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Research Article

On the Chinese Checkers Circular Inversions in the Chinese Checkers Plane

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Abstract
In present article, we introduce an inversion with respect to a Chinese Checkers circle in the Chinese Checkers plane, and prove several properties of this inversion. We also study cross ratio, harmonic conjugates and the images of lines, planes and Chinese Checkers circle in the Chinese Checkers plane.

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1. Introduction

In the game of Chinese Checkers, checkers are allowed to move in the vertical (north and south), horizontal (east and west), and diagonal (northeast, northwest, southeast and southwest) directions. Chinese Checkers geometry proposed by Krause [1] asking the question of how to develop a metric which would be similar to the movement by playing Chinese Checkers. Later Chen [2] gave such a metric for the coordinate plane.

H. Minkowski was one of the developers in "non-Euclidean" geometry and found taxicab geometry [3]. The taxicab geometry has been studied and improved by some mathematicians (for some references see [4–8]).

The inversion was introduced by Perga in his last book Plane Loci, and systematically studied and applied by Steiner about 1820s [9]. During the following decades, many physicists and mathematicians independently rediscovered inversions, proving the properties that were most useful for their particular applications (for some references see [10–12]).

Many kinds of generalizations of inversion transform have been presented in literature. The inversions with respect to the central conics in real Euclidean plane was introduced in [13]. Then the inversions with respect to ellipse was studied detailed in [14]. In three-dimensional space a generalization of the spherical inversion is given in [15]. Also, the inversions with respect to the taxicab distance, alpha-distance [16–18] or in general a p-distance [19].

The circle inversion (we define circular inversions) have been generalized in three-dimensional space by using a sphere as the circle of inversion [15, 20].

The circular inversion is one of the most important and interesting transformations in the geometry. In this article, we define a notion of inversion valid in Chinese Checkers plane. In particular, we define an inversion with respect to a Chinese Checkers circle and prove several properties of this new transformation. Also we introduce inverse points, cross ratio, harmonic conjugates and the inverse images of lines, planes and Chinese Checkers circle in the Chinese Checkers plane.

2. Chinese Checkers circular inversion

The Chinese Checkers plane $\mathbb{R}^2_C$ is almost the same as the Euclidean plane $\mathbb{R}^2$. The points and lines are the same, and the angles are measured the same way, but the distance function is different. In $\mathbb{R}^2$ the Chinese Checkers metric is defined using the distance function

$$d_C(A,B) = d_L(A,B) + \left(\sqrt{2}-1\right) d_S(A,B),$$

where

$$d_L(A,B) = \max\left\{|x_1 - x_2|, |y_1 - y_2|\right\},$$

and

$$d_S(A,B) = \min\left\{|x_1 - x_2|, |y_1 - y_2|\right\},$$

where $A = (x_1, y_1)$, $B = (x_2, y_2) \in \mathbb{R}^2$. The unit ball in $\mathbb{R}^2_C$ is the set of points (x,y) in plane which satisfy the equation

$$\max\left\{|x|, |y|\right\} + \left(\sqrt{2}-1\right) \min\left\{|x|, |y|\right\} = 1.$$

We can define the notion of inversion in $\mathbb{R}^2_C$ as an analogue of inversion in $\mathbb{R}^2$.

Definition 1. *Let $\mathscr{C}$ be a Chinese Checkers circle centered at a point O with radius r in $\mathbb{R}^2_C$. The inversion in the Chinese Checkers circle $\mathscr{C}$ or the Chinese Checkers circular inversion respect to $\mathscr{C}$ is the function such that*

$$I_{(O,r)} : \mathbb{R}^2_C - \{O\} \to \mathbb{R}^2_C - \{O\}$$

defined by $I_{(O,r)}(P) = P'$, for $P \neq O$ where P' is on the $\overrightarrow{OP}$ and

$$d_C(O,P) \cdot d_C(O,P') = r^2.$$

The point P' is said to be the Chinese Checkers circular inverse of P in $\mathscr{C}$, $\mathscr{C}$ is called the circle of inversion and O is called the center of inversion.

The Chinese Checkers circular inversions with respect to the circle, like reflections, are involutions. The fixed points of $I_{(O,r)}$ are the points on the Chinese Checkers circle $\mathscr{C}$ centered at O with radius r.

Some basic properties about circular inversion are given in the following items. Note that it is possible to extend every property of the Chinese Checkers circular inversion to Chinese Checkers spherical inversion.

Theorem 2. *Let $\mathscr{C}$ be an Chinese Checkers circle with the center O in the Chinese Checkers circular inversion $I_{(O,r)}$. If the point P is in the exterior of $\mathscr{C}$ then the point P', the inverse of P, is interior to $\mathscr{C}$, and conversely.*

Proof. Let the point P be in the exterior of $\mathscr{C}$, then $d_C(O,P) > r$. If $P' = I_{(O,r)}(P)$; then $d_C(O,P) \cdot d_C(O,P') = r^2$. Hence $r^2 = d_C(O,P) \cdot d_C(O,P') > r \cdot d_C(O,P')$ and $d_C(O,P') < r$. ∎

The inversion $I_{(O,r)}$ is undefined at the point O. However, we can add to the Chinese Checkers plane a single point at infinite O_∞, which is the inverse of the center O of Chinese Checkers inversion circle $\mathscr{C}$. So, the inversion $I_{(O,r)}$ is one-to-one map of extended Chinese Checkers circle.

Theorem 3. *Let $\mathscr{C}$ be a Chinese Checkers circle with the center $O = (0,0)$ and the radius r in $\mathbb{R}^2_C$. If $P = (x,y)$ and $P' = (x',y')$ are inverse points with respect to the Chinese Checkers circular inversion $I_{(O,r)}$, then*

$$P' = \frac{r^2}{\left(d_C(O,P)\right)^2} P.$$

Proof. The equation of $\mathscr{C}$ is $d_C(O,P) = d_L(O,P) + \left(\sqrt{2}-1\right) d_S(O,P)$. Suppose that $P = (x,y)$ and $P' = (x',y')$ are inverse points with respect to the Chinese Checkers circular inversion $I_{(O,r)}$. Since the points O, P and P' are collinear and the rays $\overrightarrow{OP}$ and $\overrightarrow{OP'}$ are same direction,

$$\overrightarrow{OP'} = k\overrightarrow{OP},\ k \in \mathbb{R}^+,$$

$$(x',y') = (kx, ky).$$

From $d_C(O,P) \cdot d_C(O,P') = r^2,\ k = \dfrac{r^2}{\left(d_C(O,P)\right)^2}$. Replacing the value of k in $(x',y') = (kx,ky)$, the equations of x' and y' are obtained. ∎

Corollary 4. *Let $\mathscr{C}$ be a Chinese Checkers circle with the center $O=(a,b)$ and the radius r in $\mathbb{R}^2_C$. If $P=(x,y)$ and $P'=(x',y')$ are inverse points with respect to the Chinese Checkers circular inversion $I_{(O,r)}$, then*

$$P'-O=\frac{r^2}{(d_C(O,P))^2}(P-O).$$

Proof. Since the translation preserve distances in the Chinese Checkers plane [21,22] by translating in $\mathbb{R}^2_C$ $(0,0)$ to (a,b) one can easily get the value of x' and y'. ■

Theorem 5. *Let P,Q and O be any three collinear different points in $\mathbb{R}^2_C$. If the Chinese Checkers circular inversion $I_{(O,r)}$ transform P and Q into P' and Q' respectively, then*

$$d_C(P',Q')=\frac{r^2 d_C(P,Q)}{d_C(O,P)d_C(O,Q)}.$$

Proof. Assume first that O,P,Q are collinear. From the Definition 1 we conclude that $d_C(O,P)\,d_C(O,P')=r^2=d_C(O,Q)\,d_C(O,Q')$. Since the ratios of the Euclidean and Chinese Checkers distances along a line are same,

$$\begin{aligned} d_C(P',Q') &= \left|d_C(O,P')-d_C(O,Q')\right| \\ &= \left|\frac{r^2}{d_C(O,P)}-\frac{r^2}{d_C(O,Q)}\right| \\ &= \frac{r^2 d_C(P,Q)}{d_C(O,P)d_C(O,Q)} \end{aligned}$$

is obtained. ■

When O, P, Q are not collinear, the theorem is not valid in Chinese Checkers plane, generally $\mathbb{R}^2_C$. For example, for $O=(0,0), P=(1,0)$, $Q=(2,1)$ and $r=2\sqrt{2}$, the inversion $I_{(O,r)}$ transform P and Q into $P'=(8,0)$ and $Q'=\left(\frac{16}{(2+\sqrt{2})^2},\frac{8}{(2+\sqrt{2})^2}\right)$. It follows that $d_C(O,P)=1$, $d_C(O,Q)=2+\sqrt{2}$ and $d_C(P,Q)=\sqrt{2}$,

$$d_C(P',Q')=\frac{8\sqrt{2}}{2+\sqrt{2}}$$

clear that the theorem is not valid.

Theorem 6. *Let P,Q and O be any three non-collinear different points in $\mathbb{R}^2_C$ and $I_{(O,r)}$ be the inversion such that transform P and Q into P' and Q' respectively. If P and Q lie on the lines with any direction $D_1=\{0,\infty\}$ or $D_2=\{+1,-1\}$ or $D_3=\left\{\sqrt{2}-1,1-\sqrt{2}\right\}$ or $D_4=\left\{-\sqrt{2}-1,1+\sqrt{2}\right\}$ through O*

$$d_C(P',Q')=\frac{r^2 d_C(P,Q)}{d_C(O,P)d_C(O,Q)}$$

is valid.

Proof. Since translations preserve the Chinese Checkers distances, it is enough to consider the center O of the inversion sphere as the origin. Let P, $Q\in D_1$ in $\mathbb{R}^2_C$. If $P=(p,0)$ and $Q=(0,q)$, the images of P and Q respect to $I_{(O,r)}$ are $P'=(\frac{r^2}{p},0)$ and $Q'=(0,\frac{r^2}{q})$. It follows that

$$d_C(P',Q')=d_L(P',Q')+\left(\sqrt{2}-1\right)d_S(P',Q'),$$

$$d_L(P',Q')=\max\left\{\left|\frac{r^2}{p}\right|,\left|\frac{r^2}{q}\right|\right\},$$

$$d_S(P',Q') = \min\left\{\left|\frac{r^2}{p}\right|, \left|\frac{r^2}{q}\right|\right\},$$

and then on D_1

$$d_C(P',Q') = \frac{r^2 d_C(P,Q)}{d_C(O,P)d_C(O,Q)}.$$

If P, $Q \in D_2$ in $\mathbb{R}^2_C$. then the images of $P=(p,p)$ and $Q=(q,-q)$ respect to $I_{(O,r)}$ are $P' = \left(\frac{r^2}{2p}, \frac{r^2}{2p}\right)$ and $Q' = \left(\frac{r^2}{2q}, -\frac{r^2}{2q}\right)$. It follows that

$$d_C(P',Q') = d_L(P',Q') + \left(\sqrt{2}-1\right) d_S(P',Q'),$$

$$d_L(P',Q') = \max\left\{\left|\frac{r^2}{2p} - \frac{r^2}{2q}\right|, \left|\frac{r^2}{2p} + \frac{r^2}{2q}\right|\right\},$$

$$d_S(P',Q') = \min\left\{\left|\frac{r^2}{2p} - \frac{r^2}{2q}\right|, \left|\frac{r^2}{2p} + \frac{r^2}{2q}\right|\right\},$$

and then on D_2

$$d_C(P',Q') = \frac{r^2 d_C(P,Q)}{d_C(O,P)d_C(O,Q)}.$$

If P, $Q \in D_3$ in $\mathbb{R}^2_C$. then the images of $P = \left(p, p\left(\sqrt{2}-1\right)\right)$ and $Q = \left(q, q\left(-\sqrt{2}+1\right)\right)$ respect to $I_{(O,r)}$ are $P' = \left(\frac{r^2}{8p\left(\sqrt{2}-1\right)^2}, \frac{r^2}{8p\left(\sqrt{2}-1\right)}\right)$ and $Q' = \left(\frac{r^2}{8q\left(\sqrt{2}-1\right)^2}, \frac{r^2}{8q\left(\sqrt{2}-1\right)}\right)$. So, we get

$$d_C(P',Q') = d_L(P',Q') + \left(\sqrt{2}-1\right) d_S(P',Q'),$$

$$d_L(P',Q') = \max\left\{\frac{r^2}{8\left(\sqrt{2}-1\right)^2}\left|\frac{1}{p} - \frac{1}{q}\right|, \frac{r^2}{8\left(\sqrt{2}-1\right)}\left|\frac{1}{p} + \frac{1}{q}\right|\right\},$$

$$d_S(P',Q') = \min\left\{\frac{r^2}{8\left(\sqrt{2}-1\right)^2}\left|\frac{1}{p} - \frac{1}{q}\right|, \frac{r^2}{8\left(\sqrt{2}-1\right)}\left|\frac{1}{p} + \frac{1}{q}\right|\right\},$$

and then on D_3

$$d_C(P',Q') = \frac{r^2 d_C(P,Q)}{d_C(O,P)d_C(O,Q)}.$$

If P, $Q \in D_4$ in $\mathbb{R}^2_C$. then the images of $P = \left(p, p\left(-\sqrt{2}-1\right)\right)$ and $Q = \left(q, q\left(\sqrt{2}+1\right)\right)$ respect to $I_{(O,r)}$ are $P' = \left(\frac{r^2}{8p}, \frac{r^2\left(-\sqrt{2}-1\right)}{8p}\right)$ and $Q' = \left(\frac{r^2}{8q}, \frac{r^2\left(\sqrt{2}+1\right)}{8q}\right)$. So, we get

$$d_C(P',Q') = d_L(P',Q') + \left(\sqrt{2}-1\right) d_S(P',Q'),$$

$$d_L(P',Q') = \max\left\{\frac{r^2}{8}\left|\frac{1}{p} - \frac{1}{q}\right|, \frac{r^2\left(\sqrt{2}+1\right)}{8}\left|\frac{1}{p} + \frac{1}{q}\right|\right\},$$

$$d_S(P',Q') = \min\left\{\frac{r^2}{8}\left|\frac{1}{p} - \frac{1}{q}\right|, \frac{r^2\left(\sqrt{2}+1\right)}{8}\left|\frac{1}{p} + \frac{1}{q}\right|\right\},$$

and then on D_4

$$d_C(P',Q') = \frac{r^2 d_C(P,Q)}{d_C(O,P)d_C(O,Q)}.$$

■

3. Cross ratio and harmonic conjugate

The Chinese Checkers directed distance from the point A to the point B along a line l in $\mathbb{R}_C^2$ is denoted by $d_C[AB]$. If the ray with initial point A containing B has the positive direction of orientation, $d_C[AB] = d_C(A,B)$ and if the ray has the opposite direction, $d_C[AB] = -d_C(A,B)$ [23]. The Chinese Checkers cross ratio is preserved by the inversion in the Chinese Checkers circle as in the taxicab plane in [16].

Now, we show the properties related to the Chinese Checkers cross ratio and harmonic conjugates in $\mathbb{R}_C^2$.

Definition 7. *Let A,B,C and D be four distinct points on an oriented line in $\mathbb{R}_C^2$. We define the their Chinese Checkers cross ratio $(AB,CD)_C$ in $\mathbb{R}_C^2$ by*

$$(AB,CD)_C = \frac{d_C[AC]}{d_C[AD]}\frac{d_C[BD]}{d_C[BC]}.$$

It is known that the cross ratio is positive if both C and D are between A and B or if neither C nor D is between A and B, whereas the cross ratio is negative if the pairs $\{A,B\}$ and $\{C,D\}$ separate each other. Also, the cross ratio is an invariant under inversion in a sphere whose center is not any of the four points A,B,C and D in the taxicab plane [23]. Similarly, this property is valid in Chinese Checkers plane.

Theorem 8. *The inversion in a Chinese Checkers circle in $\mathbb{R}_C^2$ preserves the Chinese Checkers cross ratio.*

Proof. Let A,B,C and D be four collinear points on an oriented line l with the center of the inversion $I_{(O,r)}$ in $\mathbb{R}_C^2$. Let $I_{(O,r)}$ transform A,B,C and D into A',B',C' and D', respectively. The Chinese Checkers circular inversion reverses the Chinese Checkers directed distance from the point A to the point B along a line l in $\mathbb{R}_C^2$ to the Chinese Checkers directed distance from the point B' to the point A' and preserves the separation or non separation of the pair A,B and C,D. Hence it is suffices to show that $|(A'B',C'D')_C| = |(AB,CD)_C|$. From Theorem 2 it follows that

$$\begin{aligned}
\frac{d_C\left(A',C'\right)}{d_C\left(A',D'\right)}\frac{d_C\left(B',D'\right)}{d_C\left(B',C'\right)} &= \frac{\dfrac{r^2 d_C(A,C)}{d_C(O,A)\,d_C(O,C)}\,\dfrac{r^2 d_C(B,D)}{d_C(O,B)\,d_C(O,D)}}{\dfrac{r^2 d_C(A,D)}{d_C(O,A)\,d_C(O,D)}\,\dfrac{r^2 d_C(B,C)}{d_C(O,B)\,d_C(O,C)}} \\
&= \frac{d_C(A,C)}{d_C(A,D)}\frac{d_C(B,D)}{d_C(B,C)}.
\end{aligned}$$

■

Definition 9. *Let A and B be two points on a line l in $\mathbb{R}_C^2$, any pair C and D on the line l for which $\frac{d_C[AC]}{d_C[CB]} = \frac{d_C[AD]}{d_C[DB]}$ is said to divide A and B harmonically. The points C and D are called Chinese Checkers harmonic conjugates with respect to A and B, and the Chinese Checkers harmonic set of points is denoted by $H(AB,CD)_C$.*

It is clear that two distinct points C and D are Chinese Checkers harmonic conjugates with respect to A and B if and only if $(AB,CD)_C = -1$.

Theorem 10. *Let $\mathscr{C}$ be a Chinese Checkers circle with the center O, and the line segment $[AB]$ a diameter of $\mathscr{C}$ in $\mathbb{R}_C^2$. Let P and P' be distinct points of the ray OA, which divide the segment $[AB]$ internally and externally. Then P and P' are Chinese Checkers harmonic conjugates with respect to A and B if and only if P and P' are inverse points with respect the Chinese Checkers circular inversion $I_{(O,r)}$.*

Proof. Suppose that P and P' are Chinese Checkers harmonic conjugates with respect to A and B in $\mathbb{R}_C^2$. Then

$$(AB,PP')_C = -1,$$

$$\frac{d_C[AP]}{d_C[AP']}\frac{d_C[BP']}{d_C[BP]} = -1.$$

Since P divides the line segment $[AB]$ internally and P is on the ray OB, $d_C(P,B) = r - d_C(O,P)$ and $d_C(A,P) = r + d_C(O,P)$. Since P' divides the line segment $[AB]$ externally and P' is on the ray OB, $d_C(A,P') = d_C(O,P') + r$ and $d_C(B,P') = d_C(O,P') - r$.

Hence

$$\frac{r+d_C(O,P)}{d_C(O,P')+r}\cdot\frac{d_C(O,P')-r}{d_C(O,P)-r}=-1,$$
$$(r+d_C(O,P))\left(d_C(O,P')-r\right)=\left(d_C(O,P')+r\right)\cdot(r-d_C(O,P)).$$

Simplifying the last equality, $d_C(O,P)d_C(O,P')=r^2$ is obtained. Therefore P and P' are Chinese Checkers inverse points with respect to the Chinese Checkers circular inversion $I_{(O,r)}$.

Conversely, if P and P' are Chinese Checkers inverse points with respect to the Chinese Checkers circular inversion $I_{(O,r)}$, the proof is similar. ■

4. Chinese Checkers circular inversions of lines, planes and Chinese Checkers circles

It is well known that inversions with respect to circle transform lines and circles into lines and/ or circles in Euclidean plane and Hyperbolic plane.

The following features are well known for inversion in Euclidean plane:

i. Lines passing through the inversion center are invariant.

ii. Lines that do not pass through the center of inversion transform circles passing through the center of inversion.

iii. Circles passing through the center of inversion transform lines does not pass through the center of the inversion.

iv. Circles not passing through the center of inversion transform circles does not pass through the center of the inversion.

v. Circles with center of inversion transform circles with center of inversion.

In this section, we study the Chinese Checkers circular inversion of lines, planes and Chinese Checkers circles. The Chinese Checkers circular inversion $I_{(O,r)}$ maps the lines, planes passing through O onto themselves.

The Chinese Checkers circular inversion $I_{(O,r)}$ maps Chinese Checkers circles with centered O onto Chinese Checkers circles. But the Chinese Checkers circular inversion of a circle not passing through the centre of inversion is not another Chinese Checkers circle that does not contain the centre of inversion.

Theorem 11. *Consider the inversion $I_{(O,r)}$ in a Chinese Checkers circle $\mathscr{C}$ with the centre O. Every line and plane containing O is invariant under the inversion.*

Proof. It is clear that the straight lines containing O onto themselves.

Let $\mathscr{C}$ be a Chinese Checkers circle of inversion and $P=(x,y)$ with equation $d_L(O,P)+\left(\sqrt{2}-1\right)d_S(O,P)=r$ and the plane $Mx+Ny=0$. Applying $I_{(O,r)}$ to this plane gives

$$M\frac{r^2x'}{(d_C(O,P))^2}+N\frac{r^2y'}{(d_C(O,P))^2}=0.$$

So, $Mx'+Ny'=0$ is obtained. ■

The inverse of a line not containing O is not a Chinese Checkers circle containing O.

Theorem 12. *The inverse of a Chinese Checkers circle with the centre O with respect to the Chinese Checkers circular inversion $I_{(O,r)}$ is a Chinese Checkers circle containing O.*

Proof. Since the translation preserve distance in $\mathbb{R}^2_C$, we can take a Chinese Checkers circle $\mathscr{C}$ of inversion and $P=(x,y)$ with equation $d_L(O,P)+\left(\sqrt{2}-1\right)d_S(O,P)=r$ and $\mathscr{C}$ the Chinese Checkers circle $d_L(O,P)+\left(\sqrt{2}-1\right)d_S(O,P)=k$, $k\in\mathbb{R}^+$. Applying $I_{(O,r)}$ to $\mathscr{C}$ gives

$$d_L(O,P')+\left(\sqrt{2}-1\right)d_S(O,P')=\frac{r^2}{k}.$$

Note that this is a Chinese Checkers circle with the centre O. ■

Theorem 13. *The inversion $I_{(O,r)}$ in a Chinese Checkers circle $\mathscr{C}$ with centre O. Every edge, vertice and face of Chinese Checkers circle is invariant under the inversion.*

Proof. The points of Chinese Checkers circle are mapped by $I_{(O,r)}$ back onto Chinese Checkers circle from the Definition 1. Hence every edge, vertice and face of Chinese Checkers circle is invariant under $I_{(O,r)}$. ■

5. Conclusions

In this paper, the concept of inversion with respect to the circle in Chinese Checkers plane geometry is presented. Under this inversion, the inverse properties of point, line and circle were examined and the concept of inversion was examined and has been defined. In Chinese Checkers plane point, line, plane, circle and circular inverses between them relationships are determined. In addition, double ratio and the effect of inversion on harmonic conjugate points was investigated.

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