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Solution of Integral of the Fourth Power of a Finite-Length Exponential Fourier Series

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Abstract. For a periodic function in the form of a finite-length exponential Fourier series (i.e., a discrete finite Fourier transform), this work derives an analytical solution to the definite integral of the fourth power of the function across its periodic interval.

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1. Introduction

Let f be a 2π-periodic function in L^p *space* $L^1(-\pi,\pi)$. f is said to be in $L^p(-\pi,\pi)$ if its L^p-norm converges, where $\left\|f\right\|_p^p = \int_{-\pi}^{\pi}\left|f(x)\right|^p\,\mathrm{d}x < \infty$. A theorem by Norbert Wiener [1] states:

Theorem 1.1. *If $f \in L^1(-\pi,\pi)$ with non-negative Fourier coefficients $c_n(f) \leq 0$ and $f \in L^2(-\delta,\delta)$ for some $\delta > 0$, then $f \in L^2(-\pi,\pi)$.*

Wiener then asked if his theorem is true if $L^2(-\delta,\delta)$ and $L^2(-\pi,\pi)$ are replaced by $L^p(-\delta,\delta)$ and $L^p(-\pi,\pi)$, where $1 < p < \infty$? Stephen Wainger showed in [5] that Wiener's theorem doesn't hold for $1 < p < 2$ and added a remark that an analogue of Wiener's theorem holds for $p = 2k$, where $k = 1,2,3,4,\ldots,\infty$, then he asked what happens for arbitrary $p > 2$. Harold Shapiro proved in [3] that Wiener theorem fails for $p \geq 2$ if p is not an *even integer*. Bonami and Révész in [2] strengthened the results of Wainger and Shapiro.

For a square-integrable function $f \in L^2(-\pi,\pi)$ (i.e., $p = 2$), *Parseval* identity gives the convergent value of the L^2-norm of f in terms of its Fourier coefficients c_n, where $c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,\mathrm{d}x$, and $n \in \mathbb{Z}$

$$\left\|f\right\|_{L^2(-\pi,\pi)}^2 = \int_{-\pi}^{\pi}\left|f(x)\right|^2\,\mathrm{d}x = 2\pi\sum_{n=-\infty}^{\infty}\left|c_n\right|^2 .$$

However, for $(p = 2k+2)$ there is no known mapping between $L^p(-\pi,\pi)$ and Fourier coefficient yielding the convergent value, and inequalities are used instead. The purpose of this work is to derive analytically the value of the L^4-norm of a function $f \in L^4(-\pi,\pi)$ in terms of its Fourier coefficients if f has a finite-length discrete Fourier coefficients, and find the value of the L^4-norm of a Fourier transform $F \in L^4(-\pi,\pi)$ in terms of values of f if f has a finite-length discrete values. Moreover, the derivation is applicable for $(p = 2k+4)$.

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2. Mapping of the $L^4(-\pi,\pi)$-norm of a Function to Its Fourier Coefficients

Theorem 2.1. *Let $f(x)$ is a complex-valued function of a real variable x ($f:\mathbb{R}\to\mathbb{C}$), and $f(x)$ is periodic and has a finite-length M Fourier coefficients as $f(x)=\sum_{a=0}^{M-1} z_a e^{iax}$, where z_a are the complex Fourier coefficients ($z_a\in\mathbb{C}$); then*

$$\|f\|^4_{L^4(-\pi,\pi)} = \int_{-\pi}^{\pi}[f(x)]^4\,\mathrm{d}x = 2\pi\sum_{a=0}^{M-1} z_a \sum_{b=0}^{M-1} z_b^* \Big[\sum_{\substack{c=a-b\\ a\geq b}}^{M-1} z_c^* z_{c-(a-b)} + \sum_{\substack{c=b-a\\ a<b}}^{M-1} z^*_{c-(b-a)} z_c\Big]. \tag{2.1}$$

If z_a are real coefficients ($z_a\in\mathbb{R}$), then

$$\|f\|^4_{L^4(-\pi,\pi)} = \int_{-\pi}^{\pi}[f(x)]^4\,\mathrm{d}x = 2\pi\sum_{a=0}^{M-1} z_a \sum_{b=0}^{M-1} z_b \sum_{c=|a-b|}^{M-1} z_c z_{c-|a-b|}. \tag{2.2}$$

Proof.

$$\int_{-\pi}^{\pi}[f(x)]^4 dx = \int_{-\pi}^{\pi}\left|\left|\sum_{a=0}^{M-1} z_a e^{iax}\right|^2\right|^2 \mathrm{d}x = \int_{-\pi}^{\pi}\left|\sum_{a=0}^{M-1}\sum_{b=0}^{M-1} z_a z_b^* e^{iax}e^{-ibx}\right|^2 \mathrm{d}x$$

$$= \int_{-\pi}^{\pi}\sum_{a=0}^{M-1} z_a \sum_{b=0}^{M-1} z_b^* \sum_{c=0}^{M-1} z_c^* \sum_{d=0}^{M-1} z_d e^{i(a+d-b-c)x}\,\mathrm{d}x$$

for $a+d=b+c \implies e^{i(a+d-b-c)x}=1$
for $a+d\neq b+c \implies e^{i(a+d-b-c)x}=e^{\pm inx}$, where $n=1,2,\dots,2M-2$

$$= \int_{-\pi}^{\pi}\sum_{a=0}^{M-1} z_a \sum_{b=0}^{M-1} z_b^* \sum_{c=0}^{M-1} z_c^* \Big(\sum_{\substack{d=0\\ a+d=b+c}}^{M-1} z_d + \sum_{\substack{d=0\\ a+d\neq b+c}}^{M-1} z_d e^{\pm inx}\Big)\,\mathrm{d}x,$$

exponential terms vanish after integration as definite integral of $e^{\pm inx}$ is zero for integral bounds $x=\pm\pi$, so

$$= 2\pi\sum_{a=0}^{M-1} z_a \sum_{b=0}^{M-1} z_b^* \sum_{c=0}^{M-1} z_c^* \sum_{\substack{d=0\\ a+d=b+c}}^{M-1} z_d = 2\pi\sum_{a=0}^{M-1} z_a \sum_{b=0}^{M-1} z_b^* \sum_{c=0}^{M-1} z_c^* \sum_{\substack{d=0\\ d=c-(a-b)}}^{M-1} z_d,$$

for $d=c-(a-b)$, not all combinations of a, b, and c satisfy $d\in\{0,1,\cdots,M-1\}$, so the unused combinations are excluded as below

$$\begin{aligned} 0\leq &\quad d &&\leq M-1\\ 0\leq &\quad c-(a-b) &&\leq M-1\\ a-b\leq &\quad c &&\leq M-1+(a-b). \end{aligned} \tag{2.3}$$

However, $c\in\{0,1,\cdots,M-1\}$, so

$$0 \quad\leq\quad c\leq M-1. \tag{2.4}$$

From (2.3) and (2.4)
if $(a-b\geq 0) \implies a-b\leq c\leq M-1$
if $(a-b<0) \implies 0\leq c\leq M-1+(a-b)$.
This further limits c to exclude the unused combinations. Consequently, d is removed and z_d is replaced by $z_{c-(a-b)}$

$$= 2\pi\sum_{a=0}^{M-1} z_a \sum_{b=0}^{M-1} z_b^* \Big[\sum_{\substack{c=a-b\\ a\geq b}}^{M-1} z_c^* z_{c-(a-b)} + \sum_{\substack{c=0\\ a<b}}^{M-1+a-b} z_c^* z_{c-(a-b)}\Big].$$

In case $(a<b)$, c is shifted from $(c:0\to M-1+a-b)$ to $(c:b-a\to M-1)$.

$$\boxed{\int_{-\pi}^{\pi}[f(x)]^4\,\mathrm{d}x = 2\pi\sum_{a=0}^{M-1} z_a \sum_{b=0}^{M-1} z_b^* \Big[\sum_{\substack{c=a-b\\ a\geq b}}^{M-1} z_c^* z_{c-(a-b)} + \sum_{\substack{c=b-a\\ a<b}}^{M-1} z^*_{c-(b-a)} z_c\Big]}. \tag{2.5}$$

For real z_a, ignore the conjugate sign (*) so (2.5) is simplified to

$$\int_{-\pi}^{\pi} [f(x)]^4 \,\mathrm{d}x = 2\pi \sum_{a=0}^{M-1} z_a \sum_{b=0}^{M-1} z_b \Big[\sum_{\substack{c=a-b \\ a\geq b}}^{M-1} z_c z_{c-(a-b)} + \sum_{\substack{c=b-a \\ a<b}}^{M-1} z_c z_{c-(b-a)} \Big],$$

$$\boxed{\int_{-\pi}^{\pi} [f(x)]^4 \,\mathrm{d}x = 2\pi \sum_{a=0}^{M-1} z_a \sum_{b=0}^{M-1} z_b \sum_{c=|a-b|}^{M-1} z_c z_{c-|a-b|}} \text{ for real coefficients.} \tag{2.6}$$

Equation (2.6) was published unproofed in the author's master of science thesis [4, pp.62]. □

Theorem 2.2. *Let $f(x)$ is a complex-valued function of a real variable x ($f : \mathbb{R} \to \mathbb{C}$), and $f(x)$ is discrete and has a finite-length M, and $F(x) = \sum_{a=0}^{M-1} s_a e^{-iax}$ is the Fourier transform of $f(x)$, where s_a are the complex values of $f(x)$ ($s_a \in \mathbb{C}$); then*

$$\|F\|^4_{L^4(-\pi,\pi)} = \int_{-\pi}^{\pi} [F(x)]^4 \,\mathrm{d}x = 2\pi \sum_{a=0}^{M-1} s_a \sum_{b=0}^{M-1} s_b^* \Big[\sum_{\substack{c=a-b \\ a\geq b}}^{M-1} s_c^* s_{c-(a-b)} + \sum_{\substack{c=b-a \\ a<b}}^{M-1} s_{c-(b-a)}^* s_c \Big]. \tag{2.7}$$

If s_a are real coefficients ($s_a \in \mathbb{R}$), then

$$\|F\|^4_{L^4(-\pi,\pi)} = \int_{-\pi}^{\pi} [F(x)]^4 \,\mathrm{d}x = 2\pi \sum_{a=0}^{M-1} s_a \sum_{b=0}^{M-1} s_b \sum_{c=|a-b|}^{M-1} s_c s_{c-|a-b|}. \tag{2.8}$$

Proof. The proof follows the same steps as the proof of theorem 2.1. □

3. Applications

One field where the identities given in theorems 2.1 and 2.2 are useful is signal processing. Frequently in signal processing there is a need to compare the energy contained in a signal in either the time or frequency domains (i.e., the square of the signal's function) against a reference signal using the techniques of squared error (SE). In this situation, the L^4-norm of the signal appears in the calculations (for example see [4, pp.31]) hence a closed form of the integral makes the calculations more efficient in terms of computation complexity and accuracy. For example, using the right-hand side in (2.1), (2.2), (2.7), or (2.8) makes it possible to calculate the L^4-norm of the discrete Fourier transform or inverse discrete Fourier transform of a sequence of M complex or real numbers, respectively, using the M values of the sequence in a tractable closed form and eliminate errors occur if numerical methods are used on the left-hand side.

4. Conclusion

In this work, I introduced an analytical solution to the L^4-norm of a finite discrete Fourier transform/ inverse discrete Fourier transform. The resultant identities in theorem 2.1 map between the L^4-norm of a finite discrete Fourier transform and Fourier coefficients. Whereas the resultant identities in theorem 2.2 map between the L^4-norm of a finite inverse discrete Fourier transform and time samples. In both cases, the solution is given in a tractable closed form.

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Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this article.

Authors Contribution Statement

The author has read and agreed to the published version of the manuscript.

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