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CLASSIFICATION OF THREE-DIMENSIONAL ISOPOTENT ALGEBRAS

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Dedicated to the memory of Professor Edmund R. Puczyłowski

Abstract. In the article, we perform a classification of algebras with dimensions ≤ 3 and with the property that each element is colinear with its square. The classification is complete up to properties of the ground field.

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1. Introduction

Let $\mathcal{A}$ be a linear space over a field $\mathbb{F}$ (of characteristic chr$\mathbb{F}$ and cardinal number crd$\mathbb{F}$) and $(\mathcal{A},\cdot)$ an algebra over this space.

For $\mathcal{M} \subset \mathcal{A}$, let lin$\mathcal{M}$ be the subspace spanned on this set, and alg$\mathcal{M}$ the subalgebra generated by this set.

Element $r \in \mathcal{A}$ is **isopotent** if $\exists\, \delta \in \mathbb{F} : r^2 = \delta r$. Isopotents form three disjunct classes:

- 0,
- **isotrops:** $q \neq q^2 = 0$,
- elements of the type λp where $\mathbb{F} \ni \lambda \neq 0$ and p is an **idempotent:** $p = p^2 \neq 0$.

Definition 1.1. $\mathbb{F}$-algebra $(\mathcal{A},\cdot)$ is an **isopotent algebra,** if the following (obviously equivalent) conditions are fulfilled:

➢ every element is isopotent;
➢ $\forall\, a \in \mathcal{A} :\ \mathrm{a}^2 \in \mathrm{lin}\{a\}$;
➢ $\forall\, a \in \mathcal{A} :\ \mathrm{alg}\{a\} = \mathrm{lin}\{a\}$;
➢ $\forall\, a \in \mathcal{A} :\ \dim \mathrm{alg}\{a\} \leq 1$.

Every anticommutative algebra is an isopotent algebra. According to Definition 1.1, the following easily provable lemma tells us more about this.

Lemma 1.2. $\mathbb{F}$-*algebra* $(\mathcal{A},\cdot)$ *is an isopotent algebra iff there exists such a functional* $H:\mathcal{A}\to\mathbb{F}$ *that*

$$\forall\, a\in\mathcal{A}:\ a^2 = H(a)a.$$

We may demand that $H(0)=0$ *and this functional is then unique and homogeneous:*

$$\forall\,(\lambda,a)\in\mathbb{F}\times\mathcal{A}:\ H(\lambda a)=\lambda H(a).$$

We shall call the homogeneous functional H from Lemma 1.2 **excess**. Its value $H(a)$ is then the **excess of element** a.

If the excess is nonzero linear functional, we will say that the algebra is **proper isopotent algebra**; if the excess is zero functional then the algebra is **zeropotent algebra** (anticommutative algebra in case of $\mathrm{chr}\mathbb{F}\neq 2$); and if the excess is nonlinear, we will say that the algebra is **improper isopotent algebra.**

$\cdot$	a	b
a	$\alpha\, a$	$\beta\, a+\gamma\, b$
b	$(\alpha+\beta+1)\,a+(\alpha+\gamma+1)\,b$	$\alpha\, b$

TABLE 1. Improper isopotent algebras;
$\mathbb{F}=\mathbb{Z}_2$, $(\alpha,\beta,\gamma)\in\{(0,0,0),(0,0,1),(0,1,0),(1,0,0),(1,1,1)\}$.

$\cdot$	a	b
a	0	εa
b	$-\varepsilon a$	0

TABLE 2. Zeropotent algebras, $\varepsilon\in\{0,1\}$.

$\cdot$	a	b
a	a	$\lambda\, b$
b	$(1-\lambda)\, b$	0

TABLE 3

$\cdot$	a	b
a	a	a
b	$b-a$	0

TABLE 4

Proper isopotent algebras, $\lambda\in\mathbb{F}$.

Example 1.3. All (three) algebras of dim $\mathcal{A} \leq 1$ are isopotent:

- $\{0\}$ and $(\mathbb{F}\{e\}, e \neq e^2 = 0)$ are zeropotent,
- $(\mathbb{F}\{e\}, e = e^2 \neq 0)$ is proper isopotent.

Example 1.4. 2-dimensional isopotent algebras are isomorphic to exactly one of the algebras defined in Tables 1-4. In Table 1 there are improper algebras (all over the field $\mathbb{Z}_2$), since the excess H is obviously nonlinear: $H(0) = 0,\ H(a) = H(b) = \alpha,\ H(a+b) = 1.$ In Table 2 there are two zeropotent algebras (in fact Lie algebras) and so: $H = 0$. For algebras in Tables 3 and 4, which are all proper isopotent algebras, we find $\forall(\delta, \varepsilon) \in \mathbb{F}^2 : H(\delta a + \varepsilon b) = \delta$.

Deriving all these facts is simple, although it requires quite a bit of work, and we therefore omit it.

Proposition 1.5. *Let $(\mathcal{A}, \cdot)$ be an isopotent algebra with the excess H. The algebra is power-associative and the following identities are valid for $(a, b, c) \in \mathcal{A}^3$:*

(1) $a^n = H(a)^{n-1} a \quad (n > 1),$

(2) $ab + ba = [H(a+b) - H(a)]a + [H(a+b) - H(b)]b$

(3) $[H(a+b+c) - H(a+b) - H(c+a) + H(a)]a + [H(a+b+c) - H(b+c) - H(a+b) + H(b)]b + [H(a+b+c) - H(c+a) - H(b+c) + H(c)]c = 0$

(4) $[a,a,b] + [a,b,a] + [b,a,a] = [H(a) + H(b) - H(a+b)](ab - ba)$

Proof. Straightforward, since it follows entirely from Lemma 1.2. $\square$

Proposition 1.6. *Let $(\mathcal{A}, \cdot)$ be an isopotent algebra with the excess H over a field $\mathbb{F} \neq \mathbb{Z}_2$. Then H is a linear functional and the algebra is either a zeropotent or a proper isopotent algebra.*

Proof. Because of homogeneity of H we have to prove only the additivity $H(a+b) = H(a) + H(b)$, which is obvious if a and b are colinear:

$$a = \alpha c\ ,\ b = \beta c\,.$$

From now on suppose that a, b are linearly independent. Firstly take $\mathrm{chr}\mathbb{F} \neq 2$. We will use identity (2) in Proposition 1.5 for elements a, b and $a, -b$ (respectively):

$$ab + ba = [H(a+b) - H(a)]a + [H(a+b) - H(b)]b$$
$$-ab - ba = [H(a-b) - H(a)]a - [H(a-b) - H(-b)]b$$

We add the equations:

$$[H(a+b)+H(a-b)-2H(a)]a+[H(a+b)-H(a-b)-2H(b)]b=0$$
$$H(a+b)+H(a-b)-2H(a)=0=H(a+b)-H(a-b)-2H(b).$$

By adding the last two equations we have finished the proof if $\mathrm{chr}\mathbb{F}\neq 2$.

The remaining possibility: $\mathrm{chr}\mathbb{F}=2$, $\mathbb{F}\neq\mathbb{Z}_2$. Let $\lambda\notin\{0,1\}$; then $1+\lambda\neq 0$. Let us define: $u=a+\frac{\lambda}{1+\lambda}b$, $v=\frac{1+\lambda}{\lambda}a+b$, $w=\frac{1+\lambda}{\lambda}a+\frac{\lambda}{1+\lambda}b$. We insert these elements into (3) of Proposition 1.5 instead of elements a,b,c:

$$\left[H(a+b)+\lambda^{-2}H(a)+\tfrac{1+\lambda}{\lambda}H\left(\tfrac{1+\lambda}{\lambda}a+\tfrac{\lambda}{1+\lambda}b\right)\right]a+$$
$$+\left[H(a+b)+(1+\lambda)^{-2}H(b)+\tfrac{\lambda}{1+\lambda}H\left(\tfrac{1+\lambda}{\lambda}a+\tfrac{\lambda}{1+\lambda}b\right)\right]b=0\,.$$

Both coefficients must be 0. The first equation thus obtained will be multiplied by λ^2 and the second by $(1+\lambda)^2$:

$$\lambda^2H(a+b)+H(a)+\lambda(1+\lambda)H\left(\tfrac{1+\lambda}{\lambda}a+\tfrac{\lambda}{1+\lambda}b\right)=0$$
$$(1+\lambda^2)H(a+b)+H(b)+\lambda(1+\lambda)H\left(\tfrac{1+\lambda}{\lambda}a+\tfrac{\lambda}{1+\lambda}b\right)=0.$$

We only need to add these two equations and the proof is complete. □

Proposition 1.7. *Let $(\mathcal{A},\cdot)$ be an isopotent algebra with the linear excess H. The following identities are valid for $(a,b,c)\in\mathcal{A}^3$:*

(1) $ab+ba=H(b)a+H(a)\,b$,
(2) $H(ab+ba)=2H(a)H(b)$,
(3) $H(ab)-H(a)H(b)=H(b)H(a)-H(ba)$,
(4) $H(ab-ba)=2[H(ab)-H(a)H(b)]$,
(5) $[a,a,b]+[a,b,a]+[b,a,a]=0$,
(6) $[a,b,a]=[H(ab)-H(a)H(b)]a\,$,
(7) $[a,b,c]+[c,b,a]=[H(cb)-H(c)H(b)]a\,+[H(ab)-H(a)H(b)]c.$

Proof. Straightforward. □

2. General forms of multiplication tables

Firstly, we will find the general form of multiplication table for an improper isopotent algebra. According to Proposition 1.6, the ground field is $\mathbb{F}=\mathbb{Z}_2$. Two-dimensional algebras are described in Table 1, hence we may assume that algebras are at least three-dimensional.

There must exist elements a and b such that

$$a^2=H(a)a,\ b^2=H(b)b,\ (a+b)^2=H(a+b)(a+b)$$

and $H(a+b) \neq H(a)+H(b)$, hence $H(a+b) = H(a)+H(b)+1$. These elements cannot be linearly dependent and may be a part of some basis $(a, b, c_1, c_2, \dots)$.

Introduce the following notations: $H(a) = 1+\alpha$, $H(b) = 1+\beta$, $ab = d$, $ac_i = e_i$, $bc_i = f_i$, $H(c_i) = 1+\gamma_i$, $c_j c_k = g_{jk}$ $(j < k)$. Then, from $(a+b)^2 = H(a+b)(a+b)$ we derive:

$$ba = \beta a + \alpha b + d.$$

From the expansions of $(a+c_i)^2$, $(b+c_i)^2$ and $(a+b+c_i)^2$ we get:

$$H(a+c_i) = 1+\alpha+\gamma_i = H(a)+H(c_i)+1,$$
$$H(b+c_i) = 1+\beta+\gamma_i = H(b)+H(c_i)+1,$$
$$c_i a = \gamma_i a + \alpha c_i + e_i\,,\ c_i b = \gamma_i b + \beta c_i + f_i.$$

Similarly, from $(c_j + c_k)^2$ and $(a+c_j+c_k)^2$ it follows:

$$H(c_j+c_k) = 1+\gamma_j+\gamma_k = H(\gamma_j)+H(\gamma_k)+1,$$
$$c_k c_j = \gamma_k c_j + \gamma_j c_k + g_{jk} \quad (j<k).$$

This already tells us that elements a and b are not something special and that we may write a multiplication table as Table 5 for any algebra. Elements p_{jk} and scalars λ_i are still completely arbitrary.

Simple consequence:

$$(c,d) \ \text{ linearly independent pair } \ \Leftrightarrow \ H(c+d) = H(c)+H(d)+1,$$
$$(c,d) \ \text{ linearly dependent pair } \ \Leftrightarrow \ H(c+d) = H(c)+H(d)\,.$$

$\cdot$	$\dots$	a_j	$\dots$	a_k	$\dots$
$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$
a_j	$\dots$	$(1+\lambda_j)a_j$	$\dots$	$\lambda_j a_k + p_{jk}$	$\dots$
$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$
a_k	$\dots$	$\lambda_k a_j + p_{jk}$	$\dots$	$(1+\lambda_k)a_k$	$\dots$
$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$

TABLE 5. General form of the multiplication table of an improper isopotent algebra; $\mathbb{F} = \mathbb{Z}_2$, elements p_{jk} and scalars λ_i are arbitrary.

Example 2.1. The general form of multiplication table of a 3-dimensional improper isopotent algebra is described in Table 6; elements p, q, r and scalars κ, λ, μ are completely arbitrary.

Such an algebra has only 8 elements and their excesses are

$\cdot$	a	b	c
a	$(1+\kappa)a$	$\kappa b+r$	$\kappa c+q$
b	$\lambda a+r$	$(1+\lambda)b$	$\lambda c+p$
c	$\mu a+q$	$\mu b+p$	$(1+\mu)$ c

TABLE 6. 3-dimensional improper isopotent algebra, $\mathbb{F}=\mathbb{Z}_2$.

$$H(0)=0\,,\ H(a)=1+\kappa\,,\ H(b)=1+\lambda\,,\ H(c)=1+\mu,$$
$$H(b+c)=1+\lambda+\mu\,,\ H(c+a)=1+\mu+\kappa\,,\ H(a+b)=1+\kappa+\lambda,$$
$$H(a+b+c)=1+\kappa+\lambda+\mu.$$

If we review all 8 possible choices of parameters, κ, λ, μ, we quickly see that there are only two non-isomorphic forms: Table 7 $(\kappa=\lambda=\mu=0)$ and Table 8 $(\kappa=\lambda=\mu=1.)$

$\cdot$	a	b	c
a	a	r	q
b	r	b	p
c	q	p	c

TABLE 7

$\cdot$	a	b	c
a	0	$b+r$	$c+q$
b	$a+r$	0	$c+p$
c	$a+q$	$b+p$	0

TABLE 8

Now, the proper isopotent algebras will be considered. Since H is a nonzero linear functional, we can find a basis $(a, c_1, c_2, \dots)$ such that $H(a)=1$, $H(c_n)=0$. Since we have already discussed 2-dimensional algebras in Example 1.4 (Tables 3,4), we may suppose that all algebras are at least 3-dimensional.

It is easy to find the general multiplication table from the expansion of $(a+c_n)^2$ and $(c_j+c_k)^2$. The result is Table 9, where the elements q_n and p_{jk} are arbitrary.

Example 2.2. The general form of the multiplication table of 3-dimensional proper isopotent algebra is described in Table 10; elements p, q, r are arbitrary.

3. Classification of 3-dimensional improper isopotent algebras

The first type of isopotent algebras is **zeropotent algebras**. Their classification is made in [2] and there is nothing more to add here.

$\cdot$	a	$\dots$	c_j	$\dots$	c_k	$\dots$
a	a	$\dots$	q_j	$\dots$	q_k	$\dots$
$\vdots$	$\vdots$	$\ddots$	$\vdots$		$\vdots$	
c_j	$c_j - q_j$	$\dots$	0	$\dots$	p_{jk}	$\dots$
$\vdots$	$\vdots$		$\vdots$	$\ddots$	$\vdots$	
c_k	$c_k - q_k$	$\dots$	$-p_{jk}$	$\dots$	0	$\dots$
$\vdots$	$\vdots$		$\vdots$		$\vdots$	$\ddots$

TABLE 9. General proper isopotent algebra.

$\cdot$	a	b	c
a	a	r	q
b	$b - r$	0	p
c	$c - q$	$-p$	0

TABLE 10. General 3-dimensional proper isopotent algebra.

The second type is **improper isopotent algebras.** Then $\mathbb{F} = \mathbb{Z}_2$ as we know from Proposition 1.6. From Example 2.1 we also know that these algebras are in two non-isomorphic classes: Table 7 (commutative algebras) and Table 8 (strictly non-commutative algebras).

Let us first work with the commutative case. We already know that $\forall\, d : \, d^2 = d$. We will assume the labels from Table 7.

Firstly, we will prove that it is always possible to rearrange basis so that $r = \varepsilon c$. If we can find a linearly independent triple $\hat{a}, \hat{b}, \hat{c} = \hat{a}\hat{b}$, then we adopt it as a new basis. But suppose that any triple $\hat{a}, \hat{b}, \hat{a}\hat{b}$ is linearly dependent. Then $r = \gamma a + \delta b$, If $\gamma = 0$, let the new basis be $\hat{a} = a + \delta b$, $\hat{b} = b$, $\hat{c} = c$. And if $\delta = 0$, the new basis will be $\hat{a} = a$, $\hat{b} = \gamma a + b$, $\hat{c} = c$. The remaining possibility is $ab = a + b$ and similarly $ac = a + c$, $bc = b + c$; then the new basis should be $\hat{a} = a + b$, $\hat{b} = a + c$, $\hat{c} = a$.

Suppose that the algebra has no divisors of zero. Then $ab = c = r$. After checking all possible products, we find only two possibilities $p = a + b + c$, $q = b + c$ and $p = a + c$, $q = a + b + c$, which appear to be isomorphic.

Further suppose that there exist divisors of zero, say $ab = r = 0$. We shall denote: $q = ac = \alpha a + \beta b + \gamma c$, $p = bc = \delta a + \varepsilon b + \zeta c$. Suppose that $\alpha = 1$ and

let us change the basis in the following way:
If $\gamma = 0$ then: $\hat{a} = a$, $\hat{b} = b$, $\hat{c} = a + c$;
if $\gamma = 1 \neq \varepsilon$ then: $\hat{a} = b$, $\hat{b} = a$, $\hat{c} = \zeta b + c$;
if $\gamma = \varepsilon = 1 \neq \zeta$ then: $\hat{a} = b$, $\hat{b} = a$, $\hat{c} = b + c$;
if $\gamma = \varepsilon = \zeta = 1$ then: $\hat{a} = a + b$, $\hat{b} = (1 + \delta)a + (1 + \beta)b + c$, $\hat{c} = a$.
In all these cases we find $\hat{a}\hat{c} = \hat{q} = \hat{\beta}\hat{b} + \hat{\gamma}\hat{c}$. Hence we may suppose that $\alpha = 0$.

Now, it is best to use a computer. The number of different multiplication tables (= the number of different triples r, q, p) is $2^9 = 512$. We choose a certain triple and express it in all possible 168 bases. Then we choose a new triple, that has not yet appeared before, and repeat the process. We continue until we exhaust all possible triples. In Table 11 there are all non-isomorphic types of algebra from Table 7.

	r	q	p
1	c	$b + c$	$a + b + c$
2	0	0	0
3	0	0	$b + c$
4	0	0	a
5	0	0	$a + b + c$
6	0	c	$b + c$
7	0	c	a
8	0	c	$a + b$
9	0	b	$b + c$
10	0	b	$a + b + c$

TABLE 11. Non-isomorphic algebras from Table 7.

Theorem 3.1. *In every dimension from 0 to 3, there exists (up to isomorphism) one isopotent algebra without zero divisors:*

- $\{0\}$,
- $(\mathbb{F}\{e\},\ e = e^2 \neq 0)$,
- *algebra from Table 1 with* $\alpha = \beta = \gamma = 1$,
- *algebra from Table 7 with* $r = c$, $q = b + c$, $p = a + b + c$.

The best way to classify non-commutative algebras with Table 8 as a multiplication table is again by using a computer. Namely, if we insist that the multiplication table must have zeros on its diagonal, only $a, b, c, a + b + c$ can be elements of any (24) basis. The results are in Table 12.

Theorem 3.2. *There exist 46 non-isomorphic 3-dimensional improper isopotent algebras, all over the field* $\mathbb{Z}_2$ *. Their multiplication tables are Tables 7 and 11 (commutative case) and Tables 8 and 12 (strictly non-commutative case).*

4. Classification of 3-dimensional proper isopotent algebras

A multiplication table of such an algebra is isomorphic to an algebra defined by Table 10. We shall use the following notations:

$$r = ab = \pi_0 a + \pi_1 b + \pi_2 c,$$

	r	q	p
11	0	0	0
12	0	0	c
13	0	0	$b+c$
14	0	0	a
15	0	0	$a+c$
16	0	0	$a+b+c$
17	0	c	c
18	0	c	$b+c$
19	0	c	a
20	0	c	$a+c$
21	0	c	$a+b$
22	0	c	$a+b+c$
23	0	b	$b+c$
24	0	b	a
25	0	b	$a+b$
26	0	b	$a+b+c$
27	0	$a+c$	$b+c$
28	0	$a+c$	$a+b+c$
29	c	c	c
30	c	c	b
31	c	c	$a+c$
32	c	c	$a+b$
33	c	b	a
34	c	b	$a+c$
35	c	b	$a+b+c$

36	c	$b+c$	$a+c$
37	c	$b+c$	$a+b$
38	c	$b+c$	$a+b+c$
39	c	a	b
40	c	a	$a+b$
41	c	$a+b$	$a+b$
42	b	c	$b+c$
43	b	$b+c$	$b+c$
44	b	a	c
45	b	$a+c$	b
46	b	$a+c$	$b+c$

Table 12. Non-isomorphic algebras from Table 8.

$$q = ac = \varrho_0 a + \varrho_1 b + \varrho_2 c,$$
$$p = bc = \sigma_0 a + \sigma_1 b + \sigma_2 c.$$

At the beginning we will change the basis in the following way:
$\sigma_0 \neq 0 \quad \Rightarrow \quad \hat{a} = a + \sigma_1\sigma_0^{-1}b + \sigma_2\sigma_0^{-1}c \ , \quad \hat{b} = b \ , \quad \hat{c} = \sigma_0^{-1}c;$
$\sigma_0 = 0 \neq \sigma_1 \quad \Rightarrow \quad \hat{a} = a \ , \quad \hat{b} = \sigma_1 b + \sigma_2 c \ , \quad \hat{c} = \sigma_1^{-1}c;$
$\sigma_0 = \sigma_1 = 0 \neq \sigma_2 \quad \Rightarrow \quad \hat{a} = a \ , \quad \hat{b} = \sigma_2 c \ , \quad \hat{c} = -\sigma_2^{-1}b.$
The overall result of all these transformations is

$$(\widehat{\sigma_0}, \widehat{\sigma_1}, \widehat{\sigma_2}) \in \{(0,0,0), (1,0,0), (0,1,0)\}.$$

We will continue insisting that the triple $(\sigma_0, \sigma_1, \sigma_2)$ must have only these three values. Additionally, we will make a small change in labelling:

$$\sigma_0 = \sigma \ , \ \sigma_1 = \tau \, .$$

Hence, we will begin the classification with the multiplication table in Table 13.

$\cdot$	a	b	c
a	a	$\pi_0 a + \pi_1 b + \pi_2 c$	$\varrho_0 a + \varrho_1 b + \varrho_2 c$
b	$b - \pi_0 a - \pi_1 b - \pi_2 c$	0	$\sigma a + \tau b$
c	$c - \varrho_0 a - \varrho_1 b - \varrho_2 c$	$-\sigma a - \tau b$	0

Table 13. Basic multiplication table of proper isopotent algebras; $(\sigma, \tau) \in \{(0,0), (1,0), (0,1)\}$.

A simple calculation: $(\alpha a + \beta b + \gamma c)^2 = \alpha(\alpha a + \beta b + \gamma c)$ shows that

$$H(\alpha a + \beta b + \gamma c) = \alpha \, .$$

If U is an isomorphism of two such algebras with excesses G and H, then for any element f from the first algebra we find:

$$H(U(f))U(f) = U(f)^2 = U(f^2) = U(G(f)f) = G(f)U(f),$$
$$H(U(f)) = G(f).$$

From this we conclude that any change of basis must be of the following kind:

$$\hat{a} = a + ub + vc\ ,\ \hat{b} = wb + xc\ ,\ \hat{c} = yb + zc\ ;\ wz - xy \neq 0,$$

where u, v, w, x, y, z are scalar coefficients. The transformations of structure constants are collected in System 1.

As we can see, two algebras with different pairs (σ, τ) are non-isomorphic. We will consider each of these three types separately.

$$\hat{a} = a + ub + vc\ ,\ \hat{b} = wb + xc\ ,\ \hat{c} = yb + zc$$

$$\begin{aligned}
&D = wz - xy \neq 0 = u\sigma = v\sigma = x\tau\ ,\ D\sigma = \sigma\ ,\ z\tau = \tau \\
&\widehat{\pi_0} = w\pi_0 + x\varrho_0 \\
&\widehat{\pi_1} = D^{-1}[(vy - uz)w\pi_0 + wz\pi_1 - wy\pi_2 + (vy - uz)x\varrho_0 + xz\varrho_1 - xy\varrho_2] - v\tau \\
&\widehat{\pi_2} = D^{-1}[(ux - vw)w\pi_0 - wx\pi_1 + w^2\pi_2 + (ux - vw)x\varrho_0 - x^2\varrho_1 + wx\varrho_2] \\
&\widehat{\varrho_0} = y\pi_0 + z\varrho_0 \\
&\widehat{\varrho_1} = D^{-1}[(vy - uz)y\pi_0 + yz\pi_1 - y^2\pi_2 + (vy - uz)z\varrho_0 + z^2\varrho_1 - yz\varrho_2 - (vy - uz)\tau] \\
&\widehat{\varrho_2} = D^{-1}[(ux - vw)y\pi_0 - xy\pi_1 + wy\pi_2 + (ux - vw)z\varrho_0 - xz\varrho_1 + wz\varrho_2] \\
&\hat{\sigma} = \sigma\ ,\ \hat{\tau} = \tau
\end{aligned}$$

System 1: Transformations of Table 13.

Type (σ, τ) = (0 , 0)

Suppose that $\pi_0 \neq 0$. Then we will use the transformation $u = v = 0$, $w = -\varrho_0\pi_0^{-1}$, $x = 1$, $y = 1 - \varrho_0$, $z = \pi_0$ and get $\widehat{\pi_0} = 0$. Hence we may consider $\pi_0 = 0$ as an invariant. System 1 is then simplified into System 2. From this system we immediately find out that the (non)nullity of ϱ_0 is invariant.

First suppose that $\varrho_0 \neq 0$. Then the transformation

$$u = \varrho_1\varrho_0^{-1}\ ,\ v = \varrho_2\varrho_0^{-1}\ ,\ w = 1\ ,\ x = y = 0\ ,\ z = \varrho_0^{-1}$$

provides $\widehat{\varrho_0} = 1$ and $\widehat{\varrho_1} = \widehat{\varrho_2} = 0$.

Further work is trivial. Results are algebras A_1 and A_2 in the final Table 14.

$$\hat{a} = a + ub + vc\ ,\ \hat{b} = wb + xc\ ,\ \hat{c} = yb + zc$$

$D = wz - xy \neq 0 = x\varrho_0$

$\widehat{\pi_1} = D^{-1}[wz\pi_1 - wy\pi_2 + xz\varrho_1 - xy\varrho_2]$

$\widehat{\pi_2} = D^{-1}[-wx\pi_1 + w^2\pi_2 - x^2\varrho_1 + wx\varrho_2]$

$\widehat{\varrho_0} = z\varrho_0$

$\widehat{\varrho_1} = D^{-1}[yz\pi_1 - y^2\pi_2 + (vy - uz)z\varrho_0 + z^2\varrho_1 - yz\varrho_2]$

$\widehat{\varrho_2} = D^{-1}[-xy\pi_1 + wy\pi_2 - vwz\varrho_0 - xz\varrho_1 + wz\varrho_2]$

System 2.

The second possibility is $\varrho_0 = 0$. Parameters u and v no longer play any role and we may set $u = v = 0$.

In this case it holds:

$$[\widehat{\pi_1}\ \ \widehat{\pi_2}\ \ \widehat{\varrho_1}\ \ \widehat{\varrho_2}]^T = \boldsymbol{A}\,[\pi_1\ \ \pi_2\ \ \varrho_1\ \ \varrho_2]^T\ ,\ \det \boldsymbol{A} = 1.$$

Suppose that

$$\forall (r,s) \in \mathbb{F}^2 :\ rs\pi_1 - r^2\pi_2 + s^2\varrho_1 - rs\varrho_2 = 0.$$

This property is invariant:

$$rs\widehat{\pi_1} - r^2\widehat{\pi_2} + s^2\widehat{\varrho_1} - rs\widehat{\varrho_2} =$$
$$= D^{-1}[(rw+sy)(rx+sz)\pi_1 - (rw+sy)^2\pi_2 + (rx+sz)^2\varrho_1 - (rw+sy)(rx+sz)\varrho_2] = 0$$

Consequences are obvious:

$$\pi_2 = \widehat{\pi_2} = \varrho_1 = \widehat{\varrho_1} = \varrho_2 - \pi_1 = \widehat{\varrho_2} - \widehat{\pi_1} = 0,$$

and the result is the algebra A_3 in the final Table 14.

Now suppose the opposite, namely that there exist y, z such that

$$D := yz\pi_1 - y^2\pi_2 + z^2\varrho_1 - yz\varrho_2 \neq 0.$$

This expression is the main determinant of the linear system

$$wz - xy = D,\ wz\pi_1 - wy\pi_2 + xz\varrho_1 - xy\varrho_2 = 0$$

with unknowns w, x and solution

$$w = z\varrho_1 - y\varrho_2\ ,\quad x = y\pi_2 - z\pi_1.$$

The transformation with the above parameters w, x, y, z, D gives us $\widehat{\pi_1} = 0$ and $\widehat{\varrho_1} = 1$, which we take again as invariants. This creates two conditions in System 2:

$$wz - xy = z^2 - yz\varrho_2 - y^2\pi_2$$
$$wy\pi_2 + x(y\varrho_2 - z) = 0$$

This is a linear system with unknowns w, x with solution

$$w = z - y\varrho_2,\ x = y\pi_2 .$$

Then $\widehat{\pi_2} = \pi_2$ and $\widehat{\varrho_2} = \varrho_2$. So, we have found the two-parametric family of algebras A_4 in the final Table 14.

Type (σ, τ) = (1 , 0)

System 1 is simplified into System 3. Suppose that $\pi_0 \neq 0$. Then the transformation

$$w = \varrho_0\,,\quad x = -\pi_0\,,\quad y = \pi_0^{-1}\,,\quad z = 0$$

results in $\widehat{\pi_0} = 0$. Therefore, $\pi_0 = 0$ may be taken as invariant. Then the (non)nullity of ϱ_0 is invariant.

$$\hat{a} = a\,,\quad \hat{b} = wb + xc\,,\quad \hat{c} = yb + zc\,;\quad wz - xy = 1$$

$$\begin{aligned}
\widehat{\pi_0} &= w\pi_0 + x\varrho_0\\
\widehat{\pi_1} &= wz\pi_1 - wy\pi_2 + xz\varrho_1 - xy\varrho_2\\
\widehat{\pi_2} &= -wx\pi_1 + w^2\pi_2 - x^2\varrho_1 + wx\varrho_2\\
\widehat{\varrho_0} &= y\pi_0 + z\varrho_0\\
\widehat{\varrho_1} &= yz\pi_1 - y^2\pi_2 + z^2\varrho_1 - yz\varrho_2\\
\widehat{\varrho_2} &= -xy\pi_1 + wy\pi_2 - xz\varrho_1 + wz\varrho_2
\end{aligned}$$

System 3 .

If $\varrho_0 \neq 0$, there must be $x = 0$, and we achieve $\widehat{\varrho_0} = 1$ with the choice $z = \varrho_0^{-1}$. Then y remains the only variable parameter of the transformations. The discussion ends here in three ways:

- $\pi_2 \neq 0 \quad \Rightarrow$ with the choice $y = \pi_1\pi_2^{-1}$ we get $\widehat{\pi_1} = 0$ (algebras A_5 in Table 14).
- $\pi_2 = 0 \ \wedge\ \pi_1 \neq \varrho_2 \quad \Rightarrow$ by selecting $y = \varrho_1(\varrho_2 - \pi_1)^{-1}$ we get $\widehat{\varrho_1} = 0$ (algebras A_6 in Table 14).
- $\pi_2 = 0 \ \wedge\ \pi_1 = \varrho_2$ (algebras A_7 in Table 14).

The case $\varrho_0 = 0$ remains.

First, we´ll assume that the remaining four structural constants are such that

$$\forall (r, s) \in \mathbb{F}^2 :\ rs\pi_1 - r^2\pi_2 + s^2\varrho_1 - rs\varrho_2 = 0.$$

We claim that this is an invariant property:

$rs\widehat{\pi_1} - r^2\widehat{\pi_2} + s^2\widehat{\varrho_1} - rs\widehat{\varrho_2}$

$$= (rw+sy)(rx+sz)\pi_1 - (rw+sy)^2\pi_2 + (rx+sz)^2\varrho_1$$
$$-(rw+sy)(rx+sz)\varrho_2 = 0$$

The consequences of this property are obvious.

$$\pi_2 = \widehat{\pi_2} = \varrho_1 = \widehat{\varrho_1} = \varrho_2 - \pi_1 = \widehat{\varrho_2} - \widehat{\pi_1} = 0\,.$$

The result is a new family of non-isomorphic algebras (algebras A_8 in Table 14).

Now, suppose that there exist such y, z that

$$yz\pi_1 - y^2\pi_2 + z^2\varrho_1 - yz\varrho_2 \neq 0\,.$$

This expression is the main determinant of the following linear system:

$$wz - xy = 1\,,\ wz\pi_1 - wy\pi_2 + xz\varrho_1 - xy\varrho_2 = 0$$

with unknowns w, x. It means that there exists a transformation with the result $\widehat{\pi_1} = 0 \neq \widehat{\varrho_1}$. If we take these two relations for invariants then System 3 is significantly reduced. It includes equations:

$$wz - xy = 1\,,\ -wy\pi_2 + xz\varrho_1 - xy\varrho_2 = 0$$

with the solution

$$w = (z\varrho_1 - y\varrho_2)(-y^2\pi_2 + z^2\varrho_1 - yz\varrho_2)^{-1}\,,$$
$$x = y\pi_2(-y^2\pi_2 + z^2\varrho_1 - yz\varrho_2)^{-1}\,,$$

where y, z are such that $-y^2\pi_2 + z^2\varrho_1 - yz\varrho_2 \neq 0$. Then

$$\widehat{\pi_2} = \pi_2\varrho_1(-y^2\pi_2 + z^2\varrho_1 - yz\varrho_2)^{-1}\,,$$
$$\widehat{\varrho_1} = -y^2\pi_2 + z^2\varrho_1 - yz\varrho_2\,,\ \widehat{\varrho_2} = \varrho_2\,.$$

This is the family of algebras A_9 in Table 14.

Type (σ, τ) = (0 , 1)

System 1 simplifies into System 4.

$$\hat{a} = a + ub + vc,\quad \hat{b} = wb,\quad \hat{c} = yb + c,\quad w \neq 0$$

$$\widehat{\pi_0} = w\pi_0$$
$$\widehat{\pi_1} = (vy-u)\pi_0 + \pi_1 - y\pi_2 - v$$
$$\widehat{\pi_2} = w(-v\pi_0 + \pi_2)$$
$$\widehat{\varrho_0} = y\pi_0 + \varrho_0$$
$$\widehat{\varrho_1} = w^{-1}[(vy-u)y\pi_0 + y\pi_1 - y^2\pi_2 + (vy-u)\varrho_0 + \varrho_1 - y\varrho_2 + u - vy]$$
$$\widehat{\varrho_2} = -vy\pi_0 + y\pi_2 - v\varrho_0 + \varrho_2$$

System 4 .

With the selection $u = y = 0,\ v = \pi_1,\ w = 1$ we reach $\widehat{\pi_1} = 0$, which will be invariant.

We notice that (non)nullity of π_0 is an invariant. If $\pi_0 \neq 0$, we choose $u = -\pi_0^{-2}\pi_2$, $v = \pi_0^{-1}\pi_2$, $w = \pi_0^{-1}$, $y = -\pi_0^{-1}\varrho_0$ in order to get $\widehat{\pi_0} = 1$ and $\widehat{\pi_2} = \widehat{\varrho_0} = 0$. The result is the family of algebras A_{10} in Table 14.

Finally, let $\pi_0 = 0$. There are several possibilities, which we routinely consider one after the other.

- If $\pi_2 = 0$ and we can reach $\widehat{\varrho_1} = 0$ with some suitable transformation, we get the family of algebras A_{11} in Table 14.
- If $\pi_2 = 0$ and it is impossible to reach $\widehat{\varrho_1} = 0$, we get the algebra A_{12} in Table 14.
- If $\pi_2 \neq 0$ and $\varrho_0 \neq \pm 1$ then the transformation
$$w = \pi_2^{-1},\ y = -\pi_2^{-1}\varrho_2(\varrho_0 + 1)^{-1},$$
$$u = [\varrho_1 + \pi_2^{-1}\varrho_2^2(\varrho_0 + 1)^{-2}](\varrho_0 - 1)^{-1}$$
provides the family of algebras A_{13} in Table 14.
- We use a similar procedure for the case $\pi_2 \neq 0,\ \varrho_0 = 1,\ \mathrm{chr}\,\mathbb{F} \neq 2$ (algebras A_{14}),
- for the case $\pi_2 \neq 0,\ \varrho_0 = -1,\ \mathrm{chr}\,\mathbb{F} \neq 2$ (algebras A_{15}),
- and for the case $\pi_2 \neq 0,\ \varrho_0 = 1,\ \mathrm{chr}\,\mathbb{F} = 2$ (algebras A_{16}).

Theorem 4.1. *All 3-dimensional proper isopotent algebras have a multiplication table in the form of Table 13. The classification provides the following values of structural constants as described in Table 14.*

The classification is not complete for families A_9 and A_{16} because in these two cases the structure depends on individual properties of the ground field. Let us consider these two families in the case of a finite field and the fields of complex and real numbers.

The (non)nullities of κ and μ in $A_9(\kappa, \lambda, \mu)$ are invariants, therefore we discuss separately these possibilities: $\kappa = 0 = \mu$, $\kappa = 0 \neq \mu$, and $\kappa \neq 0$. For a finite field with characteristic 2 and for the field of complex numbers the results in Table 15 are rather obvious. For the field of real numbers it is easy to show that

$$\lambda > 0 \ \wedge\ 4\kappa\lambda \leq -\mu^2 \quad \Rightarrow \quad A_9(\kappa, \lambda, \mu) \cong A_9\left(-1, 1 + \tfrac{1}{4}\mu^2, \mu\right),$$

and every other $A_9(\kappa, \lambda, \mu)$ is isomorphic to $A_9(1, \kappa\lambda, \mu)$.

The case $A_9(\kappa, \lambda, \mu)$, even for a finite ground field of characteristic $\neq 2$, is still not transparent and depends on field properties.

	π_0	π_1	π_2	ϱ_0	ϱ_1	ϱ_2	σ	τ	
A_1	0	0	1	1	0	0	0	0	
$A_2(\lambda)$	0	λ	0	1	0	0	0	0	
$A_3(\lambda)$	0	λ	0	0	0	λ	0	0	
$A_4(\lambda,\mu)$	0	0	λ	0	1	μ	0	0	
$A_5(\kappa,\lambda,\mu)$	0	0	κ	1	λ	μ	1	0	$\kappa\neq 0$
$A_6(\lambda,\mu)$	0	λ	0	1	0	μ	1	0	$\lambda\neq\mu$
$A_7(\lambda,\mu)$	0	λ	0	1	μ	λ	1	0	
$A_8(\lambda)$	0	λ	0	0	0	λ	1	0	
$A_9(\kappa,\lambda,\mu)$	0	0	κ	0	λ	μ	1	0	$\lambda\neq 0$
$A_{10}(\lambda,\mu)$	1	0	0	0	λ	μ	0	1	
$A_{11}(\lambda,\mu)$	0	0	0	λ	0	μ	0	1	
A_{12}	0	0	0	1	1	0	0	1	
$A_{13}(\lambda)$	0	0	1	λ	0	0	0	1	$\lambda\neq\pm 1$
$A_{14}(\lambda)$	0	0	1	1	λ	0	0	1	chr $\mathbb{F}\neq 2$
$A_{15}(\lambda)$	0	0	1	-1	0	λ	0	1	chr $\mathbb{F}\neq 2$
$A_{16}(\lambda,\mu)$	0	0	1	1	λ	μ	0	1	chr $\mathbb{F}=2$

TABLE 14. Classification of 3-dimensional proper isopotent algebras.

➢ Two algebras $A_9(\kappa_1,\lambda_1,\mu)$ and $A_9(\kappa_2,\lambda_2,\mu)$ are isomorphic iff $\kappa_1\lambda_1=\kappa_2\lambda_2$ and there exists such $(x,y)\in\mathbb{F}^2$ that $\lambda_2=x^2\lambda_1-xy\mu-y^2\kappa_1$.

➢ Two algebras $A_{16}(\lambda_1,\mu)$ and $A_{16}(\lambda_2,\mu)$ are isomorphic iff there exists such $x\in\mathbb{F}$ that $\lambda_2=x^2+x\mu+\lambda_1$.

The number of algebras $A_{16}(\lambda,\mu)$ can be determined in the following way. If $\mu=0$ then $x=\sqrt{\lambda_1+\lambda_2}$, which always exists; hence we can fix $\lambda_1=0$ and this is one algebra. If $\mu\neq 0$ then $x=y\mu$ and the equation $y^2+y+(\lambda_1+\lambda_2)\mu^{-2}=0$ is solvable if the trace $\mathrm{Tr}((\lambda_1+\lambda_2)\mu^{-2})=0$, which is true in $n/2$, cases ; we can select μ in $n-1$ ways and λ in two ways. The family A_{16} therefore consists of three parts:

- $A_{16}(0,0)$;
- $A_{16}(\alpha,\mu)$, $\mu\neq 0$, α is a selected scalar with the property $\mathrm{Tr}(\alpha\mu^{-2})=0$;
- $A_{16}(\beta,\mu)$, $\mu\neq 0$, β is a selected scalar with the property, $\mathrm{Tr}(\beta\mu^{-2})=1$.

	GF $(n \neq 2^m)$	GF $(n = 2^m)$	$\mathbb{C}$	$\mathbb{R}$
A_1	1	1	$A_1 \dots A_8$	$A_1 \dots A_8$
$A_2(\lambda)$	n	n		
$A_3(\lambda)$	n	n		
$A_4(\lambda, \mu)$	n^2	n^2		
$A_5(\kappa, \lambda, \mu)$	$n^3 - n^2$	$n^3 - n^2$		
$A_6(\lambda, \mu)$	$n^2 - n$	$n^2 - n$		
$A_7(\lambda, \mu)$	n^2	n^2		
$A_8(\lambda)$	n	n		
$A_9(\kappa, \lambda, \mu)$	$A_9(0, 1, \mu)$ $A_9(0, \beta, 0)^{(\#1)}$ $A_9(\kappa, \lambda, \mu)^{(\#2)}$ $n + 1 + K$	$A_9(0, 1, \mu)$ $A_9(1, \lambda, \mu)^{(\#3)}$ $n + (n^2 - n)$	$A_9(0, 1, \mu)$ $A_9(1, \lambda, \mu)^{(\#3)}$	$A_9(0, 1, \mu)$ $A_9(0, -1, 0)$ $A_9(1, \lambda, \mu)^{(\#3)}$ $A_9(-1, 1 + \mu^2, 2\mu)$
$A_{10}(\lambda, \mu)$	n^2	n^2	$A_{10} \dots A_{15}$	$A_{10} \dots A_{15}$
$A_{11}(\lambda, \mu)$	n^2	n^2		
A_{12}	1	1		
$A_{13}(\lambda)$	$n - 2$	$n - 1$		
$A_{14}(\lambda)$	n	0		
$A_{15}(\lambda)$	n	0		
$A_{16}(\lambda, \mu)$	0	$A_{16}(0, \mu)$ $A_{16}(\beta, \mu)^{(\#4)}$ $n + (n - 1)$	$\nexists$	$\nexists$

Table 15. Classifications over special fields.

(#1) β is a selected non-square.

(#2) $\kappa \neq 0$; some of these algebras may still be isomorphic; the number of non-isomorphic types is here denoted by $K : n \leq K \leq n(n-1)^2$.

(#3) $\lambda \neq 0$.

(#4) $\mu \neq 0$, β is a selected scalar with the property $\mathrm{Tr}(\beta\mu^{-2}) = 1$.

If we choose $\alpha = 0$, then the first two classes merge in one: $A_{16}(0, \mu)$, with optional μ.

From Table 15, we find the number of non-isomorphic types of algebras:
$\mathrm{chr}\,\mathrm{GF}\ (n) \neq 2 \quad \Rightarrow \quad n^3 + 4n^2 + 6n + 1 + K,$

$\operatorname{chr} \mathrm{GF}\ (n)\ = 2 \quad \Rightarrow \quad n^3 + 5n^2 + 5n.$

5. Isoproduct algebras

Lemma 5.1. *Let* $(\mathcal{A}, \cdot)$ *be an* $\mathbb{F}$*-algebra. Consider the following three statements:*

(1) $\forall (a,b) \in \mathcal{A}^2\ :\ ab \in \operatorname{lin}\{a,b\}$.
(2) $\forall (a,b) \in \mathcal{A}^2\ :\ \operatorname{alg}\{a,b\} = \operatorname{lin}\{a,b\}$.
(3) $\forall (a,b) \in \mathcal{A}^2\ :\ \dim \operatorname{alg}\ \{a,b\} \leq 2$.

Then, $(1) \Leftrightarrow (2) \Rightarrow\ (3)$. *If* $\dim \mathcal{A} \neq 2$ *then all three statements are equivalent.*

Proof. Non-trivial is only the proof of $(2) \Leftarrow (3)$ in the case of $\dim \mathcal{A} > 2$. If a, b are linearly independent, then $\dim \operatorname{lin}\ \{a,b\} = 2$, hence $\operatorname{alg}\{a,b\}= \operatorname{lin}\{a,b\}$.

Now let us take some $a \neq 0$ and prove that $\operatorname{alg}\{a\} = \mathbb{F}\{a\}$. Also let b, c be such that the triple (a,b,c) is linearly independent.

$$\begin{gathered} ab = \alpha a + \beta b\,,\ ac = \gamma a + \delta c, \\ a^2 + \alpha a + \beta b = a(a+b) = \varepsilon a + \zeta(a+b) \ \Rightarrow\ a^2 = (\dots)a + (\zeta - \beta)b, \\ a^2 + \gamma a + \delta c = a(a+c) = \eta a + \vartheta(a+c) \ \Rightarrow\ a^2 = (\dots)a + (\dots)c, \\ 0 = a^2 - a^2 = (\dots)a + (\zeta-\beta)b + (\dots)c. \end{gathered}$$

Hence $\zeta - \beta = 0$ and $a^2 = (\dots)a.$ □

Definition 5.2. An algebra $(\mathcal{A}, \cdot)$ over a field $\mathbb{F}$ is **isoproduct algebra** if

$$\forall (a,b) \in \mathcal{A}^2,\ ab \in \operatorname{lin}\{\mathrm{a,\ b}\}.$$

If there exist such linear functionals, U, V that

$$\forall (a,b) \in \mathcal{A}^2,\ ab = U(b)a + V(a)b,$$

we will say that, $(\mathcal{A}, \cdot)$ is **proper isoproduct algebra**. Otherwise, it is **improper**.

According to Lemma 5.1, in case of $\dim\ \mathcal{A} \neq 2$, this statement

$$\forall (a,b) \in \mathcal{A}^2\ :\ \dim\ \operatorname{alg}\{a,b\} \leq 2$$

may be an equivalent definition of isoproduct algebra.

Every isoproduct algebra is obviously also an isopotent algebra. Every proper isoproduct algebra is either proper isopotent algebra or zeropotent algebra.

Every isopotent algebra of $\dim \mathcal{A} \leq 2$ is isoproduct algebra.

In every proper isoproduct algebra the functionals U, V are uniquely determined, except of course in case of $\dim \mathcal{A} = 1$.

In any isoproduct algebra the following holds:

$$\forall\, \mathcal{M} \subset \mathcal{A}\ :\ \operatorname{alg} \mathcal{M}\ =\ \operatorname{lin} \mathcal{M}\,.$$

The proof is trivial.

Suppose that $(\mathcal{D},\cdot)$ is an isopotent algebra. Then its plus-algebra $(\mathcal{D},\circ)$ with the multiplication $(p,q)\mapsto p\circ q := p\cdot q + q\cdot p$ is isoproduct algebra (Proposition 1.5, (2)).

Lemma 5.3. ([1, Lemma 7]) *Let $\mathcal{L}$ be a linear space over a field $\mathbb{F}$ and $\dim\mathcal{L}\geq 2$ (with the exception of $\dim\mathcal{L}>2$ if $\mathbb{F}=\mathbb{Z}_2$). If $M:\mathcal{L}^2\to\mathcal{L}$ is a bilinear map with the property:*

$$\forall (p,q)\in\mathcal{L}^2 \;\; \exists(\alpha,\beta)\in\mathbb{F}^2 : \;\; M(p,q)=\alpha p+\beta q,$$

then there exist uniquely determined linear functionals $U,V:\mathcal{L}\to\mathbb{F}$ that:

$$\forall (p,q)\in\mathcal{L}^2 : \;\; M(p,q)=U(q)p+V(p)q.$$

Using this lemma, we immediately find the following theorem. Nothing more needs to be said about improper isoproduct algebras.

Theorem 5.4. *Suppose that $(\mathcal{A},\cdot)$ is an improper isoproduct algebra over a field $\mathbb{F}$. Then $\mathbb{F}=\mathbb{Z}_2$ and $(\mathcal{A},\cdot)$ is one of five two-dimensional algebras from Table 1. For these algebras, the functionals U and V do not exist even as non-linear functionals.*

In what follows, we will show some properties of the proper isoproduct algebras. Suppose that $(\mathcal{A},\cdot)$ ($\dim\mathcal{A}\geq 2$) is a proper isoproduct $\mathbb{F}$-algebra with multiplication

$$pq=U(q)p+V(p)q$$

and additional bilinear functional

$$W(p,q):=U(q)V(p).$$

The proofs of the next propositions A – G are straightforward.

A. The functional W is associative in the following sense:

$$\forall (p,q,r)\in\mathcal{A}^3 : W(pq,r)=W(p,qr)\,.$$

B. For the associator $[p,q,r]=(pq)r-p(qr)$ we have

$$\forall (p,q,r)\in\mathcal{A}^3 \; : \; [p,q,r]=W(p,q)r-W(q,r)p\,.$$

C. The following eight statements are equivalent:

- $(\mathcal{A},\cdot)$ is flexible.
- $(\mathcal{A},\cdot)$ is non-commutative Jordan.
- $(\mathcal{A},\cdot)$ is alternative.
- $(\mathcal{A},\cdot)$ is associative.

- ➢ $V = \lambda U$ or $U = \mu V$ for some constant λ or μ.
- ➢ $U = 0$ or $V = 0$.
- ➢ $\forall (p,q) \in \mathcal{A}^2 \,:\, W(p,q) = W(q,p)$.
- ➢ $W = 0$.

D. The following three statements are equivalent:

- ➢ $(\mathcal{A},\cdot)$ is commutative.
- ➢ $(\mathcal{A},\cdot)$ is Jordan.
- ➢ $V = U$.

E. Jacobian:

$$\mathrm{J}(p,q,r) = K(q,r)p + K(r,p)q + K(p,q)r,$$

where $K(p,q) = K(q,p) := [U(p) + V(p)][U(q) + V(q)]$.

F. The following three statements are equivalent:

- ➢ $(\mathcal{A},\cdot)$ is zeropotent.
- ➢ $(\mathcal{A},\cdot)$ is Lie.
- ➢ $V = -U$.

ω	φ	ψ
0	0	0
0	0	1
1	0	1
1	$\varphi \in \mathbb{F}$	0

TABLE 16. Theorem 5.5.

The following theorem describes the complete classification of the isoproduct algebras. The proof is simple.

Theorem 5.5. *Any proper isoproduct algebra* $(\mathcal{A},\cdot)$ *with functionals* U, V *and of* $\dim\mathcal{A} \geq 2$ *has the following structure:*

$$\mathcal{A} = \mathbb{F}\{u\} \oplus \mathbb{F}\{v\} \oplus \mathcal{B},$$

where $\mathcal{B}$ *is a subalgebra with zero multiplication,* (u,v) *linearly independent and*

$$U(\mathbb{F}\{v\} \oplus \mathcal{B}) = V(\mathcal{B}) = \{0\}, \quad \omega := U(u), \ \varphi := V(u), \quad \psi := V(v).$$

The explicit formula for multiplication (according to the direct sum above):

$$(\alpha u + \beta v + a)(\gamma u + \delta v + b) =$$
$$\gamma(\alpha\omega + \alpha\varphi + \beta\psi)u + (\beta\gamma\omega + \alpha\delta\varphi + \beta\delta\psi)v + [\gamma\omega a + (\alpha\varphi + \beta\psi)b].$$

The parameters φ, ψ, ω *have values as described in Table 16 and these values form pairwise non-isomorphic algebras.*

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Dedicated to Dr. Marjan Jerman, who passed away in 2020 at the age of 48.

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