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Research Article

On Existence and Uniqueness of Some Fractional Order Integro-Differential Equation

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Abstract

In this study, a sufficient condition for the existence and uniqueness of some fractional order Integral-Differential equations is obtained. Therefore, the fixed point method is used to solve the differential equation problem involving nonlinear degree integrals. In addition, the results found is supported by examples.

Keywords: Initial value problem, Fractional order Integro-Differential equation, Riemann-Liouville derivative, Riemann-Liouville integral, Fixed point.

Bazı Kesir Mertebe İntegro-Diferansiyel Denklemlerin Varlık ve Tekliği Üzerine

Öz

Bu çalışmada, bazı kesirli mertebeden İntegral-Diferansiyel denklemlerin varlığı ve tekliği için yeterli koşul elde edilmiştir. Böylece, doğrusal olmayan dereceli integralleri içeren diferansiyel denklem problemini çözmek için sabit nokta yöntemi kullanıldı. Ayrıca, bulunan sonuçlar örneklerle desteklenmiştir.

Anahtar Kelimeler: Başlangıç değer problemi, Kesir mertebe İntegro-Diferansiyel denklemler, Riemann-Liouville türev, Riemann-Liouville integral, Sabit nokta.

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1. Introduction

Fractional analysis is a branch of mathematics that studies derivatives and integrals of real or complex order. Differential equations involving non-integer derivatives are used to model various physical phenomena. Therefore, in addition to its applications in mathematics, it is also used in the application of many branches of science such as physics, engineering, biology and finance (see [1]- [5]). Some of the most comprehensive studies for fractional derivatives and integrals (see [6]- [7]).

The most interesting feature of fractional calculus is that it mostly contains operators. This allows a researcher to choose the most appropriate operator to describe the dynamics in a real-world problem. It has led them to discover new fractional operators, with the need to demonstrate what other fractional operators require, used to find better results each time to natural phenomena (see [16]- [18]). There are other different types of novel fractional derivatives as well (see [19]- [20]).

Also, fixed point theory is one of the most powerful and effective tools of nonlinear analysis, especially differential equations, integral equations, partial differential equations. It has a wide application area in many branches of mathematics. Fixed point theory has very fruitful applications in initial value problems, boundary value problems, and approximation problems as well as in eigenvalue problems. With the help of some Fixed Point Theorems, the existence and uniqueness of the solutions of fractional differential equations in initial value problems can be demonstrated under certain conditions. However, a method by which exact solutions of fractional differential equations can be found analytically is often not available. For this reason, a lot of extensive work has been done on the solution methods of fractional differential equations. (see [1]- [14]). In parallel with all these studies, studies have been carried out on the existence and uniqueness of the solution of fractional differential equations. Recently, some studies on this subject (see [8]-[13] and references therein).

In this study, consider the initial value problems for the following Integro-Differential equations

$$\begin{cases} \frac{du(s)}{ds} + I_{a^+}^{\alpha}\, u(s) = f\big(s\,, u(s)\big)\,, \\ \quad 0 < s < A \;,\; 0 < \alpha < 1\,, \\ \quad u(0) = 0\,, \end{cases} \tag{1}$$

where $f \in (C[0,A] \times \mathbb{R}, \mathbb{R})$,

$$\begin{cases} \frac{d^2u(s)}{ds^2} + I_{a^+}^{\alpha}\, u(s) = f\big(s\,, u(s), u'(s)\big)\,, \\ \quad 0 < s < A \;,\; 0 < \alpha < 1\,, \\ \quad u(0) = 0\,,\; u'(0) = 0\,. \end{cases} \tag{2}$$

where $f \in (C[0,A] \times \mathbb{R}^2, \mathbb{R})$.

Now, we the plan of this paper is as follows. In chapter 2, we give all the background material used in this paper. In chapter 3, we establish theorems on existence and uniqueness for initial value problems given by (1) and (2). Finally, the results found will be supported by examples.

2. Prelimneries

In this section, we present notations, definitions, and preliminary facts which are used throughout this paper.

Definition 2.1 Let $u(t) \in C([a,b])$ and $a < s < b$, $\alpha \in (-\infty, \infty)$. Riemann-Liouville fractional integral of order α

$$I_{a^+}^{\alpha} u(s) := \frac{1}{\Gamma(\alpha)} \int_a^s \frac{u(t)}{(s-t)^{1-\alpha}} dt.$$

The same definition for $\alpha \in (0,1)$ can be expressed as

$$D_{a^+}^{\alpha} u(s) := \frac{1}{\Gamma(1-\alpha)} \frac{d}{ds} \int_a^s \frac{u(t)}{(s-t)^{\alpha}} dt$$

Riemann-Liouville fractional derivative of order α. Here,

$$\Gamma(\alpha) = \int_0^{\infty} t^{\alpha-1} e^{-t} dt \,, \quad (\alpha > 0)$$

(see [7]).

Definition 2.2 Let $X \neq \emptyset$ be a set and a mapping $T: X \to X$ of set X into itself is an element $u \in X$ which is mapped onto itself. That is, $Tu = u$, the image Tu coincides with u (see [15]).

Let us give the definition of the contraction transformation, which plays an important role in determining the sufficient conditions for the existence and uniqueness of the fixed point (see [15]).

Definition 2.3 Let a mapping $T: X \to X$. If there is a positive real number $\delta < 1$ such that for all $u, v \in X$,

$$d(Tu, Tv) \leq \delta d(u, v) \,,$$

then T is called a contraction on X.

Now, let's state the Banach Fixed Point Theorem (see [15]).

The Banach fixed point theorem is an existence and uniqueness theorem for fixed points of certain mappings and also provides a better approximation to the solution of equation

$$u = Tu.$$

Here, we choose an arbitrary starting point $u_0 \in X$, determine a sequence $\{u_n\}_{n=0}^{\infty}$ defined by the relation

$$u_n = Tu_{n-1} \,, \quad n \in \mathbb{N}. \tag{3}$$

This process, which is used in many areas of applied mathematics, is called iteration (see [15]).

Theorem 2.4 If (X, d) is a complete metric space and $T: X \to X$ is a contraction mapping,

i. T has one and only one fixed point $u \in X$.
ii. For any $u_0 \in X$, iteration sequence $(T^n u_0)$ (ie iteration sequence (u_n) defined by $u_n = Tu_{n-1}$ for all $n \in \mathbb{N}$) converges to unique fixed point of T.

With the assumptions of Theorem 2.4, the sequence $\{u_n\}_{n=0}^{\infty}$ defined by recursive formula (3) with arbitrary $u_0 \in X$ converges to the unique fixed point u of the mapping T. Error estimate is a prior estimate

$$d(u_n, u) \leq \frac{\delta^n}{1-\delta} d(u_0, u_1)\,, \quad n \in \mathbb{N}$$

and a posteriori estimate

$$d(u_n, u) \leq \frac{\delta}{1-\delta} d(u_{n-1}, u_n)\,, \quad n \in \mathbb{N}$$

(see [15]).

Definition 2.5 $C[0, A] = (C[0, A], d)$ is the complete space of all continuous functions defined on the interval $[0, A]$ with the metric d defined by

$$d(u, v) = \max_{s \in [0,A]} |u(s) - v(s)| \tag{4}$$

$C^{(1)}[0, A] = (C^{(1)}[0, A], d)$ is the complete space defined on the interval $[0, A]$ with the metric d defined by

$$d(u, v) = \max_{s \in [0,A]} |u(s) - v(s)| + \max_{s \in [0,A]} |u'(s) - v'(s)| \tag{5}$$

(see [13]).

3. Main Theorem and Proof

We now consider the following initial value problem for Integro-Differential equation (1). We will establish the existence and uniqueness theorem by obtaining the sufficient condition for the existence and uniqueness of this problem.

Theorem 3.1 Let f function is continuous and bounded on the region

$$D = \{(s, u) : s \in [0, A]\,, |u - u_0| < \infty\} \subseteq \mathbb{R}^2.$$

Also, f satisfies a Lipschitz condition on D with respect to its second arguments, i.e, there is a pozitive constant L such that for arbitrary $(s, u), (s, v) \in D$

$$|f(s, u) - f(s, v)| \leq L|u - v| \tag{6}$$

is valid. Moreover, let

$$h(\alpha, A, L) = \frac{A^{\alpha+1}}{\Gamma(\alpha+2)} + LA$$

and suppose that

$$h(\alpha, A, L) < 1\,. \tag{7}$$

Then, initial value problem (1) has a unique solution $u \in C[0, A]$.

Proof. By integrating both sides of Integro-Differential equation (1), we obtain integral equation (8)

$$\int_0^s \frac{du(q)}{dq}\,dq + \int_0^s I_{0^+}^{\alpha} u(q)dq = \int_0^s f(q, u(q))dq$$

$$u(q)|_0^s + \int_0^s \left(\frac{1}{\Gamma(\alpha)} \int_0^q (q-t)^{\alpha-1} u(t)dt\right) dq = \int_0^s f(q, u(q))dq$$

$$u(s) = -\frac{1}{\Gamma(\alpha)} \int_0^s \int_0^q (q-t)^{\alpha-1} u(t)dtdq + \int_0^s f\big(q, u(q)\big)dq. \tag{8}$$

u is the limit of the iterative sequence $\{u_n\}_{n=0}^{\infty}$ defined by the recursive Picard iteration formula

$$u_n(s) = -\frac{1}{\Gamma(\alpha)} \int_0^s \int_0^q (q-t)^{\alpha-1} u_{n-1}(t)dtdq + \int_0^s f\big(q, u_{n-1}(q)\big)dq, \tag{9}$$

where $u_0(s)$ is an arbitrary continuous function. There for $h < 1$ error bounds

$$d(u_n\,, u) \le \frac{h^n}{1-h} d(u_0\,, u_1)\ ,$$

$$d(u_n\,, u) \le \frac{h}{1-h} d\big(u_{n-1}, u_n\big)\ ,\ \ n \in \mathbb{N}\,.$$

We see that initial value problem (1) can be written in the equivalent integral form (8), which is in the form $u = Tu$, where $T: C[0, A] \to C[0, A]$ is an operator defined by

$$Tu(s) = -\frac{1}{\Gamma(\alpha)} \int_0^s \int_0^q (q-t)^{\alpha-1} u(t)dtdq + \int_0^s f\big(q, u(q)\big)dq\ , \tag{10}$$

where f is continuous function on D. Since f is bounded there exists a constant $|f(s, u)| \le k$ such that $k > 0$ for all $s \in [0, A]$ and $(s, u) \in D$. Also, Under assumptions of Theorem3.1 and equation (4) we have

$$|Tu(s)| = \left|-\frac{1}{\Gamma(\alpha)} \int_0^s \int_0^q (q-t)^{\alpha-1} u(t)dt\,dq\ + \int_0^s f(q, u(q))dq\right|$$

$$\le \frac{1}{\Gamma(\alpha)} \left|\int_0^s \int_0^q (q-t)^{\alpha-1} u(t)dtdq\right| + \left|\int_0^s f(q, u(q))dq\right|$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^s\left(\int_0^q |(q-t)|^{\alpha-1}|u(t)|dt\right)dq + \int_0^s |f(q,u(q))|dq$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^s\left(\int_0^q |(q-t)|^{\alpha-1} \max_{t\in[0,A]}|u(t)|\ dt\right)dq + \int_0^s kdq$$

$$\leq \frac{\max_{t\in[0,A]}|u(t)|}{\Gamma(\alpha)} \int_0^s \frac{q^\alpha}{\alpha}\ dq + ks$$

$$= \frac{\max_{t\in[0,A]}|u(t)|}{\Gamma(\alpha)\ \alpha}\ \frac{q^{\alpha+1}}{\alpha+1}\ \Big|_0^s + ks$$

$$= \frac{\max_{t\in[0,A]}|u(t)|}{\Gamma(\alpha+1)(\alpha+1)}\ s^{\alpha+1} + ks$$

$$= \frac{\max_{t\in[0,A]}|u(t)|}{\Gamma(\alpha+2)} s^{\alpha+1} + ks.$$

Thus, $Tu \in C[0,A]$ if $u \in C[0,A]$; i.e., T maps the set $C[0,A]$ itself. Let it be shown that T is a contraction map on $C[0,A]$. If the hypotheses of Theorem3.1 are used and necessary adjustments are made, we have

$$|Tu(s) - Tv(s)| = \Bigg|-\frac{1}{\Gamma(\alpha)}\int_0^s\int_0^q (q-t)^{\alpha-1}\, u(t)dtdq + \int_0^s f\big(q,u(q)\big)dq$$

$$+\frac{1}{\Gamma(\alpha)}\int_0^s\int_0^q (q-t)^{\alpha-1} v(t)dtdq - \int_0^s f\big(q,v(q)\big)dq\Bigg|$$

$$\leq \frac{1}{\Gamma(\alpha)}\int_0^s\int_0^q |(q-t)|^{\alpha-1}|u(t)-v(t)|\ dtdq$$

$$+\int_0^s \big|f\big(q,u(q)\big) - f\big(q,v(q)\big)\big|dq$$

$$\leq \frac{1}{\Gamma(\alpha)}\ \max_{t\in[0,A]}|u(t)-v(t)| \int_0^s\int_0^q |(q-t)|^{\alpha-1}dtdq$$

$$+\int_0^s L|u(q)-v(q)|dq$$

$$\leq \frac{1}{\Gamma(\alpha)}\ \max_{t\in[0,A]}|u(t)-v(t)| \int_0^s\int_0^q |(q-t)|^{\alpha-1}dtdq$$

$$+\int_0^s L \max_{t\in[0,A]} |u(q)-v(q)| dq$$

$$\leq \frac{1}{\Gamma(\alpha)} \max_{t\in[0,A]} |u(t)-v(t)| \int_0^s \frac{q^\alpha}{\alpha}\, dq$$

$$+ \max_{q\in[0,A]} |u(q)-v(q)|\ Ls$$

$$\leq \frac{s^{\alpha+1}}{\Gamma(\alpha+2)} d(u,v) + Ls\, d(u,v)$$

$$= \left(\frac{s^{\alpha+1}}{\Gamma(\alpha+2)} + Ls\right) d(u,v)\,.$$

taking the maximum of both sides in this last inequality we have

$$d(Tu,Tv) \leq \max_{s\in[0,A]} \left(\frac{s^{\alpha+1}}{\Gamma(\alpha+2)} + Ls\right) d(u,v)$$

$$= \left(\frac{A^{\alpha+1}}{\Gamma(\alpha+2)} + LA\right)\ d(u,v)$$

$$= h(\alpha,A,L) d(u,v).$$

From (7) we see that $h(\alpha,A,L) < 1$, so

$$d(Tu,Tv) \leq hd(u,v).$$

Thus, T is a contraction on $C[0,A]$. Therefore, T has a unique fixed point $u \in C[0,A]$, that is, a continuous function on $[0,A]$ satisfying $u = Tu$. By equation (9)

$$u(s) = -\frac{1}{\Gamma(\alpha)} \int_0^s \int_0^q (q-t)^{\alpha-1} u(t) dt dq + \int_0^s f\big(q,u(q)\big) dq.$$

Theorem 3.2 Let the function f is continuous on

$$D = \{(s,u,v):\ s \in [0,A]\,, |u-u_0| < \infty, |v-v_0| < \infty\} \subseteq \mathbb{R}^3\,,$$

and there exists a constant $k > 0$ such that $|f(s,u,v)| \leq k$ for each $s \in [0,A]$. Also, f satisfies a Lipschitz condition on D with respect to its second and third arguments. Thus, for arbitrary $(s,u,z),(s,v,w) \in D$ there is a pozitive constant L such that

$$|f(s,u,z) - f(s,v,w)| \leq L(|u - v| + |z-w|). \tag{11}$$

is valid. Therefore, let

$$h(\alpha,A,L) = \frac{A^{\alpha+2}}{\Gamma(\alpha+2)} + L\frac{A^2}{2}\,,$$

and suppose that

$$h(\alpha,A,L) < 1. \tag{12}$$

Then, initial value problem (2) has a unique solution $u \in C^{(1)}[0,A]$.

Proof. By integrating both sides of Integro-Differential equation (2), we get integral equation (13)

$$\int_0^r \frac{d^2u(q)}{dq^2}\, dq + \int_0^r I_{0^+}^{\alpha} u(q)dq = \int_0^r f\big(q, u(q), u'(q)\big)dq$$

$$\frac{du(q)}{dq}\Big|_0^r + \int_0^r \left(\frac{1}{\Gamma(\alpha)}\int_0^q (q-t)^{\alpha-1}u(t)dt\right)dq = \int_0^r f(q,u(q),u'(q))dq$$

$$\frac{du(r)}{dr} + \int_0^r \left(\frac{1}{\Gamma(\alpha)}\int_0^q (q-t)^{\alpha-1}u(t)dt\right)dq = \int_0^r f(q,u(q),u'(q))dq$$

$$\int_0^s \frac{du(r)}{dr}dr + \int_0^s \int_0^r \left(\frac{1}{\Gamma(\alpha)}\int_0^q (q-t)^{\alpha-1}u(t)dt\right)\, dq = \int_0^s \int_0^r f(q,u(q),u'(q))dq$$

$$u(r)\Big|_0^s + \int_0^s \int_0^r \left(\frac{1}{\Gamma(\alpha)}\int_0^q (q-t)^{\alpha-1}u(t)dt\right)dq = \int_0^s \int_0^r f(q,u(q),u'(q))dq$$

$$u(s) = -\frac{1}{\Gamma(\alpha)}\int_0^s \int_0^r \left(\int_0^q (q-t)^{\alpha-1}u(t)dt\right)dqdr + \int_0^s \int_0^r f\big(q\,, u(q), u'(q)\big)dqdr \quad (13)$$

Actually, the u function is the limit of the iterative sequence $\{u_n\}_{n=0}^{\infty}$ defined by the recursive Picard iteration formula

$$u_n(s) = -\frac{1}{\Gamma(\alpha)}\int_0^s \int_0^r \left(\int_0^q (q-t)^{\alpha-1}u_{n-1}(t)dt\right)dqdr$$

$$+\int_0^s \int_0^r f\big(q\,, u_{n-1}(q), u_{n-1}'(q)\big)dqdr. \quad (14)$$

Error bounds are

$$d(u_n\,, u) \le \frac{h^n}{1-h} d(u_0\,, u_1)$$

$$d(u_n\,, u) \le \frac{h}{1-h} d(u_{n-1}\,, u_n) \quad , \quad n \in \mathbb{N}$$

where $u_0(s)$ is an arbitrary continuous function and $h < 1$. We see that equivalent to problem (2), the integral equation (13) can be written. Since $u = Tu$, we $T: C^{(1)}[0,A] \to C^{(1)}[0,A]$ is an operator defined by

$$Tu(s) = -\frac{1}{\Gamma(\alpha)}\int_0^s \int_0^r \left(\int_0^q (q-t)^{\alpha-1}u(t)dt\right)dqdr + \int_0^s \int_0^r f\big(q\,, u(q), u'(q)\big)dqdr \quad (15)$$

where f is a continuous function on D. Under assumptions of Theorem3.2 and equation (5), we obtain

$$|Tu(s)|$$

$$= \left| -\frac{1}{\Gamma(\alpha)} \int_0^s \int_0^r \left(\int_0^q (q-t)^{\alpha-1} u(t) dt \right) dq dr + \int_0^s \int_0^r f(q, u(q), u'(q)) dq dr \right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \left| \int_0^s \int_0^r \left(\int_0^q (q-t)^{\alpha-1} u(t) dt \right) dq dr \right| + \left| \int_0^s \int_0^r f(q, u(q), u'(q)) dq\, dr \right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^s \int_0^r \left(\int_0^q |(q-t)|^{\alpha-1} |u(t)| dt \right) dq dr$$

$$+ \int_0^s \int_0^r \left| f\big(q, u(q), u'(q)\big) \right| dq\, dr$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^s \int_0^r \left(\int_0^q |(q-t)|^{\alpha-1} \max_{t \in [0,A]} |u(t)|\, dt \right) dq dr + \int_0^s \int_0^r k\, dq\, dr$$

$$\leq \frac{\max_{t \in [0,A]} |u(t)|}{\Gamma(\alpha)} \int_0^s \int_0^r \frac{q^\alpha}{\alpha}\, dq\, dr + \int_0^s kr dr$$

$$= \frac{\max_{t \in [0,A]} |u(t)|}{\Gamma(\alpha)\alpha} \left(\frac{q^{\alpha+1}}{\alpha+1} \right) \Big|_0^s + k \left(\frac{r^2}{2} \Big|_0^s \right)$$

$$= \frac{\max_{t \in [0,A]} |u(t)|}{\Gamma(\alpha+1)(\alpha+1)} s^{\alpha+1} + k \frac{s^2}{2}$$

$$= \frac{\max_{t \in [0,A]} |u(t)|}{\Gamma(\alpha+2)} s^{\alpha+1} + \frac{ks^2}{2}.$$

Thus, $Tu \in C^{(1)}[0,A]$ if $u \in C^{(1)}[0,A]$; i.e., T maps the set $C^{(1)}[0,A]$ itself. Let it be shown that T is a contraction map on $C^{(1)}[0,A]$. If the hypotheses of Theorem3.2 are used and necessary adjustments are made, we have

$$|Tu(s) - Tv(s)|$$

$$= \left| -\frac{1}{\Gamma(\alpha)} \int_0^s \int_0^r \left(\int_0^q (q-t)^{\alpha-1} u(t) dt \right) dq dr + \int_0^s \int_0^r f\big(q, u(q), u'(q)\big) dq dr \right.$$

$$\left. -\frac{1}{\Gamma(\alpha)} \int_0^s \int_0^r \left(\int_0^q (q-t)^{\alpha-1} v(t) dt \right) dq dr - \int_0^s \int_0^r f\big(q, v(q), v'(q)\big) dq dr \right|$$

$$\leq \frac{1}{\Gamma(\alpha)} \int_0^s \int_0^r \left(\int_0^q |(q-t)|^{\alpha-1} |u(t) - v(t)|\, dt \right) dq dr$$

$$+\int_0^s\int_0^r \left|f\big(q,u(q),u'(q)\big)-f\big(q,v(q),v'(q)\big)\right|\,dqdr$$

$$\leq \frac{1}{\Gamma(\alpha)}\max_{t\in[0,A]}|u(t)-v(t)|\int_0^s\int_0^r\int_0^q |(q-t)|^{\alpha-1}dtdq$$

$$+\int_0^s\int_0^r L(|u(q)-v(q)|+|u'(q)-v'(q)|)dqdr$$

$$\leq \frac{1}{\Gamma(\alpha)}\max_{t\in[0,A]}|u(t)-v(t)|\int_0^s\int_0^r\left(\int_0^q |(q-t)|^{\alpha-1}dt\right)dqdr$$

$$+\int_0^s\int_0^r L\left(\max_{q\in[0,A]}|u(q)-v(q)|+\max_{q\in[0,A]}|u'(q)-v'(q)|\right)dqdr$$

$$\leq \frac{1}{\Gamma(\alpha)}\max_{t\in[0,A]}|u(t)-v(t)|\frac{s^{\alpha+1}}{\alpha(\alpha+1)}$$

$$+\left(\max_{q\in[0,A]}|u(q)-v(q)|+\max_{q\in[0,A]}|u'(q)-v'(q)|\right)L\frac{s^2}{2}$$

$$\leq \frac{s^{\alpha+2}}{\Gamma(\alpha+2)}d(u,v)+L\frac{s^2}{2}\,d(u,v)$$

$$=\left(\frac{s^{\alpha+2}}{\Gamma(\alpha+2)}+L\frac{s^2}{2}\right)d(u,v).$$

If the maximum is taken from both sides in this last inequality we have

$$d(Tu,Tv)\leq \max_{s\in[0,A]}\left(\frac{s^{\alpha+2}}{\Gamma(\alpha+2)}+L\frac{s^2}{2}\right)d(u,v)$$

$$=\left(\frac{A^{\alpha+2}}{\Gamma(\alpha+2)}+L\frac{A^2}{2}\right)d(u,v)$$

$$=h(\alpha,A,L)d(u,v).$$

From (12) we see that $h(\alpha,A,L)<1$, so

$$d(Tu,Tv)\leq hd(u,v).$$

Thus, T is a contraction on $C^{(1)}[0,A]$. Therefore, T has a unique fixed point $u\in C^{(1)}[0,A]$, that is, a continuous function on $[0,A]$ satisfying $u=Tu$. By equation (14)

$$u(s)=-\frac{1}{\Gamma(\alpha)}\int_0^s\int_0^r\left(\int_0^q(q-t)^{\alpha-1}u(t)dt\right)dqdr+\int_0^s\int_0^r f\big(q,u(q),u'(q)\big)dqdr.$$

Example 3.3 Solve initial value problem for Integro-Differential equation

$$\begin{cases} \frac{d^2u(s)}{ds^2} + I_{0^+}^{\frac{1}{2}}u(s) = 2 + \frac{1}{12\sqrt{\pi}}s^2u(s) + u'(s), \\ 0 < s < A, \\ u(0) = 0 \ , \ u'(0) = 0\,. \end{cases} \tag{16}$$

by the iteration method.

Solution: Firstly, since $f(s, u(s), u'(s)) = 2 + \frac{1}{12\sqrt{\pi}}s^2u(s) + u'(s), 0 < s < 1 \ , \ \alpha = \frac{1}{2}$ we have that $L = \frac{1}{12\sqrt{\pi}} > 0$. Thus, we have

$$h(\alpha, A, L) = \frac{A^{\alpha+2}}{\Gamma(\alpha+2)} + L\frac{A^2}{2}$$

$$h(\alpha, A, L) = \left(\frac{1}{2}\ , 1, \frac{1}{12\sqrt{\pi}}\right)$$

$$= \frac{1^{\frac{5}{2}}}{\Gamma\left(\frac{7}{2}\right)} + \frac{1}{12\sqrt{\pi}} \cdot \frac{1^2}{2}$$

$$< \frac{8}{15\sqrt{\pi}} + \frac{1}{12\sqrt{\pi}} \cdot \frac{1}{2}$$

$$= \frac{69}{120\sqrt{\pi}} < 1\,.$$

Then, by Theorem3.2 initial value problem (16) has a unique solution on $[0,1]$ for value of $\alpha = \frac{1}{2}$ satisfying condition (16).

$$u(s) = -\frac{1}{\Gamma(\alpha)}\int_0^s\int_0^r\left(\int_0^q (q-t)^{\alpha-1}u(t)dt\right)dqdr + \int_0^s\int_0^r\left(2 + \frac{1}{12\sqrt{\pi}}q^2u(q) + u'(q)\right)dqdr.$$

Thus, $u_n(s)$ is defined by formula,

$$u_n(s) = -\frac{1}{\Gamma(\alpha)}\int_0^s\int_0^r\left(\int_0^q (q-t)^{\alpha-1}u_{n-1}(t)dt\right)dqdr + \int_0^s\int_0^r\big(f(q), u_{n-1}(q), u'_{n-1}(q)\big)dqdr,$$

for $n \in \mathbb{N}$. Since $u_0(0) = 0,\ u'_0(0) = 0$ we have

$$u_1(s) = -\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^s\int_0^r\left(\int_0^q (q-t)^{-\frac{1}{2}}u_0(t)dt\right)dqdr + \int_0^s\int_0^r\big(f(q), u_0(q), u'_0(q)\big)dqdr$$

$$= -\frac{1}{\sqrt{\pi}}\int_0^s\int_0^r\left(\int_0^q (q-t)^{-\frac{1}{2}}\,0\;\,dt\right)dqdr + \int_0^s\int_0^r\left(2+\frac{1}{12\sqrt{\pi}}q^2\cdot 0+0\right)dqdr$$

$$= s^2,$$

$$u_2(s) = -\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^s\int_0^r\left(\int_0^q (q-t)^{-\frac{1}{2}}u_1(t)dt\right)dqdr + \int_0^s\int_0^r\left(f(q),u_1(q),u_1'(q)\right)dqdr$$

$$= \frac{1}{\sqrt{\pi}}\int_0^s\int_0^r\left(\int_0^q (q-t)^{-\frac{1}{2}}\,t^2dt\right)dqdr + \int_0^s\int_0^r\left(2+\frac{1}{12\sqrt{\pi}}q^2\cdot 0+0\right)dqdr$$

$$= \frac{1}{\sqrt{\pi}}\int_0^s\int_0^r \frac{q^4}{12}dqdr + \frac{1}{\sqrt{\pi}}\int_0^s\int_0^r \frac{q^4}{12}dqdr + \int_0^s\int_0^r 2dqdr$$

$$= s^2,$$

$$u_3(s) = -\frac{1}{\Gamma\left(\frac{1}{2}\right)}\int_0^s\int_0^r\left(\int_0^q (q-t)^{-\frac{1}{2}}u_2(t)dt\right)dqdr + \int_0^s\int_0^r\left(f(q),u_2(q),u_2'(q)\right)dqdr$$

$$= s^2\,.$$

If it continues like this, it will be shown

$$u_n(s) = s^2\,,\;\; n \in \mathbb{N}$$

Hence,

$$u(s) = \lim_{n\to\infty} u_n(s) = \lim_{n\to\infty} s^2 = s^2.$$

Conclusion

Fixed point and operator theory play an important role in different areas of mathematics, differential equation, physics, game theory and dynamic programming. In this study, the initial value problem for the Integral-Differential equation is discussed. A sufficient condition for the existence and uniqueness of this problem is obtained. This approach allows using the fixed point iteration method to solve the differential equation problem involving fractional order integrals. The work can be applied to different classes of fractional order operators, which will be the subject of future articles.

Ethics in Publishing

There are no ethical issues regarding the publication of this study.

Author Contributions

The authors contributed equally.

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