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Research Article

Dominions and closed varieties of bands

Shabnam Abbas, Wajih Ashraf*, Noor Mohammad Khan

Department of Mathematics,
Aligarh Muslim University, Aligarh-202002, India

Abstract

We show that all subvarieties of the variety of rectangular bands are closed in the variety of n-nilpotent extension of bands. Ahanger, Nabi and Shah in [1] have proved that the variety of regular bands is closed. In this paper, we improve this result and provide its simple and shorter proof. Finally, we show that all subvarieties of the variety of normal bands are closed in the variety of left [right] semiregular bands.

Mathematics Subject Classification (2020). 20M07

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1. Introduction and preliminaries

Fennemore, in [7], had described the lattice of all varieties of bands and had given its diagram. After that, Petrich [12, Theorem II.5.1] had classified an identity on bands in atmost three variables. Though the varieties of semilattices and left [right] zero semigroups are absolutely closed, but the varieties of rectangular bands and right [left] normal bands are not absolutely closed (see Higgins [8, Chapter 4]). Therefore, it is worth to find subvarieties of the variety of all semigroups that are closed in itself or closed in the containing varieties of semigroups. As a first step in this direction, one attempts to find those varieties of semigroups that are closed in itself. Encouraged by the fact that Scheiblich [13] had shown that the variety of normal bands was closed, Alam and Khan in [3–5] had shown that the varieties of left [right] regular bands, left [right] quasi-normal bands and left [right] semi-normal bands were closed. In [2], Ahanger and Shah had proved a stronger fact that the variety of left [right] regular bands was closed in the variety of all bands.

Let U be a subsemigroup of a semigroup S. Then an element $d \in S$ is said to be dominated by U if for every semigroup P and for all homomorphisms $\gamma, \delta : S \longrightarrow P$ and $u\gamma = u\delta$ for every $u \in U$ implies $d\gamma = d\delta$. The set of all such elements of S is called the *dominion* of U in S and will be denoted by $Dom(U, S)$. It is well known that $Dom(U, S)$ is a subsemigroup of S containing U. If $Dom(U, S) = U$, then U is called closed in S, and if $Dom(U, S) = U$ in every containing semigroup S, then U is called absolutely closed. Let $\mathcal{V}_1$ and $\mathcal{V}_2$ be any varieties of semigroups such that $\mathcal{V}_1 \subseteq \mathcal{V}_2$. Then the variety $\mathcal{V}_1$

*Corresponding Author.
Email addresses: shabnamabbas.25@gmail.com (S. Abbas), swashraf81@gmail.com (W. Ashraf), noormohammad.khan@gmail.com (N.M Khan)
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is said to be closed in the variety $\mathcal{V}_2$ if whenever a semigroup $U \in \mathcal{V}_1$ is a subsemigroup of a member S of $\mathcal{V}_2$, then U is closed in S. Obviously, if $\mathcal{V}_1$ is closed in $\mathcal{V}_2$, then all subvarieties of $\mathcal{V}_1$ are closed in containing subvarieties of $\mathcal{V}_2$. A variety $\mathcal{V}$ will be called absolutely closed if all its members are absolutely closed. Let S^n, for each $n \geq 1$, denote the set of all products of n elements of any semigroup S. If S^n belongs to a class $\mathcal{C}$ of semigroups for some $n \geq 1$, then we say that S is an n-nilpotent extension of a semigroup in $\mathcal{C}$.

The following definitions and results are necessary for our investigations.

Result 1.1. ([10, Theorem 2.3]). Let U be a subsemigroup of a semigroup S and let $d \in S$. Then $d \in Dom(U,S)$ if and only if $d \in U$ or there exists a series of factorizations of d as follows:

$$d = a_0t_1 = y_1a_1t_1 = y_1a_2t_2 = y_2a_3t_2 = \cdots = y_ma_{2m-1}t_m = y_ma_{2m} \tag{1.1}$$

where $m \geq 1$, $a_i \in U$ $(i = 0, 1, \ldots, 2m)$, $y_i, t_i \in S$ $(i = 1, 2, \ldots, m)$, and

$$a_0 = y_1a_1, \qquad a_{2m-1}t_m = a_{2m},$$
$$a_{2i-1}t_i = a_{2i}t_{i+1}, \qquad y_ia_{2i} = y_{i+1}a_{2i+1} \qquad (1 \leq i \leq m-1).$$

Such a series of factorizations is called a *zigzag* in S over U with value d, length m and spine $a_0, a_1, \ldots, a_{2m}$.

Result 1.2. ([11, Result 3]). Let U and S be semigroups with U as a subsemigroup of S. Assume $d \in S \backslash U$ is such that $d \in Dom(U,S)$. If (1.1) is a zigzag of shortest possible length m over U with value d, then $t_j, y_j \in S \backslash U$ for all $j = 1, 2, \ldots, m$.

Result 1.3. ([2, Lemma 2.1]). Let U and S be any two bands with U as a subband of S. If any $d \in Dom(U,S) \backslash U$ has zigzag equations of type (1.1) in S over U of length m, then

$$a_0a_2 = a_0a_2y_2a_3a_0a_2.$$

Definition 1.4. A semigroup S is said to be a band ($\mathcal{B}$) if S satisfies the identity $a^2 = a$ for all $a \in S$.

Definition 1.5. A semigroup S is said to be an n-nilpotent extension of band ($\mathcal{B}^n$) if S^n is a band for some $n \in \mathbf{N}$.

Definition 1.6. A band S is said to be a rectangular band ($\mathcal{RB}$) if S satisfies the identity $a = axa$ for all $a, x \in S$.

Definition 1.7. A band S is said to be a normal band ($\mathcal{N}$) if S satisfies the identity $axya = ayxa$ for all $a, x, y \in S$.

Definition 1.8. A band S is said to be a regular band ($\mathcal{R}$) if S satisfies the identity $axya = axaya$ for all $a, x, y \in S$.

Definition 1.9. A band S is said to be a left semiregular band ($\mathcal{LSR}$) if S satisfies the identity $axy = axyayxy$ for all $a, x, y \in S$.

From the definitions 1.4 – 1.9, the following connection between them is clear.

Remark 1.10. $\mathcal{RB} \subset \mathcal{N} \subset \mathcal{R} \subset \mathcal{LSR} \subset \mathcal{B} \subset \mathcal{B}^n$.

The semigroup-theoretic notations and conventions of Clifford and Preston [6] and Howie [9] will be used throughout without explicit mention.

2. Closedness of rectangular bands

In [1], Ahanger, Nabi and Shah had proved that all subvarieties of the variety of rectangular bands are closed in the variety of bands. In this section, we extended this result and show that all subvarieties of the variety of rectangular bands are closed in the variety of n-nilpotent extension of bands by using Isbell's zigzag equations that characterize dominions.

Lemma 2.1. *Let S be an n-nilpotent extension of a band and U be a rectangular band such that $U \subseteq S$ and let $d \in Dom(U,S) \backslash U$. If d has a zigzag of type (1.1) in S over U of length m, then, for all $a, b \in U$ and $k = 1, 2, \ldots, m$,*

$$ab = ay_k b.$$

Proof. As S is an n-nilpotent extension of a band, S^n is a band for some $n \in \mathbf{N}$. We prove this lemma by applying induction on k. First note that

$$ay = ayay \quad \text{for all } a \in U \text{ and } y \in S. \tag{$*$}$$

First we will show that the result is true for $k = 1$, we have

$$\begin{aligned}
ab &= a(a_0)b \text{ (as } U \text{ is a rectangular band)}\\
&= (ay_1)a_1 b \text{ (by zigzag equations)}\\
&= (a)y_1 a y_1 a_1 b \text{ (by (*))}\\
&= a((ba)y_1)ay_1a_1b \text{ (as } U \text{ is a rectangular band)}\\
&= abay_1b(ay_1ay_1)a_1b \text{ (by (*))}\\
&= abay_1ba(y_1a_1)b \text{ (by (*))}\\
&= (aba)y_1(b(aa_0)b) \text{ (by zigzag equations)}\\
&= ay_1b \text{ (as } U \text{ is a rectangular band).}
\end{aligned}$$

Thus the result holds for $k = 1$. Assume inductively that the result is true for $k = j$. Finally we will show that the result also holds for $k = j + 1$. Now

$$\begin{aligned}
ab &= (a)y_jb \text{ (by the inductive hypothesis)}\\
&= a((a_{2j}a)y_j)b \text{ (as } U \text{ is a rectangular band)}\\
&= aa_{2j}a(y_ja_{2j})ay_jb \text{ (by (*))}\\
&= (aa_{2j}a)y_{j+1}a_{2j+1}ay_jb \text{ (by zigzag equations)}\\
&= (a)y_{j+1}a_{2j+1}ay_jb \text{ (as } U \text{ is a rectangular band)} \qquad (**)\\
&= a((ba)y_{j+1})a_{2j+1}ay_jb \text{ (as } U \text{ is a rectangular band)}\\
&= abay_{j+1}b(ay_{j+1}a_{2j+1}ay_jb) \text{ (by (*))}\\
&= (aba)y_{j+1}(bab) \text{ (by (**))}\\
&= ay_{j+1}b \text{ (as } U \text{ is a rectangular band).}
\end{aligned}$$

Therefore the result holds for $k = j + 1$ and, hence, by induction, the proof of the lemma is complete. □

Theorem 2.2. *Rectangular bands are closed in n-nilpotent extension of bands.*

Proof. Let U be any rectangular band and S be any n-nilpotent extension of band containing U. We show that $Dom(U,S) = U$. To show this, take any $d \in Dom(U,S) \backslash U$ such that d has a zigzag of type (1.1) in S over U with value d of shortest possible length m.

Now

$$\begin{aligned} d &= (a_0)t_1 \text{ (by zigzag equations)} \\ &= a_0(a_0t_1) \text{ (as } U \text{ is a band)} \\ &= (a_0y_ma_{2m}) \text{ (by zigzag equations)} \\ &= a_0a_{2m} \text{ (by Lemma 2.1)} \\ &\in U. \end{aligned}$$

Therefore $Dom(U, S) = U$, and, hence, the theorem is proved. □

The following corollaries are immediate consequences of Theorem 2.2

Corollary 2.3. *The variety of all rectangular bands is closed in the variety of all n-nilpotent extension of bands.*

Corollary 2.4. *The variety of all rectangular bands is closed in the variety of all bands.*

The following problem still remains open.
Problem 2.5. Is the variety of normal bands closed in the variety of all n-nilpotent extension of bands ?

3. Closedness of regular bands

In [1], Ahanger, Nabi and Shah had proved that the variety of regular bands is closed. In this section, we gave a new, simple and shorter proof of closedness of the variety of regular bands by using Isbell's zigzag equations that characterize dominions.

Lemma 3.1. *Let U and S be regular bands with U as a subband. If $d \in Dom(U, S)\backslash U$ has a zigzag of type (1.1) in S over U of length m, then*

$$a_0a_2 \cdots a_{2i} = a_0a_2 \cdots a_{2i}y_{i+1}a_{2i+1}a_0a_2 \cdots a_{2i} \ (i = 1, 2, 3, \ldots, m-1).$$

Proof. We prove it by induction on i. For i=1, we have

$$a_0a_2 = a_0a_2y_2a_3a_0a_2 \text{ (by Result 1.3)}.$$

Thus the result is true for i=1. Assume, next, inductively that the result is true for i=k $(1 \leq k < m-1)$. Then, we have

$$a_0a_2 \cdots a_{2k} = a_0a_2 \cdots a_{2k}y_{k+1}a_{2k+1}a_0a_2 \cdots a_{2k}. \tag{3.1}$$

We now show that the result also holds for i=k+1. To show this, let

$$s_i = a_0a_2a_4 \cdots a_{2i-2}a_{2i} \ (i \in \{k-1, k, k+1\}). \tag{3.2}$$

Then, equation (3.1) becomes

$$s_k = s_ky_{k+1}a_{2k+1}s_k. \tag{3.3}$$

With this notation, we need to show that

$$s_{k+1} = s_{k+1}y_{k+2}a_{2k+3}s_{k+1}.$$

Now

$$\begin{aligned}
s_{k+1} &= s_k a_{2k+2} \text{ (by equation (3.2))}\\
&= s_k a_{2k+2}(s_k)a_{2k+2} \text{ (as } U \text{ is a band)}\\
&= s_k(a_{2k+2}s_k y_{k+1})a_{2k+1}s_k a_{2k+2} \text{ (by equation (3.3))}\\
&= s_k(a_{2k+2}s_k y_{k+1}a_{2k+2})s_k y_{k+1}a_{2k+1}s_k a_{2k+2} \text{ (as } S \text{ is a band)}\\
&= s_k a_{2k+2}s_k a_{2k+2}(y_{k+1}a_{2k+2})s_k y_{k+1}a_{2k+1}s_k a_{2k+2} \text{ (as } S \text{ is a regular band)}\\
&= (s_k a_{2k+2}s_k a_{2k+2})y_{k+2}a_{2k+3}s_k y_{k+1}a_{2k+1}s_k a_{2k+2} \text{ (by zigzag equations)}\\
&= (s_k a_{2k+2})y_{k+2}a_{2k+3}s_k y_{k+1}a_{2k+1}s_k a_{2k+2} \text{ (as } U \text{ is a band)}\\
&= s_{k+1}y_{k+2}a_{2k+3}(s_k y_{k+1}a_{2k+1}s_k)a_{2k+2} \text{ (by equation (3.2))}\\
&= s_{k+1}y_{k+2}a_{2k+3}(s_k a_{2k+2}) \text{ (by equation (3.3))}\\
&= s_{k+1}y_{k+2}a_{2k+3}s_{k+1} \text{ (by equation (3.2))}.
\end{aligned}$$

This shows that the result holds for $i = k+1$. Hence, by induction, the lemma follows. □

Theorem 3.2. *The variety of regular bands is closed.*

Proof. Let U and S be regular bands with U as a subband of S. Then we have to show that $Dom(U,S) = U$. Take any $d \in Dom(U,S)\backslash U$. Suppose that d has a zigzag of type (1.1) in S over U with value d of shortest possible length m. Now

$$\begin{aligned}
d &= d^m \text{ (as } S \text{ is a band)}\\
&= \prod_{i=1}^{m}(y_i a_{2i-1}t_i) \text{ (by zigzag equations)}\\
&= y_1(a_1)t_1\prod_{i=2}^{m}(y_i a_{2i-1}t_i)\\
&= y_1a_1a_1t_1y_2a_3t_2\prod_{i=3}^{m}(y_i a_{2i-1}t_i) \text{ (as } U \text{ is a band)}\\
&= (a_0a_2)t_2y_2a_3t_2\prod_{i=3}^{m}(y_i a_{2i-1}t_i) \text{ (by zigzag equations)}\\
&= a_0a_2y_2(a_3(a_0a_2)(t_2y_2)a_3)t_2\prod_{i=3}^{m}(y_i a_{2i-1}t_i) \text{ (by Lemma 3.1)}\\
&= (a_0a_2y_2a_3a_0a_2)a_3t_2y_2a_3t_2\prod_{i=3}^{m}(y_i a_{2i-1}t_i) \text{ (as } S \text{ is a regular band)}\\
&= a_0a_2a_3t_2\prod_{i=2}^{m}(y_i a_{2i-1}t_i) \text{ (by Lemma 3.1)}\\
&\;\vdots\\
&= a_0a_2a_4\cdots a_{2m-4}(a_{2m-3}t_{m-1})y_m a_{2m-1}t_m\\
&= (a_0a_2a_4\cdots a_{2m-4}a_{2m-2})t_m y_m a_{2m-1}t_m \text{ (by zigzag equations)}\\
&= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}y_m(a_{2m-1}(a_0a_2a_4\cdots a_{2m-4}a_{2m-2})(t_m y_m)a_{2m-1})t_m\\
&\quad \text{(by Lemma 3.1)}
\end{aligned}$$

$$= a_0a_2a_4\cdots a_{2m-4}a_{2m-2}y_m(a_{2m-1}(a_0a_2a_4\cdots a_{2m-4}a_{2m-2})a_{2m-1}(t_my_m)a_{2m-1})t_m$$
(as S is a regular band)
$$= (a_0a_2\cdots a_{2m-2}y_ma_{2m-1}a_0a_2\cdots a_{2m-2})a_{2m-1}t_my_ma_{2m-1}t_m$$
$$= a_0a_2\cdots a_{2m-2}a_{2m-1}t_my_ma_{2m-1}t_m \text{ (by Lemma 3.1)}$$
$$= a_0a_2\cdots a_{2m-2}(a_{2m-1}t_m)y_ma_{2m-1}t_m$$
$$= a_0a_2\cdots a_{2m-2}a_{2m}y_ma_{2m-1}t_m \text{ (by zigzag equations)}$$
$$= a_0a_2\cdots a_{2m-2}a_{2m}y_m(a_{2m-1}t_m)$$
$$= a_0a_2\cdots a_{2m-2}a_{2m}(y_ma_{2m-1}t_m)a_{2m-1}t_m \text{ (as } S \text{ is a band)}$$
$$= a_0a_2\cdots a_{2m-2}a_{2m}y_{m-1}(a_{2m-3}t_{m-1})a_{2m-1}t_m \text{ (by zigzag equations)}$$
$$= a_0a_2\cdots a_{2m-2}a_{2m}(y_{m-1}a_{2m-3}t_{m-1})a_{2m-3}t_{m-1}a_{2m-1}t_m$$
(as S is a band)
$$= a_0a_2\cdots a_{2m-2}a_{2m}y_{m-2}a_{2m-5}t_{m-2}a_{2m-3}t_{m-1}a_{2m-1}t_m$$
(by zigzag equations)

$\vdots$

$$= a_0a_2\cdots a_{2m-2}a_{2m}y_1(a_1)t_1a_3t_2\cdots a_{2m-1}t_m$$
$$= a_0a_2\cdots a_{2m-2}a_{2m}y_1a_1a_1t_1a_3t_2\cdots a_{2m-1}t_m \text{ (as } U \text{ is a band)}$$
$$= (a_0a_2)a_4\cdots a_{2m-2}a_{2m}a_0a_2t_2a_3t_2\cdots a_{2m-1}t_m \text{ (by zigzag equations)}$$
$$= a_0a_2y_2(a_3(a_0a_2a_4\cdots a_{2m-2}a_{2m}a_0a_2)t_2a_3)t_2\cdots a_{2m-1}t_m \text{ (by Lemma 3.1)}$$
$$= a_0a_2y_2(a_3(a_0a_2a_4\cdots a_{2m-2}a_{2m}a_0a_2)a_3t_2a_3)t_2\cdots a_{2m-1}t_m$$
(as S is a regular band)
$$= (a_0a_2y_2a_3a_0a_2)a_4\cdots a_{2m-2}a_{2m}a_0a_2a_3t_2a_3t_2\cdots a_{2m-1}t_m$$
$$= a_0a_2a_4\cdots a_{2m-2}a_{2m}a_0a_2a_3t_2a_3t_2\cdots a_{2m-1}t_m \text{ (by Lemma 3.1)}$$
$$= a_0a_2\cdots a_{2m-2}a_{2m}a_0a_2(a_3t_2)a_5t_3\cdots a_{2m-1}t_m \text{ (as } S \text{ is a band)}$$
$$= a_0a_2\cdots a_{2m-2}a_{2m}a_0a_2a_4t_3a_5t_3\cdots a_{2m-1}t_m \text{ (by zigzag equations)}$$

$\vdots$

$$= a_0a_2\cdots a_{2m-2}a_{2m}(a_0a_2a_4\cdots a_{2m-2})t_ma_{2m-1}t_m$$
$$= a_0a_2\cdots a_{2m-2}a_{2m}(a_0a_2a_4\cdots a_{2m-2}y_ma_{2m-1}a_0a_2a_4\cdots a_{2m-2})t_ma_{2m-1}t_m$$
(by Lemma 3.1)
$$= a_0a_2\cdots a_{2m-2}a_{2m}a_0a_2\cdots a_{2m-2}y_m(a_{2m-1}(a_0a_2\cdots a_{2m-2})t_ma_{2m-1})t_m$$
$$= a_0a_2\cdots a_{2m-2}a_{2m}a_0a_2\cdots a_{2m-2}y_m(a_{2m-1}(a_0a_2\cdots a_{2m-2})a_{2m-1}t_ma_{2m-1})t_m$$
(as S is a regular band)
$$= a_0a_2\cdots a_{2m-2}a_{2m}(a_0a_2\cdots a_{2m-2}y_ma_{2m-1}a_0a_2\cdots a_{2m-2})a_{2m-1}t_ma_{2m-1}t_m$$
$$= a_0a_2\cdots a_{2m-2}a_{2m}a_0a_2\cdots a_{2m-2}a_{2m-1}t_ma_{2m-1}t_m \text{ (by Lemma 3.1)}$$
$$= a_0a_2\cdots a_{2m-2}a_{2m}a_0a_2\cdots a_{2m-2}(a_{2m-1}t_m) \text{ (as } S \text{ is a band)}$$
$$= a_0a_2\cdots a_{2m-2}a_{2m}a_0a_2\cdots a_{2m-2}a_{2m} \text{ (by zigzag equations)}$$
$$= a_0a_2\cdots a_{2m-2}a_{2m} \text{ (as } U \text{ is a band)}$$
$$\in U.$$

Therefore $Dom(U,S) = U$, and, hence, the theorem is proved. □

The following problem still remains open.

Problem 3.3. Is the variety of left [right] semiregular bands closed?

4. Closedness of normal bands

In [13], Scheiblich had shown that the variety of normal bands was closed. In this section, we extend this result and show that all subvarieties of the variety of normal bands are closed in the variety of left semiregular bands by using Isbell's zigzag equations that characterize dominions.

Lemma 4.1. *Let U and S be any two bands with U as a subband of S. Assume that $d \in Dom(U,S)\backslash U$. If (1.1) is a zigzag in S over U with value d of length m, then, for all $k = 2, 3, \ldots, m$,*

$$d = (\prod_{i=1}^{k-1} y_i a_{2i-1} a_{2i}) y_k a_{2k-1} t_k.$$

Proof. We prove this lemma by applying induction on k. First we will show that the result is true for $k = 2$, we have

$$\begin{aligned}
d &= y_1(a_1)t_1 \text{ (by zigzag equations)}\\
&= y_1a_1(a_1t_1) \text{ (as } U \text{ is a band)}\\
&= (y_1a_1a_2)t_2 \text{ (by zigzag equations)}\\
&= y_1a_1a_2y_1a_1(a_2t_2) \text{ (as } S \text{ is a band)}\\
&= y_1a_1a_2y_1(a_1a_1)t_1 \text{ (by zigzag equations)}\\
&= y_1a_1a_2y_1a_1t_1 \text{ (as } U \text{ is a band)}\\
&= y_1a_1a_2y_2a_3t_2 \text{ (by zigzag equations).}
\end{aligned}$$

Thus the result holds for $k = 2$. Assume inductively that the result is true for $k = j$. Finally we will show that the result also holds for $k = j + 1$. Now

$$\begin{aligned}
d &= (\prod_{i=1}^{j-1} y_i a_{2i-1} a_{2i}) y_j (a_{2j-1}) t_j \text{ (by inductive hypothesis)}\\
&= (\prod_{i=1}^{j-1} y_i a_{2i-1} a_{2i}) y_j a_{2j-1} (a_{2j-1} t_j) \text{ (as } U \text{ is a band)}\\
&= (\prod_{i=1}^{j-1} y_i a_{2i-1} a_{2i}) (y_j a_{2j-1} a_{2j}) t_{j+1} \text{ (by zigzag equations)}\\
&= (\prod_{i=1}^{j-1} y_i a_{2i-1} a_{2i}) y_j a_{2j-1} a_{2j} y_j a_{2j-1} (a_{2j} t_{j+1}) \text{ (as } S \text{ is a band)}\\
&= (\prod_{i=1}^{j} y_i a_{2i-1} a_{2i}) y_j (a_{2j-1} a_{2j-1}) t_j \text{ (by zigzag equations)}
\end{aligned}$$

$$= (\prod_{i=1}^{j} y_i a_{2i-1} a_{2i}) y_j a_{2j-1} t_j \text{ (as } U \text{ is a band)}$$
$$= (\prod_{i=1}^{j} y_i a_{2i-1} a_{2i}) y_{j+1} a_{2j+1} t_{j+1} \text{ (by zigzag equations).}$$

This shows that the result holds for $k = j+1$. Hence, by induction, the lemma follows. □

Lemma 4.2. *Let U be any normal band and S be any left semiregular band containing U. Assume that $d \in Dom(U,S) \backslash U$. If (1.1) is a zigzag in S over U with value d of length m, then*

$$y_1 a_1 a_2 y_2 a_3 a_4 \cdots y_{m-1} a_{2m-3} a_{2m-2} y_m a_{2m-1} a_{2m} = y_1 a_1 a_2 \cdots a_{2m-2} a_{2m}.$$

Proof.

$y_1 a_1 a_2 y_2 a_3 a_4 \cdots y_{m-1} a_{2m-3} a_{2m-2} y_m a_{2m-1} a_{2m}$
$= y_1 a_1 a_2 \cdots y_{m-1} a_{2m-3} a_{2m-2} y_{m-1} (a_{2m-2} a_{2m-1} t_m)$ (by zigzag equations)
$= y_1 a_1 a_2 \cdots y_{m-1} a_{2m-3} a_{2m-2} y_{m-1} a_{2m-2} (a_{2m-1} t_m)(a_{2m-2} t_m)(a_{2m-1} t_m)$
(as S is a left semiregular band)
$= y_1 a_1 a_2 \cdots y_{m-1} a_{2m-3} a_{2m-2} y_{m-1} (a_{2m-2}) a_{2m} (a_{2m-3}) t_{m-1} a_{2m}$ (by zigzag equations)
$= y_1 a_1 a_2 \cdots y_{m-1} a_{2m-3} a_{2m-2} y_{m-1} (a_{2m-2} (a_{2m-2} a_{2m}) a_{2m-3} a_{2m-3}) t_{m-1} a_{2m}$
(as U is a band)
$= y_1 a_1 a_2 \cdots y_{m-2} a_{2m-5} a_{2m-4} (y_{m-1} a_{2m-3} a_{2m-2} y_{m-1} a_{2m-2} a_{2m-3} a_{2m-2}) a_{2m} a_{2m-3} t_{m-1}$
a_{2m}(as U is a normal band)
$= y_1 a_1 a_2 \cdots y_{m-2} a_{2m-5} a_{2m-4} y_{m-1} a_{2m-3} a_{2m-2} (a_{2m})(a_{2m-3} t_{m-1})(a_{2m})$
(as S is a left semiregular band)
$= y_1 a_1 a_2 \cdots y_{m-1} a_{2m-3} (a_{2m-2} a_{2m-1} t_m a_{2m-2} t_m a_{2m-1} t_m)$ (by zigzag equations)
$= y_1 a_1 a_2 \cdots (y_{m-1} a_{2m-3}) a_{2m-2} a_{2m-1} t_m$ (as S is a left semiregular band)
$= y_1 a_1 a_2 \cdots y_{m-2} ((a_{2m-4} a_{2m-2}) a_{2m-1} t_m)$ (by zigzag equations)
$= y_1 a_1 a_2 \cdots y_{m-2} a_{2m-4} a_{2m-2} (a_{2m-1} t_m) a_{2m-4} (a_{2m-2} t_m)(a_{2m-1} t_m)$
(as S is a left semiregular band)
$= y_1 a_1 a_2 \cdots y_{m-2} a_{2m-4} a_{2m-2} ((a_{2m} a_{2m-4}) a_{2m-3} (t_{m-1} a_{2m}))$ (by zigzag equations)
$= y_1 a_1 a_2 \cdots y_{m-2} (a_{2m-4} (a_{2m-2} a_{2m}) a_{2m-4} a_{2m-3}) t_{m-1} a_{2m} a_{2m} a_{2m-4} t_{m-1} a_{2m} a_{2m-3}$
$t_{m-1} a_{2m}$(as S is a left semiregular band)
$= y_1 a_1 a_2 \cdots y_{m-2} a_{2m-4} a_{2m-4} a_{2m-2} a_{2m} a_{2m-3} t_{m-1} (a_{2m} a_{2m})(a_{2m-4} t_{m-1}) a_{2m} a_{2m-3}$
$t_{m-1} a_{2m}$(as U is a normal band)
$= y_1 a_1 a_2 \cdots y_{m-2} a_{2m-4} a_{2m-4} a_{2m-2} (a_{2m})(a_{2m-3} t_{m-1})(a_{2m}) a_{2m-5} t_{m-2} a_{2m} a_{2m-3}$
$t_{m-1} a_{2m}$(as U is a band and by zigzag equations)
$= y_1 a_1 a_2 \cdots y_{m-2} a_{2m-4} a_{2m-4} (a_{2m-2} a_{2m-1} t_m a_{2m-2} t_m a_{2m-1} t_m) a_{2m-5} t_{m-2} a_{2m} a_{2m-3}$
$t_{m-1} a_{2m}$ (by zigzag equations)

$= y_1 a_1 a_2 \cdots y_{m-2} a_{2m-4} a_{2m-4} a_{2m-2} a_{2m-1} t_m (a_{2m-5}) t_{m-2} a_{2m} a_{2m-3} t_{m-1} a_{2m}$
(as S is a left semiregular band)
$= y_1 a_1 a_2 \cdots y_{m-2} a_{2m-4} a_{2m-4} a_{2m-2} (a_{2m-1} t_m) a_{2m-5} (a_{2m-5} t_{m-2}) a_{2m} a_{2m-3}$
$t_{m-1} a_{2m}$ (as U is as band)
$= y_1 a_1 a_2 \cdots y_{m-2} (a_{2m-4} (a_{2m-4} a_{2m-2} a_{2m}) a_{2m-5} a_{2m-4}) t_{m-1} a_{2m} a_{2m-3} t_{m-1} a_{2m}$
(by zigzag equations)
$= y_1 a_1 a_2 \cdots (y_{m-2} a_{2m-5} a_{2m-4} y_{m-2} a_{2m-4} a_{2m-5} a_{2m-4}) a_{2m-2} a_{2m} a_{2m-4} t_{m-1} a_{2m}$
$a_{2m-3} t_{m-1} a_{2m}$ (as U is a normal band)
$= y_1 a_1 a_2 \cdots y_{m-2} a_{2m-5} a_{2m-4} a_{2m-2} (a_{2m}) a_{2m-4} t_{m-1} a_{2m} a_{2m-3} t_{m-1} a_{2m}$
(as S is a left semiregular band)
$= y_1 a_1 a_2 \cdots y_{m-2} a_{2m-5} a_{2m-4} a_{2m-2} (a_{2m}) a_{2m} a_{2m-4} t_{m-1} a_{2m} a_{2m-3} t_{m-1} a_{2m}$
(as U is as band)
$= y_1 a_1 a_2 \cdots y_{m-2} a_{2m-5} ((a_{2m-4} a_{2m-2}) a_{2m-1} (t_m a_{2m})) a_{2m-4} t_{m-1} a_{2m} a_{2m-3} t_{m-1}$
a_{2m}(by zigzag equations)
$= y_1 a_1 a_2 \cdots y_{m-2} a_{2m-5} a_{2m-4} a_{2m-2} (a_{2m-1} t_m a_{2m}) a_{2m-4} (a_{2m-2} t_m) (a_{2m} a_{2m-1} t_m$
$a_{2m}) a_{2m-4} t_{m-1} a_{2m} a_{2m-3} t_{m-1} a_{2m}$ (as S is a left semiregular band)
$= y_1 a_1 a_2 \cdots y_{m-2} a_{2m-5} a_{2m-4} a_{2m-2} a_{2m} (a_{2m-4} a_{2m-3} (t_{m-1} a_{2m}) a_{2m-4} t_{m-1} a_{2m}$
$a_{2m-3} t_{m-1} a_{2m})$ (by zigzag equations and as U is a band)
$= y_1 a_1 a_2 \cdots y_{m-2} a_{2m-5} a_{2m-4} a_{2m-2} a_{2m} a_{2m-4} (a_{2m-3} t_{m-1}) (a_{2m})$
(as S is a left semiregular band)
$= y_1 a_1 a_2 \cdots y_{m-2} a_{2m-5} (a_{2m-4} (a_{2m-2} a_{2m}) a_{2m-4} a_{2m-2}) t_m a_{2m-1} t_m$
(by zigzag equations)
$= y_1 a_1 a_2 \cdots y_{m-2} a_{2m-5} (a_{2m-4} a_{2m-4}) a_{2m-2} (a_{2m}) a_{2m-2} t_m a_{2m-1} t_m$
(as U is a normal band)
$= y_1 a_1 a_2 \cdots y_{m-2} a_{2m-5} a_{2m-4} (a_{2m-2} a_{2m-1} t_m a_{2m-2} t_m a_{2m-1} t_m)$
(as U is a band and by zigzag equations)
$= y_1 a_1 a_2 \cdots y_{m-2} a_{2m-5} a_{2m-4} a_{2m-2} (a_{2m-1} t_m)$ (as S is a left semiregular band)
$= y_1 a_1 a_2 \cdots y_{m-2} a_{2m-5} a_{2m-4} a_{2m-2} a_{2m}$ (by zigzag equations)
$\vdots$
$= y_1 a_1 a_2 a_4 \cdots a_{2m-2} a_{2m}.$

Thus the proof of the lemma is completed. □

Theorem 4.3. *Normal bands are closed in left semiregular bands.*

Proof. Let U be any normal band and S be any left semiregular band containing U. We show that $Dom(U, S) = U$. To show this, take any $d \in Dom(U, S) \backslash U$ such that d has a zigzag of type (1.1) in S over U with value d of shortest possible length m. Now

$$d = (\prod_{i=1}^{m-1} y_i a_{2i-1} a_{2i}) y_m a_{2m-1} t_m \text{ (by Lemma 4.1 for } k = m)$$

$= (\prod_{i=1}^{m-1} y_i a_{2i-1} a_{2i}) y_m a_{2m-1} a_{2m}$ (as U is a band and by zigzag equations)
$= y_1 a_1 a_2 \cdots y_{m-1} a_{2m-3} a_{2m-2} y_m a_{2m-1} a_{2m}$
$= (y_1 a_1) a_2 \cdots a_{2m}$ (by Lemma 4.2)
$= a_0 a_2 \cdots a_{2m}$ (by zigzag equations)
$\in U.$

Therefore $Dom(U, S) = U$, and, hence, the theorem is proved. □

Finally we propose the following related open problem.

Problem 4.4. Is the variety of normal bands closed in the variety of bands ?

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