^2}}(d_1\cos t + c_1 \sin t) + d_2 s + d_3\Big),\end{aligned}$$

where $c_i, d_i \in \mathbb{R}$, $i = 1,2,3$.

Theorem 19.4. *There are no tubular surface given by (19.1) in $\mathbb{G}_3$ with harmonic Gauss map ν, i.e., $\Delta\nu = 0$.*

Theorem 19.5. *Let M be a tubular surface given by (19.1) in $\mathbb{G}_3$. If the Gauss map ν of M satisfies $\Delta\nu = A\nu$ for a 3×3 matrix A, then the following statements hold:*

(1) *The torsion of the curve γ is constant.*
(2) *The Gauss map ν is of 1-type.*

Theorem 19.6. *Let M be a tubular surface in $\mathbb{G}_3$. If M has pointwise 1-type Gauss map of the first kind, then the Gauss map of M is of usual 1-type.*

Theorem 19.7. *There is no a tubular surface in $\mathbb{G}_3$ with pointwise 1-type Gauss map of the second kind.*

19.2. Surfaces in pseudo-Galilean space $\mathbb{G}_3^1$

The pseudo-Galilean geometry is one of the real Cayley-Klein geometries of projective signature $(0,0,+,-)$. The geometry of $\mathbb{G}_3^1$ is similar to $\mathbb{G}_3$, but not the same.

The absolute figure of the Galilean space is the ordered triple $\{w, f, I\}$, where w is an ideal (absolute) plane, f is a line (absolute line) in w, and I is a fixed hyperbolic involution of points of f. The group of motion of $\mathbb{G}_3^1$ takes the form:

$$\begin{aligned}\overline{x} &= a_1 + x,\\ \overline{y} &= a_2 + a_3 x + y\cosh\theta + z\sinh\theta,\\ \overline{z} &= a_4 + a_5 x + y\sinh\theta + z\cosh\theta,\end{aligned}$$

where a_1, a_2, a_3, a_4, a_5, and θ are real numbers. The scalar product of two vectors $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ in $\mathbb{G}_3^1$ is given by

$$\langle \mathbf{v}, \mathbf{w}\rangle = \begin{cases} v_1 w_1, & \text{if } v_1 \neq 0 \text{ or } w_1 \neq 0,\\ v_2 w_2 - v_3 w_3, & \text{if } v_1 = 0 \text{ and } w_1 = 0\,.\end{cases}$$

The length of the vector $\mathbf{v} = (v_1, v_2, v_3)$ is

$$\|v\| = \begin{cases} v_1, & \text{if } v_1 \neq 0, \\ \sqrt{v_2^2 - v_3^2}, & \text{if } v_1 = 0. \end{cases}$$

The group of motion of $\mathbb{G}_3^1$ leaves invariant the pseudo-Galilean length of the vector. The distance between $p = (p_1, p_2, p_3)$ and $q = (q_1, q_2, q_3)$ in $\mathbb{G}_3^1$ is given by

$$d(p,q) = \begin{cases} |q_1 - p_1| & \text{if } p_1 \neq q_1, \\ \sqrt{(q_2 - p_2)^2 - (q_3 - p_3)^2} & \text{if } p_1 = q_1. \end{cases}$$

M. Choi and D. W. Yoon studied in [60] surfaces of revolution with pointwise 1-type Gauss map in $\mathbb{G}_3^1$, and proved the following.

Theorem 19.8. *[60] (Classification) Let M be a surface of revolution generated by an isotropic curve in $\mathbb{G}_3^1$. If M has pointwise 1-type Gauss map of the first kind, then the Gauss map of M is of usual 1-type.*

In [60], M. Choi and D. W. Yoon also gave a theorem for pointwise 1-type Gauss map of the second kind.
D. W. Yoon, Y. H. Kim, and J. S. Jung considered in [158] rotation surfaces with L_1-pointwise 1-type Gauss map in pseudo-Galilean space, and they proved the following.

Theorem 19.9. *[158] Let M be a rotation surface defined by*

$$\phi(u, v) = (u, g(u) \cosh v, g(u) \sinh v),$$

where $g(u)$ is a positive function on $\mathbb{G}_3^1$. Then M has L_1-pointwise 1-Gauss map of the first kind if and only if M is an open part of one of the following surfaces:

$$\phi_1(u, v) = (u, a \cos(ku + b) \cosh v, a \cos(ku + b) \sinh v), \;\; c = -k^2,$$
$$\phi_2(u, v) = (u, a \cosh(ku + b) \cosh v, a \cosh(ku + b) \sinh v), \;\; c = k^2,$$

where $a, b, k \in \mathbb{R}$.

D. W. Yoon, Y. H. Kim, and J. S. Jung also obtained in [158] a result for L_1-pointwise 1-type Gauss map of the second kind.

Let $\gamma(s)$ be a space-like (time-like) curve in $\mathbb{G}_3^1$ whose position vector is given by

$$\gamma(s) = (s, p(s), q(s)). \tag{19.2}$$

Denote the Frenet frame of $\gamma(s)$ by $(\mathbf{t}(s), \mathbf{n}(s), \mathbf{b}(s))$. In [149], Y. Tunçer, M. K. Karacan and D. W. Yoon called a tubular surface in $\mathbb{G}_3^1$ to be *of type I* if it is parametrized by

$$\phi^\epsilon(s, t) = \gamma(s) + r \cosh(t)\mathbf{n}(s) + r \sinh(t)\mathbf{b}(s), \tag{19.3}$$

where r is a positive number and

$$\epsilon = \begin{cases} +1, & \text{if } M \text{ is a space-like canal surface with space-like centered curve,} \\ -1 & \text{if } M \text{ is a time-like canal surface with space-like centered curve.} \end{cases}$$

They proved in [149] the following result.

Theorem 19.10. *Let M be a type-1 tubular surface in $\mathbb{G}_3^1$ given by (19.3). Then:*

(1) There are no type-I tubular surface given by (19.3) with harmonic Gauss map, i.e., $\Delta\nu = 0$,
(2) All type-I tubular surfaces satisfy $\Delta\nu = \lambda\nu$ with $0 \neq \lambda \in \mathbb{R}$,
(3) All type-I tubular surface satisfy $\Delta\nu = A\nu$, where $A = \frac{1}{\epsilon r^2} I_3$.

19.3. Surfaces in sol space Sol_3

The space Sol_3 is a simply connected homogenous 3-dimensional manifold whose isometry group has dimension 3 and it is one of the eight models of geometry of W. Thurston [148]. This space can be identified with $\mathbb{R}^3$ on which the group operation

$$(x, y, z) * (\bar{x}, \bar{y}, \bar{z}) = (x + \bar{x}e^{-z}, y + \bar{y}e^{z}, z + \bar{z})$$

and the left invariant Riemannian metric is given by $g = e^{2z}dx^2 + e^{-2z}dy^2 + dz^2$, where (x, y, z) are the standard coordinates in $\mathbb{R}^3$. The corresponding Killing vector fields associated to these families of isometrics are given by

$$F_1 = \frac{\partial}{\partial x}, \quad F_2 = \frac{\partial}{\partial y}, \quad F_3 = -x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \frac{\partial}{\partial z}.$$

A surface M in Sol_3 is called T_i*-invariant* if it is invariant under the action of 1-parameter subgroup of isometric generated by the Killing vector fields F_i with $i = 1, 2, 3$.

D. W. Yoon studied in [157] T_1-invariant surfaces in Sol_3 with pointwise 1-type Gauss map. He proved the following results.

Theorem 19.11. *Let M be a T_1-invariant surface in Sol_3. Then it has pointwise 1-type Gauss map of the first kind if and only if it is a surface given by*

$$\phi(s, t) = (t, y(s), z(s)),$$

where

$$y(s) = \int \exp\left(-\frac{1}{a}\cos(as + b)\right)\cos(as + b)ds \ \text{ and } \ z(s) = \frac{1}{a}\cos(as + b).$$

Theorem 19.12. *Let M be a T_1-invariant surface in Sol_3. Then it has pointwise 1-type Gauss map of the second kind if and only if it is given by $\phi(s, t) = (t, a, \pm s + b)$, with constant a, b.*

Theorem 19.13. *Let M be a T_1-invariant surface in Sol_3. Then it has L_1-pointwise 1-type Gauss map of the first kind if and only if it has constant Gauss curvature.*

In [157], D. W. Yoon also obtained necessary and sufficient conditions for a T_1-invariant surface in Sol_3 having L_1-pointwise 1-type Gauss map of the second kind.

20. Some related results

In [94], E. Güler, M. Magid, and Y. Yaylı considered helicoidal hypersurfaces and their Gauss map in $\mathbb{E}^4$, and they gave the following result.

Theorem 20.1. [94] *Let $\phi : M^3 \longrightarrow \mathbb{E}^4$ be a helicoidal hypersurface given by*

$$\phi(u, v, w) = (u\cos v\cos w, u\sin v\cos w, u\sin w, \varphi(u) + av + bw)$$

with $u, a, b \in \mathbb{R}\backslash\{0\}$ and $0 \leq v, w \leq 2\pi$. Then $\Delta\phi = A\phi$ holds for some 4×4 matrix A if and only if M^3 has zero mean curvature.

In [91], E. Güler studied helical hypersurfaces in Minkowski geometry $\mathbb{E}^4_1$, and he proved the following.

Theorem 20.2. *Let $\phi : M^3 \rightarrow \mathbb{E}^4_1$ be a time-like helical hypersurface given by*

$$\phi(u, v, w) = (u\cos v\cos w, u\sin v\cos w, u\sin w, \varphi(u) + av + bw)$$

with $u, a, b \in \mathbb{R}\backslash\{0\}$ and $0 \leq v, w \leq 2\pi$. Then $\Delta\phi = A\phi$ holds for some 4×4 matrix A if and only if M^3 has zero mean curvature.

In [114], D.-S. Kim, J. R. Kim, and Y. H. Kim studied Gauss map ν of surfaces of revolution in 3-dimensional Euclidean space $\mathbb{E}^3$ and proved the following.

Theorem 20.3. *[114] Let M be a surface of revolution given by*

$$\phi(s,t) = (x(s)\cos t, x(s)\sin t, z(s)).$$

Then the Gauss map ν of M satisfies $L_1\nu = A\nu$ for some 3×3 matrix A if and only if M is an open part of one of the following surfaces: (1) a plane, (2) a right circular cone, (3) a circular cylinder, or (4) a sphere.

In [95], E. Güler and N. C. Turgay proved the following.

Theorem 20.4. *Let M be a rotational hypersurface in Euclidean four-space $\mathbb{E}^4$ given by*

$$\phi(u,v,w) = (f(u)\cos v\cos w, f(u)\sin v\cos w, f(u)\sin w, \varphi(u)),$$

where $u \in \mathbb{R}\backslash\{0\}$ and $0 \le v, w \le 2\pi$. Then the Gauss map ν of M satisfies $L_1\nu = A\nu$ for some 4×4 matrix A if and only if M is an open part of the following four types of hypersurfaces: (1) a hyperplane, (2) a right circular hypercone, (3) a circular hypercylinder, or (4) a hypersphere.

E. Güler also considered generalized helical hypersurfaces having time-like axis in Minkowski spacetime $\mathbb{E}^4_1$, and gave the following two results.

Theorem 20.5. *[92] The Laplacian of the generalized helical hypersurface*

$$\phi(u,v,w) = (f(u)\cos v\cos w, f(u)\sin v\cos w, f(u)\sin w, g(u) + av + bw), \tag{20.1}$$

is given by $\Delta\phi = 3H_1\nu$, where H_1 is the mean curvature and ν is the Gauss map.

Theorem 20.6. *[92] Let $\phi : M^3 \longrightarrow \mathbb{E}^4_1$ be a helicoidal hypersurface with time-like axis given by (20.1). Then $\Delta\phi = A\phi$ for some 4×4 matrix A if and only if M^3 has zero mean curvature.*

In [93], E. Güler, H. H. Hacısalihoğlu, and Y. H. Kim studied Δ^{III} for rotational hypersurfaces. They gave the following.

Theorem 20.7. [93] *Let M be a rotational hypersurface in $\mathbb{E}^4$ given by*

$$\phi(u,v,w) = (u\cos v\cos w, u\sin v\cos w, u\sin w, \varphi(u)),$$

where $u \in \mathbb{R}\backslash\{0\}$ and $0 \le v, w \le 2\pi$. Then M satisfies $\Delta^{III}\phi_i = f_i\phi_i$ for some functions f_i for $i = 1,2,3,4$ and $\phi = (\phi_1,\ldots,\phi_4)$.

For some further results related to $\Delta^J\phi = A\phi$ (*J=I, II, III*), see [97, 98, 99].

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Author's contributions

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