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THE DUAL OF INFINITESIMAL UNITARY HOPF ALGEBRAS AND PLANAR ROOTED FORESTS

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Dedicated to the memory of Professor Edmund R. Puczyłowski

Abstract. We study the infinitesimal (in the sense of Joni and Rota) bialgebra H_{RT} of planar rooted trees introduced in a previous work of two of the authors, whose coproduct is given by deletion of a vertex. We prove that its dual H_{RT}^* is isomorphic to a free non unitary algebra, and give two free generating sets. Giving H_{RT} a second product, we make it an infinitesimal bialgebra in the sense of Loday and Ronco, which allows to explicitly construct a projector onto its space of primitive elements, which freely generates H_{RT}.

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Keywords: Infinitesimal bialgebra, planar rooted trees, graftings, 1-cocycle

1. Introduction

Rooted trees and planar rooted trees have a very rich algebraic structure: Grossman and Larson [10] first gave rooted trees a structure of noncommutative and cocommutative algebraic structure, closely related to the Butcher group of Runge-Kutta methods [2]; then, in order to algebraically treat the process of renormalization in quantum field theory, Connes and Kreimer introduced a commutative, noncocommutative Hopf algebra of rooted trees [3], and it was proved that the Connes-Kreimer and the Grossman-Larson Hopf algebra are in duality [12,16]. A self-dual noncommutative version of the Connes-Kreimer Hopf algebra was simultaneously introduced by Foissy and Holtkamp [5,13], and this object was deformed as an infinitesimal bialgebra in the sense of Loday and Ronco [15] in [7].

Recently, Gao and Wang introduced another infinitesimal coproduct Δ_{RT} on planar rooted trees, where the usual 1-cocycle compatibility between the operator B^+ (see paragraph 2.1 below) and the coproduct is modified: in the Foissy-Holtkamp

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case, this is the 1-cocycle condition

$$\Delta_{\mathcal{T}} \circ B^+(x) = B^+(x) \otimes \mathbb{1} + (\mathrm{id} \otimes B^+) \circ \Delta_{\mathcal{T}}(x),$$

whereas in the Gao-Wang case, this is:

$$\Delta_{RT} \circ B^+(x) = x \otimes \mathbb{1} + (\mathrm{id} \otimes B^+) \circ \Delta_{RT}(x).$$

All these coproducts on planar rooted trees are different; for example, in the "classical" Foissy-Holtkamp case:

$$\Delta(\vee) = \vee \otimes \mathbb{1} + \mathbb{1} \otimes \vee + 2 \bullet \otimes \mathrm{I} + \bullet\bullet \otimes \bullet,$$

whereas in the "infinitesimal" Foissy-Holtkamp case:

$$\Delta_{\mathcal{T}}(\vee) = \vee \otimes \mathbb{1} + \mathbb{1} \otimes \vee + \bullet \otimes \mathrm{I} + \bullet\bullet \otimes \bullet,$$

and in the Gao-Wang case:

$$\Delta_{RT}(\vee) = \bullet\bullet \otimes \mathbb{1} + \mathbb{1} \otimes \mathrm{I} + \bullet \otimes \bullet\bullet.$$

The Foissy-Holtkamp coproducts are described with the help of different families of admissible cuts, whereas the Gao-Wang coproduct is combinatorially described by the deletion of a vertex, separating the planar tree into two forests, see (7) below.

Our aim in this paper is to understand of this infinitesimal Hopf algebra H_{RT}, as well as its dual. We start by giving a combinatorial description of the product $\diamond$ of H_{RT}^* in the dual basis (Z_F) of the basis of forests of H_{RT} in terms of particular graftings in 3.13. We deduce that $(H_{RT}^*, \diamond)$ is a free nonunitary algebra, freely generated by the elements Z_F indexed by forests of even length. As a consequence, the infinitesimal bialgebra H_{RT}^* is both, a free nonunitary algebra, and a cofree counitary coalgebra; by duality, H_{RT} is both, a free unitary algebra (an immediate result), and a cofree noncounitary coalgebra.

With its usual product, H_{RT} is an infinitesimal bialgebra in the sense of Joni and Rota [14]:

$$\Delta_{RT}(xy) = x \cdot \Delta_{RT}(y) + \Delta_{RT}(x) \cdot y, \ \forall x, y \in H_{RT}.$$

With a second product $\star$, we make it an infinitesimal bialgebra in the sense of Loday and Ronco [15] (Proposition 4.1):

$$\Delta_{RT}(x \star y) = x \star \Delta_{RT}(y) + \Delta_{RT}(x) \star y + x \otimes y, \ \forall x, y \in H_{RT}.$$

As a consequence, we obtain a projector θ on the space of primitive elements of H_{RT} in Theorem 4.2, for which we give a cancellation-free expression in Corollary 4.8. As a consequence, we prove in Corollary 4.6 that $(H_{RT}, \star, \Delta_{RT})$ is isomorphic, as an infinitesimal bialgebra, to a nonunitary free algebra, with the concatenation product and deconcatenation coproduct. Dualizing these results, we describe the transposition ▲ of the product $\star$ and obtain by transposition a projector θ^* on the space of primitive elements of H_{RT}^*; as a consequence, we obtain a second set of free generators of $(H_{RT}^*, \diamond)$, namely the elements Z_F indexed by forests with no tree reduced to a single root (Corollary 4.14).

This paper is organized as follows: Section 2 contains reviews on planar rooted trees and forests, the Gao and Wang infinitesimal Hopf algebra H_{RT} and the description of the coproduct Δ_{RT}. In Section 3, we describe the dual product $\diamond$ of H_{RT}^* in terms of graftings, with also results on the number of such graftings, and we deduce the freeness of H_{RT}^*. We define and study the second product $\star$ on H_{RT} in Section 4, as well as the associated projector θ and its transpose.

Notation. In this paper, we will be working over a unitary commutative base ring $\mathbf{k}$. By an algebra we mean an associative algebra (possibly without unit) and by a coalgebra we mean a coassociative coalgebra (possibly without counit), unless otherwise stated. Linear maps and tensor products are taken over $\mathbf{k}$. For any algebra A, we view $A \otimes A$ as an A-bimodule via

$$a \cdot (b \otimes c) := ab \otimes c \text{ and } (b \otimes c) \cdot a := b \otimes ca. \tag{1}$$

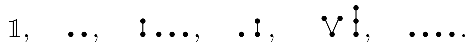
2. The infinitesimal unitary Hopf algebras of planar rooted forests

In this section, we first recall some basic notations used throughout the paper.

2.1. Planar rooted forests. We expose some concepts and notations on planar rooted forests from [11,17]. Let $\mathcal{T}$ denote the set of planar rooted trees and $M(\mathcal{T})$ the free monoid generated by $\mathcal{T}$ in which the multiplication is the concatenation, denoted by m_{RT} and usually suppressed. Thus an element F in $M(\mathcal{T})$, called a **planar rooted forest**, is a noncommutative product of planar rooted trees in $\mathcal{T}$. The empty tree $\mathbb{1}$ is the unity of $M(\mathcal{T})$.

Here are some examples of elements of $\mathcal{T}$ where the root is on the bottom:

Here are some examples of elements of $M(\mathcal{T})$:



Let $H_{RT} := \mathbf{k}M(\mathcal{T})$ be the free $\mathbf{k}$-module spanned by $M(\mathcal{T})$. Denote by

$$B^+ : H_{RT} \to H_{RT}$$

the grafting map sending $\mathbb{1}$ to $\bullet$ and sending a planar rooted forest in H_{RT} to its grafting on a new root, and by m_{RT} the concatenation on H_{RT}. Then H_{RT} is closed under the concatenation m_{RT} [17]. Here are some examples of B^+ on H_{RT}:

$B^+(\mathbb{1}) = \bullet$, $B^+(\bullet) =$, $B^+($ $\bullet) =$ .

For $F = T_1 \cdots T_m \in M(\mathcal{T})$ with $T_1, \cdots, T_m \in \mathcal{T}$, we define $\mathrm{bre}(F) := m$ to be the **breadth** of F. Here we use the convention that $\mathrm{bre}(\mathbb{1}) = 0$ when $m = 0$. The **depth** $\mathrm{dep}(T)$ of a rooted tree is the maximal length of linear chains from the root to the leaves of the tree. For $F = T_1 \cdots T_m \in M(\mathcal{T})$ with $m \geqslant 0$, we define

$$\mathrm{dep}(F) := \max\{\mathrm{dep}(T_i) \mid i = 1, \ldots, m\}.$$

2.2. Infinitesimal unitary Hopf algebras of planar rooted forests. In order to provide an algebraic framework for the calculus of divided differences, Joni and Rota [14] introduced the concept of an infinitesimal bialgebra.

Definition 2.1. [14] An **infinitesimal bialgebra** is a triple (A, m, Δ) where (A, m) is an associative algebra, (A, Δ) is a coassociative coalgebra and for each $a, b \in A$,

$$\Delta(ab) = a \cdot \Delta(b) + \Delta(a) \cdot b = \sum_{(b)} ab_{(1)} \otimes b_{(2)} + \sum_{(a)} a_{(1)} \otimes a_{(2)} b. \qquad (2)$$

If (A, m, Δ) is an infinitesimal bialgebra, the space of its primitive elements is $\mathrm{Prim}(A) = \ker(\Delta)$.

Note that we do not require that (A, m) is unitary, nor that (A, Δ) is counitary. The concept of an infinitesimal Hopf algebra was introduced by Aguiar in order to develop and study infinitesimal bialgebras [1]. If A is an infinitesimal bialgebra, then the space $\mathrm{Hom}_{\mathbf{k}}(A, A)$ is still an algebra under convolution:

$$f * g := m\,(f \otimes g)\,\Delta,$$

but possibly without unity with respect to the convolution $*$ [1]. Therefore, it is impossible to consider antipode. To solve this difficulty, Aguiar equipped the space $\mathrm{Hom}_{\mathbf{k}}(A, A)$ with circular convolution $\circledast$ given by

$$f \circledast g := f * g + f + g, \text{ that is, } (f \circledast g)(a) := \sum_{(a)} f(a_{(1)}) g(a_{(2)}) + f(a) + g(a) \text{ for } a \in A.$$

Note that $f \circledast 0 = f = 0 \circledast f$ and so $0 \in \mathrm{Hom}_{\mathbf{k}}(A, A)$ is the unity with respect to the circular convolution $\circledast$.

With the help of the circular convolution, one can describe infinitesimal Hopf algebras.

Definition 2.2. [1] An infinitesimal bialgebra (A, m, Δ) is called an **infinitesimal Hopf algebra** if the identity map $\mathrm{id} \in \mathrm{Hom}_{\mathbf{k}}(A, A)$ is invertible with respect to the circular convolution. In this case, its inverse $S \in \mathrm{Hom}_{\mathbf{k}}(A, A)$ is called the **antipode** of A. It is characterized by the equations

$$\sum_{(a)} S(a_{(1)})a_{(2)} + S(a) + a = 0 = \sum_{(a)} a_{(1)}S(a_{(2)}) + S(a) + a \ \text{ for } \ a \in A, \tag{3}$$

where $\Delta(a) = \sum_{(a)} a_{(1)} \otimes a_{(2)}$.

Now we recall the infinitesimal Hopf algebraic structure on top of planar rooted forests defined in [9]. The coproduct Δ_{RT} on H_{RT} is defined recursively on depth. Let F be a forest in H_{RT}. For the initial step of $\mathrm{dep}(F) = 0$, we define

$$\Delta_{RT}(F) := \Delta_{RT}(\mathbb{1}) = 0. \tag{4}$$

For the induction step of $\mathrm{dep}(F) \geqslant 1$, we reduce to the induction on $\mathrm{bre}(F) \geqslant 1$. If $\mathrm{bre}(F) = 1$, then $F = B^+(\overline{F})$ for some $\overline{F} \in M(\mathcal{T})$ and define

$$\Delta_{RT}(F) := \Delta_{RT}B^+(\overline{F}) := \overline{F} \otimes \mathbb{1} + (\mathrm{id} \otimes B^+)\Delta_{RT}(\overline{F}), \tag{5}$$

that is, $\Delta_{RT}B^+ = \mathrm{id} \otimes \mathbb{1} + (\mathrm{id} \otimes B^+)\Delta_{RT}$. Here the coproduct $\Delta_{RT}(\overline{F})$ is defined by the induction hypothesis on depth. If $\mathrm{bre}(F) \geqslant 2$, then $F = T_1T_2 \cdots T_m$ with $\mathrm{bre}(F) = m \geqslant 2$ and define

$$\Delta_{RT}(F) := T_1 \cdot \Delta_{RT}(T_2 \cdots T_m) + \Delta_{RT}(T_1) \cdot (T_2 \cdots T_m). \tag{6}$$

Remark 2.3. Foissy [7] also studied another kind of infinitesimal Hopf algebras on planar rooted forests, using a different coproduct $\Delta_{\mathcal{T}}$ given by

$$\Delta_{\mathcal{T}}(F) := \begin{cases} \mathbb{1} \otimes \mathbb{1}, & \text{if } F = \mathbb{1}, \\ F \otimes \mathbb{1} + (\mathrm{id} \otimes B^+)\Delta_{\mathcal{T}}(\overline{F}), & \text{if } F = B^+(\overline{F}), \\ F_1 \cdot \Delta_{\mathcal{T}}(F_2) + \Delta_{\mathcal{T}}(F_1) \cdot F_2 - F_1 \otimes F_2, & \text{if } F = F_1F_2. \end{cases}$$

We give some examples to expose the differences between these two coproducts Δ_{RT} and $\Delta_{\mathcal{T}}$. On the one hand,

$$\Delta_{RT}(\tun) = \mathbb{1} \otimes \mathbb{1};$$

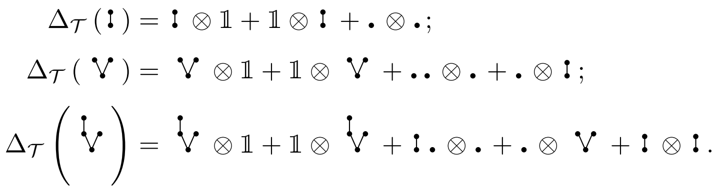
$$\Delta_{RT}(\tdeux) = \tun \otimes \mathbb{1} + \mathbb{1} \otimes \tun;$$

$$\Delta_{RT}(\ttroisun) = \tun\tun \otimes \mathbb{1} + \tun \otimes \tun + \mathbb{1} \otimes \tdeux;$$

$$\Delta_{RT}\left(\tquatretrois\right) = \tdeux\tun \otimes \mathbb{1} + \tdeux \otimes \tun + \tun \otimes \tdeux + \mathbb{1} \otimes \ttroisun.$$

On the other hand,

$$\Delta_{\mathcal{T}}(\tun) = \tun \otimes \mathbb{1} + \mathbb{1} \otimes \tun;$$



Let us recall the combinatorial description of Δ_{RT} given in [9], in terms of an order on the set $V(F)$ of vertices of a forest F [5,7].

Definition 2.4. Let $F = T_1 \cdots T_m \in M(\mathcal{T})$ with $T_1, \ldots, T_m \in \mathcal{T}$ and $m \geqslant 1$, and let $u, v \in V(F)$ be two vertices. Then

(a) $u \leqslant_h v$ (**being higher**) if there exists a (directed) path from u to v in F, the edges of F being oriented from roots to leaves;

(b) $u \leqslant_\ell v$ (**being more on the left**) if u and v are not comparable for $\leqslant_h$ and one of the following assertions is satisfied:

  (i) u is a vertex of T_i and v is a vertex of T_j with $1 \leqslant j < i \leqslant m$.

  (ii) u and v are vertices of the same T_i, and $u \leqslant_\ell v$ in the forest obtained from T_i by deleting its root;

(c) $u \leqslant_{h,\ell} v$ (**being higher or more on the left**) if $u \leqslant_h v$ or $u \leqslant_\ell v$.

As usual, we denote $u <_{h,\ell} v$ (resp. $u <_\ell v$, $u <_h v$) if $u \leqslant_{h,\ell} v$ (resp. $u \leqslant_\ell v$, $u \leqslant_h v$) but $u \neq v$. The **induced subgraph** in G by V is the graph whose vertex set is V and whose edge set consists of all of the edges in G that have both endpoints in V [4].

Let $F \in M(\mathcal{T})$ be a planar rooted forest. For each vertex $v \in V(F)$, denote by B_v the induced subgraph in F by the set $\{u \in V(F) \mid v <_{h,\ell} u\}$, and by R_v the induced subgraph in F by the set $V(F) \setminus (V(B_v) \cup \{v\})$. Equivalently, R_v is the induced subgraph in F by the set $\{u \in V(F) \mid u <_{h,\ell} v\}$. Note that both B_v and R_v are planar rooted forests in $M(\mathcal{T})$, not containing the vertex v. Then by [9, eq. (8)],

$$\Delta_{RT}(F) = \sum_{v \in V(F)} B_v \otimes R_v \quad \text{for } F \in M(\mathcal{T}). \tag{7}$$

Lemma 2.5. [9]

(a) *The quadruple* $(H_{RT}, m_{RT}, \mathbb{1}, \Delta_{RT})$ *is an infinitesimal unitary bialgebra.*

(b) *The quadruple* $(H_{RT}, m_{RT}, \mathbb{1}, \Delta_{RT})$ *is an infinitesimal unitary Hopf algebra.*

3. The dual of infinitesimal unitary Hopf algebra on planar rooted forests

In this section, we show that the dual $H_{RT}^* = (H_{RT}^*, \Delta_{RT}^*, m_{RT}^*, \mathbb{1}^*)$ of H_{RT} is a free algebra. Let us first recall some fundamental facts.

Lemma 3.1. [8] *Let $V = \bigoplus_{n=1}^{\infty} V^{(n)}$ be a graded vector space, with finite-dimensional homogeneous components. Then*

(a) *The graded dual $V^* := \bigoplus_{n=1}^{\infty} \left(V^{(n)}\right)^*$ is also a graded vector space, and $V^{**} \simeq V$.*
(b) *$V \otimes V$ is also a graded vector space with $(V \otimes V)^{(n)} = \sum_{i=0}^{n} V^{(i)} \otimes V^{(n-i)}$ for all $n \in \mathbb{N}$. Moreover, $(V \otimes V)^* \simeq V^* \otimes V^*$.*

The Hopf algebra H_{RT} can be graded by the number of vertices. Denote by

$$H_{RT}(n) := \mathbf{k}\left\{F \in M(\mathcal{T}) \;\middle|\; |F| = n-1\right\} \quad \text{for} \quad n \geqslant 1,$$

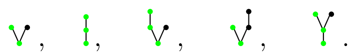
where $|F|$ is the number of vertices of F. Then

$$H_{RT} = \bigoplus_{n=1}^{\infty} H_{RT}(n) \text{ and } H_{RT}^* = \bigoplus_{n=1}^{\infty} (H_{RT}(n))^* .$$

We now give a combinatorial description of the dual of the coproduct Δ_{RT}. Let us propose the following concepts as a preparation.

Definition 3.2. Let T be a planar rooted tree. The **left path** $\mathrm{LP}(T)$ of T is defined to be the path from the root to the left most leaf of T.

Example 3.3. The following paths in green are left paths of planar rooted trees, respectively.



Definition 3.4. Let T and T' be two planar rooted trees. A **left grafting** of T' over T is a planar rooted tree obtained by grafting T' to a vertex v of the left path $\mathrm{LP}(T)$ by connecting v and the root of T', such that T' is on the left of T. Denote by $\mathcal{L}(T', T)$ the set of all left graftings of T' over T.

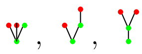
Example 3.5. Consider $T' = \bullet$ and $T =$. Then  and  are the two left graftings of T' over T.

In general, we propose

Definition 3.6. Let $F = T_1 \cdots T_m$ be a planar rooted forest with $T_1, \ldots, T_m \in \mathcal{T}$ and T a planar rooted tree. A **left grafting** of F over T is a planar rooted tree obtained by left grafting each T_i in a vertex v_i of $\mathrm{LP}(T)$ such that $v_i \leqslant_{h,\ell} v_j$ when $i < j$. Denoted by $\mathcal{L}(F, T)$ the set of all left grafting of F over T.

Notice that the T_i and T_j may be grafted in the same vertex.

Example 3.7. Let $F = \bullet\bullet$ and $T =$. Then

$$\mathcal{L}(F, T) = \Big\{$$  $$\Big\}.$$

Definition 3.8. Let F be a planar rooted forest and T a planar rooted tree.

Step 1: Decompose $F = F_1F_2$ and $F_1B^+(F_2) = F_1'F_2'$, where $F_1, F_2, F_1', F_2' \in M(\mathcal{T})$.

Step 2: Left graft F_2' over T to obtain an $\tilde{F} \in \mathcal{L}(F_2', T)$, and concatenate F_1' and $\tilde{F}$ to get $F_1'\tilde{F}$, and call the concatenation $F_1'\tilde{F}$ a **grafting** of F over T.

Denote by $\mathcal{G}(F, T)$ the set of all graftings of F over T.

Let us compute explicitly an example for better understanding of Definition 3.8.

Example 3.9. Let $F = \bullet\bullet$ and $T =$. Then the decomposition $F = F_1F_2$ as concatenation product can be

$$F = \mathbb{1}(\bullet\bullet) = (\bullet)(\bullet) = (\bullet\bullet)\mathbb{1}.$$

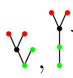
Case 1. $F_1 = \mathbb{1}$ and $F_2 = \bullet\bullet\bullet$. Then $B^+(F_2) =$  and $F_1B^+(F_2) =$ . The decomposition $F_1B^+(F_2) = F_1'F_2'$ can be

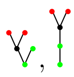
$F_1B^+(F_2) = ($  $)\mathbb{1} = \mathbb{1}($  $).$

We have two subcases.

Subcase 1.1. $F_1' =$  and $F_2' = \mathbb{1}$. Then  is the only one grafting of F over T in this subcase.

Subcase 1.2. $F_1' = \mathbb{1}$ and $F_2' =$ . Then

$\mathcal{L}(F_2', T) = \mathcal{L}($  $) = \Big\{$  $\Big\}$

and  are two graftings of F over T in this subcase.

Case 2. $F_1 = \bullet$ and $F_2 = \bullet\bullet$. Then $B^+(F_2) =$ and $F_1B^+(F_2) =$ . The decomposition $F_1B^+(F_2) = F_1'F_2'$ can be

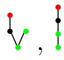
$F_1B^+(F_2) = ($  $)\mathbb{1} = (\bullet)($  $) = \mathbb{1}($  $).$

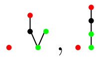
We have the following three subcases.

Subcase 2.1. $F_1' =$  and $F_2' = \mathbb{1}$. Then  is the only one grafting of F over T in this subcase.

Subcase 2.2. $F_1' = \bullet$ and $F_2' =$. Then

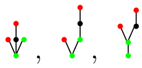
$$\mathcal{L}(F_2', T) = \mathcal{L}(\ \cdot\ ) = \left\{\ \cdot\ \right\}$$

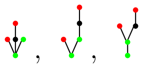
 

and  are two graftings of F over T in this subcase.

Subcase 2.3. $F_1' = \mathbb{1}$ and $F_2' =$ . Then

$$\mathcal{L}(F_2', T) = \mathcal{L}(\ \cdot\ ) = \left\{\ \cdot\ \right\}$$

and  are three graftings of F over T in this subcase.

Case 3. $F_1 = \bullet\bullet$ and $F_2 = \mathbb{1}$. Then $B^+(F_2) = \bullet$ and $F_1B^+(F_2) = \bullet\bullet\bullet$. The decomposition $F_1B^+(F_2) = F_1'F_2'$ can be

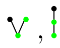
$$F_1B^+(F_2) = (\bullet\bullet\bullet)\mathbb{1} = (\bullet\bullet)(\bullet) = (\bullet)(\bullet\bullet) = \mathbb{1}(\bullet\bullet\bullet).$$

There are four subcases.

Subcase 3.1. $F_1' = \bullet\bullet\bullet$ and $F_2' = \mathbb{1}$. Thus  is the only one grafting of F over T in this subcase.

Subcase 3.2. $F_1' = \bullet\bullet$ and $F_2' = \bullet$. Then

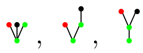
$$\mathcal{L}(F_2', T) = \mathcal{L}(\bullet,\ \cdot\ ) = \left\{\ \cdot\ \right\}$$

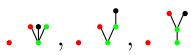
 

and  are two graftings of F over T in this subcase.

Subcase 3.3. $F_1' = \bullet$ and $F_2' = \bullet\bullet$. We have

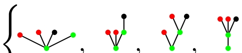
$$\mathcal{L}(F_2', T) = \mathcal{L}(\bullet\bullet,\ \cdot\ ) = \left\{\ \cdot\ \right\}.$$

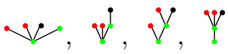
 

Then  are three graftings of F over T in this subcase.

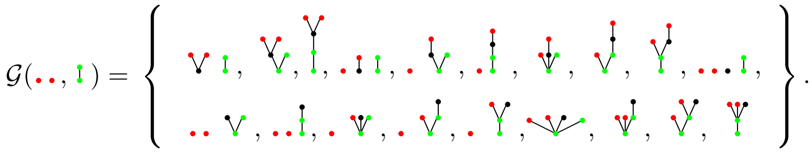
Subcase 3.4. $F_1' = \mathbb{1}$ and $F_2' = \bullet\bullet\bullet$. Then

$$\mathcal{L}(F_2', T) = \mathcal{L}(\bullet\bullet\bullet,\ \cdot\ ) = \left\{\ \cdot\ \right\}$$

and  are four graftings of F over T in this subcase.

In summary, there are nineteen graftings of F over T:



Proposition 3.10. *Let F be a planar rooted forest and T be a planar rooted tree. We put* $\mathrm{bre}(F) := k$ *and we denote by l the cardinality of $LP(T)$. Then:*

$$a_{k,l} := |\mathcal{G}(F,T)| = \binom{k+l+2}{l+1} - 1.$$

Moreover, in $\mathbb{Q}[[X,Y]]$*:*

$$\sum_{k,l=0}^{\infty} a_{k,l} X^k Y^l = \frac{1}{(1-X)(1-Y)(1-X-Y)}.$$

Proof. As $\mathrm{bre}(F) = k$, if we put $F = T_1 \cdots T_k$, with $T_1, \ldots, T_k$ planar rooted trees, there are exactly $k+1$ possibilities for writing $F = F_1 F_2$, which are $(F_1^{(i)}, F_2^{(i)}) = (T_1 \cdots T_{i-1}, T_i \cdots T_k)$, with $1 \leqslant i \leqslant k+1$. Note that $G^{(i)} = F_1^{(i)} B^+(F_2^{(i)})$ is a forest of breadth i. Therefore, there are exactly $\binom{i+l}{i}$ ways to write it as $G^{(i)} = G_0 \cdots G_l$, so there are exactly $\binom{i+l}{l}$ ways to proceed in the second step of Definition 3.8. Hence:

$$|\mathcal{G}(F,T)| = \sum_{i=1}^{k+1} \binom{i+l}{l} = \sum_{i=0}^{k+1} \binom{i+l}{l} - 1 = \sum_{j=l}^{k+l+1} \binom{j}{l} - 1 = \binom{k+l+2}{l+1} - 1.$$

Moreover:

$$\begin{aligned}
\sum_{k,l=0}^{\infty} a_{k,l} X^k Y^l &= \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} \binom{k+l+2}{l+1} X^k Y^l - \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} X^k Y^l \\
&= \sum_{k=0}^{\infty} \frac{X^k}{Y} \sum_{l=1}^{\infty} \binom{k+l+1}{l} Y^l - \frac{1}{(1-X)(1-Y)} \\
&= \sum_{k=0}^{\infty} \frac{X^k}{Y} \left(\frac{1}{(1-Y)^{k+2}} - 1 \right) - \frac{1}{(1-X)(1-Y)} \\
&= \frac{1}{Y(1-Y)^2} \sum_{k=0}^{\infty} \left(\frac{X}{1-Y} \right)^k - \frac{1}{Y} \sum_{k=0}^{\infty} X^k - \frac{1}{(1-X)(1-Y)} \\
&= \frac{1}{Y(1-Y)^2} \frac{1}{1 - \frac{X}{1-Y}} - \frac{1}{Y(1-X)} - \frac{1}{(1-X)(1-Y)} \\
&= \frac{1}{(1-X)(1-Y)(1-X-Y)}.
\end{aligned}$$

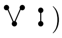
□

Note that for any $k, l \geqslant 1$, $a_{k,l} = a_{l,k}$.

In general, we propose

Definition 3.11. Let F and $F' = TF_1$ be two planar rooted forests with $T \in \mathcal{T}$. We call $\tilde{F}F_1$ a **grafting** of F over F', where $\tilde{F}$ is a grafting of F over T given in Definition 3.8. Let $\mathcal{G}(F, F')$ be the set of all graftings of F over F'.

For $F, F', F'' \in M(\mathcal{T})$, denote by $n'(F, F'; F'')$ the number of ways of grafting of F over F' to obtain F''. For example,

$n'(\bullet\bullet,$ $;\bullet\bullet) = 0,$ $n'(\bullet\bullet,$ $;\bullet\bullet\bullet\bullet\bullet) = 0,$ $n'(\bullet\bullet,$ $;$  $) = 1$

and $n'(\bullet\bullet,$ $;$  $) = 1.$

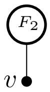
For each $F \in M(\mathcal{T})$, we define

$$Z_F : \begin{cases} H_{RT} & \longrightarrow & \mathbf{k}, \\ F' & \mapsto & \delta_{F,F'} \text{ for } F' \in M(\mathcal{T}), \end{cases} \tag{8}$$

where $\delta_{F,F'}$ is the Kronecker function. Then $\left\{ Z_F \;\middle|\; F \in M(\mathcal{T}) \right\}$ is a basis of H_{RT}^*. We denote by $n(F, F'; F'')$ the coefficient of $F \otimes F'$ in $\Delta_{RT}(F'')$, where $F, F', F'' \in M(\mathcal{T})$. The following result gives the relation between $n(F, F'; F'')$ and $n'(F, F'; F'')$.

Lemma 3.12. *Let $F, F', F'' \in M(\mathcal{T})$. Then $n'(F, F'; F'') = n(F, F'; F'')$. Moreover, this coefficient is* 0 *or* 1.

Proof. We first show that $n'(F, F'; F'') \leqslant n(F, F'; F'')$. Let $F'' \in \mathcal{G}(F, F')$ be a grafting of F over F' determinated by the decompositions $F = F_1F_2$ and $F_1B^+(F_2) = F_1'F_2'$, in which the new vertex added by B^+ is denoted by v. Graphically,

$F_1B^+(F_2) = F_1$  .

Then $v \in V(F'')$ and

$$V(F_1) = \{u \in V(F'') \mid v <_\ell u\} \text{ and } V(F_2) = \{u \in V(F'') \mid v <_h u\}.$$

By the combinatorial description of the coproduct $\Delta_{RT}(F'')$, we have

$$B_v = \{u \in V(F'') \mid v <_{h,\ell} u\} = F_1F_2 = F.$$

Since $V(F'') = V(F) \sqcup V(F') \sqcup \{v\}$, we get

$$R_v = \text{the induced subgraph of } F'' \text{ by } V(F'') \setminus (\{v\} \sqcup V(B_v)) = F'.$$

Therefore a grafting F'' of F over F' induces a term $B_v \otimes R_v = F \otimes F'$ in $\Delta_{RT}(F'')$ and so $n'(F, F'; F'') \leqslant n(F, F'; F'')$.

Next we show that $n'(F, F'; F'') \geqslant n(F, F'; F'')$. By Eq. (7), we may let $B_v \otimes R_v$ be an item in $\Delta_{RT}(F'')$ for some $v \in V(F'')$. Write $F'' = T_1 \cdots T_m$ with $T_1, \ldots, T_m \in \mathcal{T}$, and assume $v \in V(T_i)$ for some $1 \leqslant i \leqslant m$. Let $F := B_v$, $F' := R_v$,

$$F_1 := \text{the induced subgraph of } F'' \text{ by } \{u \in V(F'') \mid v <_\ell u\}$$

and

$$F_2 := B_v \setminus F_1 = \text{the induced subgraph of } F'' \text{ by } \{u \in V(F'') \mid v <_h u\}.$$

Since $F = B_v$ is the induced subgraph of F'' by $\{u \in V(F'') \mid v <_{h,\ell} u\}$, it follows from Definition 2.4 that $F = B_v = F_1 F_2$. Let $F_1' := T_1 \cdots T_{i-1}$ and

$$F_2' := \text{the induced subgraph of } T_i \text{ by } \{u \in V(T_i) \mid v <_{h,\ell} u\} \sqcup \{v\}.$$

Then F'' is a grafting of F over F' determinate by the decomposition $F_1 B^+(F_2) = F_1' F_2'$ (see Fig.1).

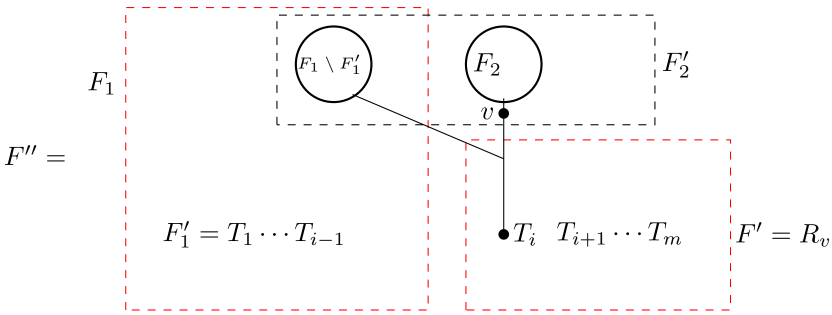


Fig. 1 The illustration of the grafting F'' of F over F'.

Hence, if $F \otimes F'$ is a term of $\Delta_{RT}(F'')$, then we obtain a grafting F'' of F over F' and so $n'(F, F'; F'') \leqslant n(F, F'; F'')$. Therefore $n'(F, F'; F'') = n(F, F'; F'')$.

If $n(F, F'; F'') \geqslant 2$, then there exist two different vertices u and v such that $B_u \otimes R_u = F \otimes F' = B_v \otimes R_v$ by Eq. (7). Thus $B_u = B_v$ and so $u = v$ by the definition of B_u and B_v, a contradiction. □

The following result gives a combinatorial description of the multiplication in H_{RT}^*, dual of the coproduct Δ_{RT}, which we denote by $\diamond = \Delta_{RT}^*$.

Proposition 3.13. *Let $F, F' \in M(\mathcal{T})$. The product of Z_F and $Z_{F'}$ is*

$$Z_F \diamond Z_{F'} = \sum_{F'' \in M(\mathcal{T})} n'(F, F'; F'') Z_{F''} = \sum_{F'' \in \mathcal{G}(F, F')} Z_{F''}. \tag{9}$$

Proof. The second equation follows from Lemma 3.12. We are left to show the first equation. For any $F, F' \in M(\mathcal{T})$, we suppose

$$Z_F \diamond Z_{F'} = \sum_{F'' \in M(\mathcal{T})} a_{F,F'}^{F''} Z_{F''}.$$

For any $F_1 \in M(\mathcal{T})$, we have

$$\sum_{F'' \in M(\mathcal{T})} a_{F,F'}^{F''} Z_{F''}(F_1) = a_{F,F'}^{F_1} Z_{F_1}(F_1) = a_{F,F'}^{F_1},$$

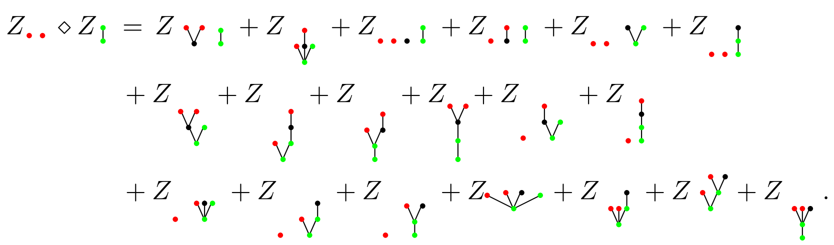
and

$$\begin{aligned}
Z_F \diamond Z_{F'}(F_1) = \; & \Delta_{RT}^*(Z_F \otimes Z_{F'})(F_1) = (Z_F \otimes Z_{F'})(\Delta_{RT}(F_1)) \\
= \; & (Z_F \otimes Z_{F'}) \left(\sum_{(F_1)} F_{1(1)} \otimes F_{1(2)} \right) \\
= \; & \sum_{(F_1)} Z_F(F_{1(1)}) \otimes Z_{F'}(F_{1(2)}) \\
= \; & \sum_{(F_1)} \delta_{F, F_{1(1)}} \otimes \delta_{F', F_{1(2)}} \\
= \; & n(F, F'; F_1) \quad \text{(by Definition of } n(F, F'; F_1)) \\
= \; & n'(F, F'; F_1) \quad \text{(by Lemma 3.12)}.
\end{aligned}$$

Thus $a_{F,F'}^{F_1} = n'(F, F'; F_1)$, as required. □

Let us expose an example.

Example 3.14. Let F, F' be the planar rooted forests in Example 3.9. Then the product of Z and Z is



Remark 3.15. (a) We define the degree of a forest as $\deg(F) = |F| + 1$ for $F \in M(\mathcal{T})$. Since a new vertex is added by the grafting operator B^+ in the second step of Definition 3.8, the multiplication

$$\diamond : H_{RT}^* \otimes H_{RT}^* \to H_{RT}^*$$

is homogeneous of degree 0 by Proposition 3.13.

(b) Let $F, F' = T\overline{F'}$ be two planar rooted forests. Then by Definition 3.11 and Proposition 3.13,

$$Z_F \diamond Z_{F'} = \sum_{F'' \in \mathcal{G}(F,F')} Z_{F''} = \sum_{F_1F_2=F} \sum_{F_1'F_2'=F_1B^+(F_2)} \sum_{\tilde{F} \in \mathcal{L}(F_2',T)} Z_{F_1'\tilde{F}\overline{F'}},$$

where $F_1, F_2, F_1', F_2' \in M(\mathcal{T})$.

The following result characterizes the coproduct on H_{RT}^*.

Lemma 3.16. *Let $T_1 \cdots T_n \in M(\mathcal{T})$. The coproduct of $Z_{T_1\cdots T_n} \in H_{RT}^*$ is*

$$m_{RT}^*(Z_{T_1\cdots T_n}) = \sum_{i=0}^{n} Z_{T_1\cdots T_i} \otimes Z_{T_{i+1}\cdots T_n}, \tag{10}$$

with the convention that $Z_{T_1T_0} = \mathbb{1}$ and $Z_{T_{n+1}T_n} = \mathbb{1}$.

Proof. Suppose

$$m_{RT}^*(Z_{T_1\cdots T_n}) = \sum_{F',F'' \in M(\mathcal{T})} c_{F',F''} Z_{F'} \otimes Z_{F''}.$$

For any $F_1, F_2 \in M(\mathcal{T})$, we have

$$m_{RT}^*(Z_{T_1\cdots T_n})(F_1 \otimes F_2) = Z_{T_1\cdots T_n}(m_{RT}(F_1 \otimes F_2)) = Z_{T_1\cdots T_n}(F_1F_2) = \delta_{T_1\cdots T_n, F_1F_2},$$

and

$$\begin{aligned}\left(\sum_{F',F'' \in M(\mathcal{T})} c_{F',F''}(Z_{F'} \otimes Z_{F''})\right)(F_1 \otimes F_2) &= \sum_{F',F'' \in M(\mathcal{T})} c_{F',F''} Z_{F'}(F_1) \otimes Z_{F''}(F_2) \\ &= \sum_{F',F'' \in M(\mathcal{T})} c_{F',F''} \delta_{F',F_1} \otimes \delta_{F'',F_2} \\ &= c_{F_1,F_2}.\end{aligned}$$

Thus $\delta_{T_1\cdots T_n, F_1F_2} = c_{F_1,F_2}$ and so $c_{F_1,F_2} = 1$ if $T_1 \cdots T_n = F_1F_2$ and $c_{F_1,F_2} = 0$ otherwise. This completes the proof. □

For example, we have

$$m_{RT}^*(Z_{\bullet\,\substack{\bullet\\|\\\bullet}}) = Z_{\bullet\,\substack{\bullet\\|\\\bullet}} \otimes \mathbb{1} + \mathbb{1} \otimes Z_{\bullet\,\substack{\bullet\\|\\\bullet}} + Z_{\bullet} \otimes Z_{\substack{\bullet\\|\\\bullet}}.$$

Now we are ready for our main result in this section. Denote by

$$S := \{Z_F \mid F \in M(\mathcal{T}) \text{ such that } \mathrm{bre}(F) \text{ is even}\}.$$

Theorem 3.17. *The algebra $(H_{RT}^*, \diamond)$ is the free non unitary algebra on S.*

Proof. We first prove that S generates H^*_{RT}. Let A be the subalgebra of $(H^*_{RT}, \diamond)$ generated by S and let $Z_{T_1\cdots T_m} \in H^*_{RT}$ be a basis element with $T_1, \ldots, T_m \in \mathcal{T}$. If m is even, then $Z_{T_1\cdots T_m} \in A$. If m is odd, we prove that $Z_{T_1\cdots T_m} \in A$ by induction on $|T_1| \geqslant 1$. If $|T_1| = 1$, then $T_1 = \bullet$ and

$$Z_{\mathbb{1}} \diamond Z_{T_2\cdots T_m} = Z_{\bullet T_2\cdots T_m} + \sum_{F' \in \mathcal{L}(\bullet, T_2)} Z_{F'T_3\cdots T_m} \quad \text{(by Item (b) of Remark 3.15).}$$

Since $Z_{\mathbb{1}}, Z_{T_2\cdots T_m} \in A$ and $Z_{F'T_3\cdots T_m} \in A$ by $\mathrm{bre}(F'T_3 \cdots T_m) = m-1$, we have $Z_{T_1T_2\cdots T_m} = Z_{\bullet T_2\cdots T_m} \in A$. If $|T_1| \geqslant 2$, then $T_1 = B^+(F)$ for some $F \in H_{RT}$. We have

$$\begin{aligned} Z_F \diamond Z_{T_2\cdots T_m} &= \sum_{F_1F_2 = F} \sum_{F_1'F_2' = F_1B^+(F_2)} \sum_{\tilde{F} \in \mathcal{L}(F_2', T_2)} Z_{F_1'\tilde{F}T_3\cdots T_m} \quad \text{(by Item (b) of Remark 3.15)} \\ &= Z_{B^+(F)T_2\cdots T_m} + \sum_{F_1F_2 = F} \sum_{\substack{F_1'F_2' = F_1B^+(F_2) \\ F_1' = \mathbb{1}}} \sum_{\tilde{F} \in \mathcal{L}(F_2', T_2)} Z_{\tilde{F}T_3\cdots T_m} \\ &+ \sum_{\substack{F_1F_2 = F \\ F_1 \neq \mathbb{1}}} \sum_{\substack{F_1'F_2' = F_1B^+(F_2) \\ F_1' \neq \mathbb{1}}} \sum_{\tilde{F} \in \mathcal{L}(F_2', T_2)} Z_{F_1'\tilde{F}T_3\cdots T_m}. \end{aligned}$$

Since $\mathrm{bre}(\tilde{F}T_3 \cdots T_m) = m-1$, we have $Z_{T_2\cdots T_m}, Z_{\tilde{F}T_3\cdots T_m} \in A$ by the definition of A. Moreover, $Z_F, Z_{F_1'\tilde{F}T_3\cdots T_m} \in A$ by the induction hypothesis. Hence $Z_{T_1\cdots T_m} \in A$ and $A = H^*_{RT}$.

Next, we prove that $(H^*_{RT}, \diamond)$ is the free algebra on S. By [6, Proposition 8], the formal series of $H_{RT} = \bigoplus_{n=1}^{\infty} H_{RT}(n)$ is

$$\mathbf{F}(x) = \sum_{i=1}^{\infty} \dim H_{RT}(i) x^i = x + x^2 + 2x^3 + 5x^4 + \cdots = \frac{1 - \sqrt{1-4x}}{2},$$

which is also the formal series of $H^*_{RT} = \bigoplus_{n=1}^{\infty} (H_{RT}(n))^*$. Thus $\mathbf{F}^2(x) = \mathbf{F}(x) - x$. Since each planar rooted tree is a grafting operation B^+ of a planar rooted forest F and vice-versa, the formal series of planar rooted trees is the same as the one of planar rooted forests, that is,

$$\mathbf{T}(x) = \sum_{i=1}^{\infty} a_i x^i = \mathbf{F}(x) = x + x^2 + 2x^3 + 5x^4 + \cdots,$$

where a_i is the number of trees with i vertices. So the formal series of forests with even breadth is

$$\sum_{i=0}^{+\infty} b_i x^i = 1 + \mathbf{T}(x)\mathbf{T}(x) + \mathbf{T}(x)\mathbf{T}(x)\mathbf{T}(x)\mathbf{T}(x) + \cdots = \sum_{i=0}^{+\infty} \mathbf{T}^{2i}(x) = \frac{1}{1 - \mathbf{T}^2(x)} = \frac{1}{1 - \mathbf{F}^2(x)},$$

where b_i is the number of forests of even breadth with i vertices. Let $\mathbf{G}(x) := \sum_{i=1}^{+\infty} g_i x^i$, where g_i is the number of forests of even breadth with degree i. Then

$$\begin{aligned}\mathbf{G}(x) &= x\sum_{i=1}^{\infty} g_i x^{i-1} = x\sum_{i=0}^{\infty} g_{i+1}x^i = x\sum_{i=0}^{\infty} b_i x^i \quad (\text{by } \deg(F) = |F|+1)\\ &= \frac{x}{1-\mathbf{F}^2(x)} = \frac{\mathbf{F}(x)}{1+\mathbf{F}(x)} \quad (\text{by } x = \mathbf{F}(x) - \mathbf{F}^2(x)).\end{aligned}$$

Let $T(S)$ be the free algebra on S. Since H_{RT}^* is generated by S, there exists a surjective algebra morphism $\phi : T(S) \twoheadrightarrow H_{RT}^*$. By Item (a) of Remark 3.15, the formal series of $T(S)$ is

$$\mathbf{G}(x)+\mathbf{G}(x)\mathbf{G}(x)+\mathbf{G}(x)\mathbf{G}(x)\mathbf{G}(x)+\mathbf{G}(x)\mathbf{G}(x)\mathbf{G}(x)\mathbf{G}(x)+\cdots = \frac{\mathbf{G}(x)}{1-\mathbf{G}(x)} = \frac{\dfrac{\mathbf{F}(x)}{1+\mathbf{F}(x)}}{1-\dfrac{\mathbf{F}(x)}{1+\mathbf{F}(x)}} = \mathbf{F}(x).$$

Thus H_{RT}^* and $T(S)$ have the same formal series and so ϕ is injective. Hence $T(S)$ is isomorphic to H_{RT}^*, as required. □

4. Primitive elements of H_{RT}

In this section, we first construct a second product $\star$ on H_{RT}, making H_{RT} a unital infinitesimal graded bialgebra in the sense of Loday and Ronco [15]. Then we give a projection of H_{RT} on its primitive elements. Finally, we characterize the dual of $\star$.

4.1. A second product on H_{RT}. The following result gives H_{RT} a second product $\star$.

Proposition 4.1. *We define a product $\star$ on H_{RT} by*

$$x \star y = x \cdot \bullet \cdot y,\ \forall x, y \in H_{RT}.$$

Then $\star$ is associative (and not unitary), and for any $x, y \in H_{RT}$:

$$\Delta_{RT}(x \star y) = x \star \Delta_{RT}(y) + \Delta_{RT}(x) \star y + x \otimes y.$$

Moreover, $(H_{RT}, \star)$ is a graded algebra (recall that for any $n \geqslant 1$, $H_{RT}(n)$ is the subspace generated by forests with $n-1$ vertices).

Proof. For any $x, y, z \in H_{RT}$:

$$(x \star y) \star z = x \cdot \bullet \cdot y \cdot \bullet \cdot z = x \star (y \star z),$$

so $\star$ is associative. As $\Delta_{RT}(\bullet) = \mathbb{1} \otimes \mathbb{1}$, for any $x, y \in H_{RT}$:

$$\Delta_{RT}(x \star y) = \Delta_{RT}(x \cdot \bullet \cdot y)$$

$$
\begin{aligned}
&= \Delta_{RT}(x \cdot \bullet) \cdot y + x_\bullet \cdot \Delta_{RT}(y) \\
&= (x \cdot \Delta_{RT}(\bullet)) \cdot y + (\Delta_{RT}(x) \cdot \bullet) \cdot y + (x \cdot \bullet) \cdot \Delta_{RT}(y) \\
&= x \cdot (\mathbb{1} \otimes \mathbb{1}) \cdot y + \Delta_{RT}(x) \cdot (\bullet \cdot y) + (x \cdot \bullet) \cdot \Delta_{RT}(y) \\
&= x \otimes y + \Delta_{RT}(x) \star y + x \star \Delta_{RT}(y).
\end{aligned}
$$

Let $x \in H_{RT}(k)$ and $y \in H_{RT}(l)$, with $k, l \geqslant 1$. Then x is a linear span of forests with $k-1$vertices and y is a linear span of forests with $l-1$ vertices. By definition of $\star$, $x \star y$ is a linear span of forests with $k-1+l-1+1 = k+l-1$ vertices, so belongs to $H_{RT}(k+l)$. □

4.2. A projection on primitive elements. The following result gives a projection of H_{RT} on its primitive elements.

Theorem 4.2. *We define an operator θ on H_{RT} by:*

$$\theta(x) := \sum_{k=1}^{\infty} (-1)^{k+1} \star^{(k-1)} \circ \Delta^{(k-1)}(x),\ \forall x \in H_{RT},$$

where $\Delta^{(l)} : H_{RT} \longrightarrow H_{RT}^{\otimes(l+1)}$ and $\star^{(l)} : H_{RT}^{\otimes(l+1)} \longrightarrow H_{RT}$ are inductively defined:

$$
\begin{aligned}
\Delta^{(0)} &= \mathrm{id}_{H_{RT}}, & \star^{(0)} &= \mathrm{id}_{H_{RT}}, \\
\Delta^{(l+1)} &= (\Delta^{(l)} \otimes \mathrm{id}_{H_{RT}}) \circ \Delta_{RT}, & \star^{(l+1)} &= \star \circ (\star^{(l)} \otimes \mathrm{id}_{H_{RT}}).
\end{aligned}
$$

Then θ is a projector on $Prim(H_{RT}) = Ker(\Delta_{RT})$. The kernel of θ is

$$Ker(\theta) = H_{RT} \star H_{RT} = \mathbf{k}\{F \in M(\mathcal{T}) \text{ with at least one tree equal to } \bullet\}.$$

Proof. For any forest F with n vertices, $\Delta^{(k)}(F) = 0$ if $k \geqslant n$, so θ is well-defined. If $x \in Prim(H_{RT})$, $\Delta^{(k)}(x) = 0$ if $k \geqslant 1$, so $\theta(x) = x + 0 = x$.

Let $k \geqslant 2$ and $x_1, \ldots, x_k \in H_{RT}$. By the compatibility between the product $\star$ and the coproduct Δ_{RT}:

$$
\begin{aligned}
\Delta_{RT} \circ \star^{(k-1)}(x_1 \otimes \cdots \otimes x_k) = &\sum_{i=1}^{k} \sum_{(x_i)} x_1 \star \cdots \star x_{i-1} \star x_i^{(1)} \otimes x_i^{(2)} \star x_{i+1} \star \cdots \star x_k \\
&+ \sum_{i=1}^{k-1} x_1 \star \cdots \star x_i \otimes x_{i+1} \star \cdots \star x_k.
\end{aligned}
$$

Hence, using Sweedler's notation:

$$
\begin{aligned}
\Delta_{RT} \circ \theta(x) = x^{(1)} \otimes x^{(2)} &+ \sum_{k=2}^{\infty} (-1)^{k+1} \sum_{(x)} \sum_{i=1}^{k} x^{(1)} \star \cdots \star x^{(i)} \otimes x^{(i+1)} \star \cdots \star x^{(k+1)} \\
&+ \sum_{k=2}^{\infty} (-1)^{k+1} \sum_{(x)} \sum_{i=1}^{k-1} x^{(1)} \star \cdots \star x^{(i)} \otimes x^{(i+1)} \star \cdots \star x^{(k)}
\end{aligned}
$$

$$
\begin{aligned}
&= \sum_{k=1}^{\infty}(-1)^{k+1}\sum_{(x)}\sum_{i=1}^{k} x^{(1)}\star\cdots\star x^{(i)}\otimes x^{(i+1)}\star\cdots\star x^{(k+1)}\\
&+ \sum_{k=1}^{\infty}(-1)^{k}\sum_{(x)}\sum_{i=1}^{k} x^{(1)}\star\cdots\star x^{(i)}\otimes x^{(i+1)}\star\cdots\star x^{(k+1)}\\
&= 0,
\end{aligned}
$$

so θ is a projector on $Prim(H_{RT})$.

By the definition of θ, for any $x \in H_{RT}$, $\theta(x) - x \in H_{RT} \star H_{RT}$, so:

$$H_{RT} = Prim(H_{RT}) + H_{RT} \star H_{RT}.$$

Let $x, y \in H_{RT}$. By the compatibility between the product $\star$ and the coproduct Δ_{HR}, for any $k \geqslant 2$:

$$
\begin{aligned}
\Delta^{(k-1)}(x\star y) &= \sum_{i+j=k}\sum_{(x)}\sum_{(y)} x^{(1)}\otimes\cdots\otimes x^{(i)}\otimes y^{(1)}\otimes\cdots\otimes y^{(j)}\\
&+ \sum_{i+j=k+1}\sum_{(x)}\sum_{(y)} x^{(1)}\otimes\cdots\otimes x^{(i-1)}\otimes x^{(i)}\star y^{(1)}\otimes y^{(2)}\otimes\cdots\otimes y^{(j)}.
\end{aligned}
$$

Therefore:

$$
\begin{aligned}
\theta(x\star y) &= x\star y + \sum_{k=2}^{\infty}(-1)^{k+1}\sum_{i+j=k}\sum_{(x)}\sum_{(y)} x^{(1)}\star\cdots\star x^{(i)}\star y^{(1)}\star\cdots\star y^{(j)}\\
&+ \sum_{k=2}^{\infty}(-1)^{k+1}\sum_{i+j=k+1}\sum_{(x)}\sum_{(y)} x^{(1)}\star\cdots\star x^{(i)}\star y^{(1)}\star\cdots\star y^{(j)}\\
&= \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{(x)}\sum_{(y)}(-1)^{i+j-1} x^{(1)}\star\cdots\star x^{(i)}\star y^{(1)}\star\cdots\star y^{(j)}\\
&+ \sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\sum_{(x)}\sum_{(y)}(-1)^{i+j} x^{(1)}\star\cdots\star x^{(i)}\star y^{(1)}\star\cdots\star y^{(j)}\\
&= 0.
\end{aligned}
$$

Consequently, $Prim(H_{RT}) \cap (H_{RT} \star H_{RT}) = (0)$, so:

$$H_{RT} = Prim(H_{RT}) \oplus (H_{RT} \star H_{RT}),$$

and the projection on $Prim(H_{RT})$ in this direct sum is θ. □

Remark 4.3. $(H_{RT}, \star)$ is not a unitary algebra. If we consider the unitary $\overline{H}_{RT} = \mathbf{k} \oplus H_{RT}$, with the coproduct $\overline{\Delta}_{RT}$ defined by $\overline{\Delta}_{RT}(1) = 1 \otimes 1$ and

$$\overline{\Delta}_{RT}(x) = \underbrace{x\otimes 1}_{\in H_{RT}\otimes\mathbf{k}} + \underbrace{1\otimes x}_{\in\mathbf{k}\otimes H_{RT}} + \underbrace{\Delta_{RT}(x)}_{\in H_{RT}\otimes H_{RT}}, \forall x \in H_{RT},$$

then $(\overline{H}_{RT}, \star, \overline{\Delta}_{RT})$ is a unitary infinitesimal bialgebra in the sense of Loday and Ronco [15], and θ is its antipode, defined as the idempotent e in [15].

By definition of the coproduct Δ_{RT}:

Proposition 4.4. *Let F be a planar rooted forest. We denote its vertices according to the order $\leqslant_{h,\ell}$:*

$$v_1 \leqslant_{h,\ell} \cdots \leqslant_{h,\ell} v_n.$$

Then:

$$\theta(F) = \sum_{k=0}^{n} \sum_{1\leqslant i_1<\cdots<i_k\leqslant n} (-1)^k F_{|\{v_{i_k}+1,\ldots,v_n\}} \bullet F_{|\{v_{i_{k-1}}+1,\ldots,v_{i_k}-1\}} \bullet \cdots \bullet F_{|\{v_{i_1}+1,\ldots,v_{i_2}-1\}} \bullet F_{|\{v_1,\ldots,v_{i_1}-1\}}. \tag{11}$$

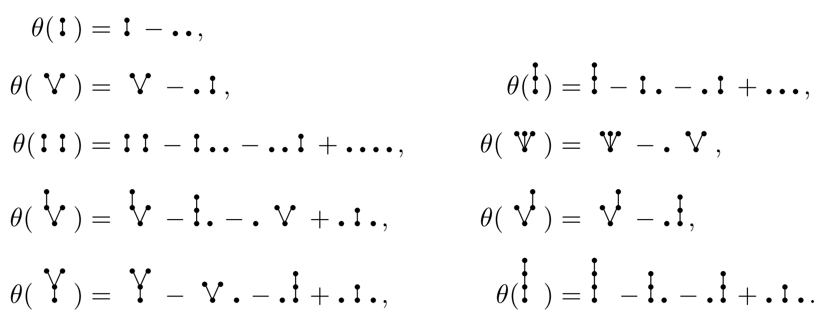
Example 4.5. Applying this formula for a given n, we obtain, after simplifications:

if $n = 2$, $\quad \theta(F) = F - \bullet\bullet,$

if $n = 3$, $\quad \theta(F) = F - F_{|\{v_2,v_3\}}\bullet - \bullet F_{|\{v_1,v_2\}} + \bullet\bullet\bullet,$

if $n = 4$, $\quad \theta(F) = F - F_{|\{v_2,v_3,v_4\}}\bullet - \bullet F_{|\{v_1,v_2,v_3\}} + \bullet F_{|\{v_2,v_3\}}\bullet.$

Consequently:



Corollary 4.6. *Let V be the vector space $Prim(H_{RT})$. We put:*

$$T_+(V) = \bigoplus_{n\geqslant 1} V^{\otimes n}.$$

We give it the concatenation product m_{conc} and the deconcatenation coproduct Δ_{dec}:

$$\Delta_{dec}(v_1\cdots v_n) = \sum_{i=1}^{n-1} v_1\cdots v_i \otimes v_{i+1}\cdots v_n,\ \forall v_1,\ldots,v_n \in V.$$

Then the following map is an algebra and a coalgebra isomorphism:

$$\Upsilon : \begin{cases} (T_+(V), m_{conc}, \Delta_{dec}) & \longrightarrow & (H_{RT}, \star, \Delta_{RT}) \\ v_1\cdots v_n & \mapsto & v_1\star\cdots\star v_n. \end{cases}$$

Proof. Obviously, Υ is an algebra morphism. By the compatibility between the product $\star$ and the coproduct Δ_{RT}, for any $v_1, \ldots, v_n \in Prim(H)$:

$$\Delta_{RT}(v_1 \star \cdots \star v_n) = \sum_{i=1}^{n} v_1 \star \cdots \star v_i \otimes v_{i+1} \star \cdots \star v_n.$$

Consequently, θ is a coalgebra morphism.

The gradation of H_{RT} induces a gradation of $Prim(H_{RT}) = V$, which in turn gives a gradation of $T_+(V)$:

$$T_+(V)_n = \bigoplus_{k=1}^{n} \bigoplus_{n_1+\cdots+n_k=n} V_{n_1} \otimes \cdots \otimes V_{n_k}, \ \forall n \geqslant 1.$$

As the product $\star$ is homogeneous of degree 0, Υ is homogeneous of degree 0. Let us assume that Υ is not injective, and let $x \in Ker(\Upsilon)$, nonzero, of minimal degree n. Then:

$$0 = \Delta_{RT} \circ \Upsilon(x) = (\Upsilon \otimes \Upsilon) \circ \Delta_{dec}(x).$$

Moreover,

$$\Delta_{dec}(x) \in \sum_{k=1}^{n-1} T_+(V)_k \otimes T_+(V)_{n-k}.$$

By definition of n, $\Upsilon_{|T_+(V)_k}$ is injective if $k < n$, so $\Delta_{dec}(x) = 0$, and $x \in Ker(\Delta_{dec}) = V$. Therefore, $\Upsilon(x) = x = 0$: this is a contradiction, and Υ is injective.

In order to prove that Υ is surjective, it is enough to prove that $Prim(H_{RT})$ generates the algebra $(H_{RT}, \star)$. Let F be a planar rooted forest with n vertices, let us prove that it belongs to the subalgebra A generated by $Prim(H)$ by induction on n. If $n = 1$, then $F = \bullet \in V$ and this is obvious. Otherwise, let us put $y = F - \Upsilon(F)$. By definition of $\Upsilon(F)$, y is a linear span of forests of the form $G = G_1 \bullet G_2$, with n vertices. By the induction hypothesis, $G_1, G_2 \in A$, so $G = G_1 \star G_2 \in A$ and finally $y \in A$. As $\Upsilon(F) \in V \subseteq A$, $F = y + \Upsilon(F) \in A$. $\Box$

Let us simplify the writing of $\theta(F)$, in order to avoid the simplifications we observed in the examples.

Definition 4.7. Let us define a sequence of scalars $(c(n))_{n\geqslant 0}$ by:

$$c(n) = \begin{cases} 1, & \text{if } n \equiv 0 \pmod 3, \\ 0, & \text{if } n \equiv 1 \pmod 3, \\ -1, & \text{if } n \equiv 2 \pmod 3. \end{cases}$$

Recall that a composition is a finite sequence of positive integers. If $(n_1,\ldots,n_k)$ is a composition, we can write it as:

$$(n_1,\ldots,n_k)=(\underbrace{1,\ldots,1}_{\alpha_0},b_1,\underbrace{1,\ldots,1}_{\alpha_1},\ldots,\underbrace{1,\ldots,1}_{\alpha_{p-1}},b_p,\underbrace{1,\ldots,1}_{\alpha_p}),$$

where $p\geqslant 0$, $b_1,\ldots,b_p\geqslant 2$, $\alpha_0,\ldots,\alpha_p\geqslant 0$. This is abbreviated as

$$(n_1,\ldots,n_k)=1^{\alpha_0}b_1 1^{\alpha_1}\cdots 1^{\alpha_{p-1}}b_p 1^{\alpha_p}.$$

We then put:

$$c(n_1,\ldots,n_k)=\begin{cases} c(\alpha_0), & \text{if } p=0,\\ c(\alpha_0+2)c(\alpha_1+1)\cdots c(\alpha_{p-1}+1)c(\alpha_p+2), & \text{if } p\geqslant 1.\end{cases}$$

Corollary 4.8. *Let F be a planar rooted forest. We denote its vertices according to the order $\leqslant_{h,\ell}$:*

$$v_1\leqslant_{h,\ell}\cdots\leqslant_{h,\ell} v_n.$$

Then:

$$\theta(F)=\sum_{n_1+\cdots+n_k=n} c(n_1,\ldots,n_k)F_{|\{v_{n_1+\cdots+n_{k-1}+1},\ldots,v_{n_1+\cdots+n_k}\}}\cdots F_{|\{v_1,\ldots,v_{n_1}\}}.$$

Proof. Interpreting the trees $\bullet$ in (11) as $F_{|\{v_i\}}$, we can rewrite, for any forest F with n vertices:

$$\theta(F)=\sum_{n_1+\cdots+n_k=n} a(n_1,\ldots,n_k)F_{|\{v_{n_1+\cdots+n_{k-1}+1},\ldots,v_{n_1+\cdots+n_k}\}}\cdots F_{|\{v_1,\ldots,v_{n_1}\}}, \tag{12}$$

for a certain family of scalars $a(n_1,\ldots,n_k)$, independent of F. Let us prove that $a(n_1,\ldots,n_k)=c(n_1,\ldots,n_k)$ for any composition $(n_1,\ldots,n_k)$.

First case. We consider the case $p=0$, that is to say $(n_1,\ldots,n_k)=1^n$. The term $F_{|\{v_n\}}\cdots F_{|\{v_1\}}$ in (12) comes from the terms in (11) indexed by $1\leqslant i_1<\cdots<i_k\leqslant n$ with:

- $i_1\leqslant 2$.
- $i_k\geqslant n-1$.
- If $2\leqslant p\leqslant k$, then $i_p\leqslant i_{p-1}+2$.

Any such $(i_1,\ldots,i_k)$ contributes with $(-1)^k$. Note that, in $\mathbb{Q}[[X]]$:

$$\begin{aligned}\frac{1}{X}\sum_{l=1}^{\infty}(-1)^{l-1}(X+X^2)^l&=\sum_{l=1}^{\infty}(-1)^{l-1}\sum_{j_1,\ldots,j_l\in\{1,2\}}X^{j_1+\cdots+j_l-1}\\&=\sum_{m=1}^{\infty}\left(\sum_{\substack{j_1,\ldots,j_l\in\{1,2\},\\ j_1+\cdots+j_l=m}}(-1)^{l-1}\right)X^{m-1}\end{aligned}$$

$$= \sum_{m=1}^{\infty} \left(\sum_{\substack{1 \leqslant i_1 < \cdots < i_{l-1} \leqslant m-1, \\ i_1 \leqslant 2,\, i_{l-1} \geqslant m-2, \\ i_p \leqslant i_{p-1}+2 \text{ if } 2 \leqslant p \leqslant m-1}} (-1)^{l-1} \right) X^{m-1}$$

$$= \sum_{n=0}^{\infty} \left(\sum_{\substack{1 \leqslant i_1 < \cdots < i_k \leqslant n, \\ i_1 \leqslant 2,\, i_k \geqslant n-1, \\ i_p \leqslant i_{p-1}+2 \text{ if } 2 \leqslant p \leqslant n}} (-1)^{k} \right) X^{n}$$

$$= \sum_{n=0}^{\infty} a(1^n) X^n,$$

where in the third equality, $i_p = j_1 + \cdots + j_p$ for any p and $(k, n) = (l-1, m-1)$ for the fourth one. Therefore:

$$\sum_{n=0}^{\infty} a(1^n)X^n = \frac{1}{X}\sum_{l=1}^{\infty}(-1)^{l-1}(X+X^2)^l = -\frac{1}{X}\sum_{l=1}^{\infty}(-X-X^2)^l = \frac{1+X}{1+X+X^2} = \frac{X^2-1}{X^3-1}.$$

Hence:

$$a(1^0) = 1, \qquad a(1^1) = 0, \qquad a(1^2) = -1, \qquad a(1^n) = a(1^{n-3}) \text{ if } n \geqslant 3.$$

So $a(1^n) = c(n)$ for any $n \geqslant 0$.

Second case. We now assume that $p \geqslant 1$. For any $1 \leqslant i_1 < \cdots < i_k \leqslant n$ contributing in (12) to $a(n_1, \ldots, n_k)$, necessarily:

- If $\alpha_0 \geqslant 1$, α_0 belongs to $\{i_1, \ldots, i_k\}$.
- If $\alpha_p \geqslant 1$, $\alpha_0 + \cdots + \alpha_{p-1} + b_1 + \cdots + b_p + 1$ belongs to $\{i_1, \ldots, i_k\}$.
- If $1 \leqslant i \leqslant p-1$ and $\alpha_i \geqslant 1$, then $\alpha_0 + \cdots + \alpha_{i-1} + b_1 + \cdots + b_{i-1} + 1$ and $\alpha_0 + \cdots + \alpha_i + b_1 + \cdots + b_{i-1}$ belong to $\{i_1, \ldots, i_k\}$.

Separating the study for each block of 1 in $(n_1, \ldots, n_k)$, we observe that $a(n_1, \ldots, n_k)$ can be written as a product

$$a(n_1, \ldots, n_k) = a^{(0)}(\alpha_0) \cdots a^{(p)}(\alpha_p).$$

Mimicking the study of the first case:

$$a^{(0)}(\alpha_0) = \begin{cases} 1, & \text{if } \alpha_0 = 0, \\ -1, & \text{if } \alpha_0 = 1, \\ -a(1^{\alpha_0 - 1}), & \text{if } \alpha_0 \geqslant 2. \end{cases}$$

In all cases, we obtain that $a^{(0)}(\alpha_0) = -a(1^{\alpha_0+2}) = -c(\alpha_0+2)$. Similarly, $a^{(p)}(\alpha_p) = -c(\alpha_p+2)$. If $1 \leqslant i \leqslant p-1$:

$$a^{(i)}(\alpha_i) = \begin{cases} 0, & \text{if } \alpha_i = 0,\\ -1, & \text{if } \alpha_i = 1,\\ a(1^{\alpha_i-2}), & \text{if } \alpha_i \geqslant 2. \end{cases}$$

In all cases, we obtain that $a^{(i)}(\alpha_i) = a(1^{\alpha_0+1}) = c(\alpha_0+1)$. As a consequence, $a(n_1,\dots,n_k) = c(n_1,\dots,n_k)$. □

For any composition $(n_1,\dots,n_k)$, $c(n_1,\dots,n_k) \in \{-1,0,1\}$. Let us denote by t_n the number of compositions $(n_1,\dots,n_k)$, with $n_1+\dots+n_k = n$ and $c(n_1,\dots,n_k) \neq 0$.

Proposition 4.9. *In* $\mathbb{Q}[[X]]$*:*

$$\sum_{n=0}^{\infty} t_n X^n = \frac{1-X+2X^2}{1-X-2X^3}.$$

As a consequence, for any $n \geqslant 3$,

$$t_n = t_{n-1} + 2t_{n-3}.$$

Proof. By definition, t_n is the number of compositions $(n_1,\dots,n_k) = 1^{\alpha_0}b_1 1^{\alpha_1}\cdots 1^{\alpha_{p-1}}b_p 1^{\alpha_p}$ with $n_1+\dots+n_k = n$, such that:

- If $p=0$, then $\alpha_i \equiv 0[3]$ or $\alpha_i \equiv 2[3]$.
- if $p \geqslant 1$:
  - $\alpha_0 \equiv 0[3]$ or $\alpha_0 \equiv 1[3]$.
  - $\alpha_p \equiv 0[3]$ or $\alpha_p \equiv 1[3]$.
  - If $1 \leqslant i \leqslant p-1$, $\alpha_i \equiv 1[3]$ or $\alpha_i \equiv 2[3]$.

We shall use the following formal series:

$$P_0(X) = \sum_{k=0}^{\infty} X^{3k} + \sum_{k=0}^{\infty} X^{3k+2} = \frac{1+X^2}{1-X^3},$$

$$P_2(X) = \sum_{k=0}^{\infty} X^{3k} + \sum_{k=0}^{\infty} X^{3k+1} = \frac{1+X}{1-X^3},$$

$$P_1(X) = \sum_{k=0}^{\infty} X^{3k+1} + \sum_{k=0}^{\infty} X^{3k+2} = XP_2(X).$$

Then:

$$\sum_{n=0}^{\infty} t_n X^n = P_0(X) + \sum_{k=1}^{\infty} P_2(X)\left(\frac{X^2}{1-X}P_1(X)\right)^{k-1}\frac{X^2}{1-X}P_2(X)$$

$$= P_0(X) + \sum_{k=1}^{\infty} P_2(X)^{k+1} X^{k-1} \left(\frac{X^2}{1-X}\right)^k$$

$$= P_0(X) + \frac{P_2(X)}{X} \sum_{k=1}^{\infty} \left(\frac{X^3 P_2(X)}{1-X}\right)^k$$

$$= P_0(X) + \frac{P_2(X)}{X} \frac{\frac{X^3 P_2(X)}{1-X}}{1 - \frac{X^3 P_2(X)}{1-X}}$$

$$= \frac{1 - X + 2X^2}{1 - X - 2X^3}.$$

□

Here are the first values of t_n:

n	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
t_n	1	0	2	4	4	8	16	24	40	72	120	200	344	584	984	1672

Example 4.10. Let us consider the case $n = 5$. There are 8 contributing terms in (12), corresponding to the compositions:

$$(5), \quad (4,1), \quad (1,4), \quad (1,3,1), \quad (2,1,2), \quad (2,1,1,1), \quad (1,1,1,2), \quad (1,1,1,1,1).$$

Therefore, if F has 5 vertices:

$$\theta(F) = F - \bullet F_{|\{1,2,3,4\}} - F_{|\{2,3,4,5\}}\bullet + \bullet F_{|\{2,3,4\}}\bullet - F_{|\{4,5\}}\bullet F_{|\{1,2\}}$$
$$+ \bullet\bullet\bullet F_{|\{1,2\}} + F_{|\{4,5\}}\bullet\bullet\bullet - \bullet\bullet\bullet\bullet\bullet.$$

4.3. Dual coproduct of $\star$. The product $\star$ on H_{RT} induces by duality a coproduct $\blacktriangle$ on H_{RT}^*.

Lemma 4.11. *Let $T_1, \ldots, T_k \in \mathcal{T}$. Then, in H_{RT}^*:*

$$\blacktriangle(Z_{T_1 \cdots T_k}) = \sum_{i=1}^{k} \delta_{T_i, \bullet} Z_{T_1 \cdots T_{i-1}} \otimes Z_{T_{i+1} \cdots T_k}. \tag{13}$$

Proof. Let $F, G \in M(\mathcal{T})$. Then

$$\blacktriangle(Z_{T_1 \cdots T_k})(F \otimes G) = Z_{T_1 \cdots T_k}(F \star G)$$
$$= Z_{T_1 \cdots T_k}(F \bullet G)$$
$$= \delta_{T_1 \cdots T_k, F \bullet G}$$
$$= \sum_{i=1}^{k} \delta_{T_1 \cdots T_{i-1}, F} \delta_{T_i, \bullet} \delta_{T_{i+1} \cdots T_k, G}$$

$$= \left(\sum_{i=1}^{k} \delta_{T_i, \bullet} Z_{T_1 \cdots T_{i-1}} \otimes Z_{T_{i+1} \cdots T_k} \right) (F \otimes G),$$

which implies (13). □

Dualizing Proposition 4.1:

Proposition 4.12. *For any $f, g \in H_{RT}^*$:*

$$\blacktriangle(f \diamond g) = f \diamond \blacktriangle(g) + \blacktriangle(f) \diamond g + f \otimes g.$$

Proof. Let $f, g \in H_{RT}^*$. For any $x, y \in H_{RT}$:

$$\begin{aligned}
\blacktriangle(f \diamond g)(x \otimes y) &= (f \diamond g)(x \star y) \\
&= (f \otimes g)\Delta_{RT}(x \star y) \\
&= (f \otimes g)(x \star \Delta_{RT}(y) + \Delta_{RT}(x) \star y + x \otimes y) \\
&= (\blacktriangle(f) \diamond g + f \diamond \blacktriangle(g) + f \otimes g)(x \otimes y),
\end{aligned}$$

which implies the result. □

Let us then dualize Theorem 4.2:

Theorem 4.13. *The transpose of θ is the map θ^* given by:*

$$\theta^*(f) := \sum_{k=1}^{\infty} (-1)^{k+1} \diamond^{(k-1)} \circ \blacktriangle^{(k-1)}(f), \, \forall f \in H_{RT}^*,$$

where $\blacktriangle^{(l)} : H_{RT}^* \longrightarrow (H_{RT}^*)^{\otimes(l+1)}$ and $\diamond^{(l)} : (H_{RT}^*)^{\otimes(l+1)} \longrightarrow H_{RT}^*$ are inductively defined:

$$\blacktriangle^{(0)} = \mathrm{id}_{H_{RT}^*}, \qquad \diamond^{(0)} = \mathrm{id}_{H_{RT}^*},$$
$$\blacktriangle^{(l+1)} = (\blacktriangle^{(l)} \otimes \mathrm{id}_{H_{RT}^*}) \circ \blacktriangle_{RT}, \qquad \diamond^{(l+1)} = \diamond \circ (\diamond^{(l)} \otimes \mathrm{id}_{H_{RT}^*}).$$

Then θ^* is a projector on $Ker(\blacktriangle) = \mathbf{k}\{Z_F$, no tree of F is equal to $\bullet\}$. The kernel of θ^* is

$$Ker(\theta^*) = H_{RT}^* \diamond H_{RT}^*.$$

Proof. The description of θ^* comes from $\diamond = \Delta_{RT}^*$ and $\blacktriangle = \star^*$. As θ is the projection on $Ker(\Delta_{RT})$ which vanishes on $Im(\star)$, θ^* is the projection on $Im(\star)^\perp$ which vanishes on $Ker(\Delta_{RT})^\perp$, and:

$$Im(\star)^\perp = Ker(\star^*) = Ker(\blacktriangle),$$
$$Ker(\Delta_{RT})^\perp = Im(\Delta_{RT}^*) = Im(\diamond) = H_{RT}^* \diamond H_{RT}^*.$$

The description of $Ker(\blacktriangle)$ is immediate. □

Consequently,

$$H_{RT}^* = Ker(\blacktriangle) \oplus H_{RT}^* \diamond H_{RT}^*.$$

As $(H_{RT}^*, \diamond)$ is a free non unitary algebra, any complement of $H_{RT}^* \diamond H_{RT}^*$ freely generates it as an algebra. Hence:

Corollary 4.14. *The algebra* $(H_{RT}^*, \diamond)$ *is freely generated by the elements* Z_F*, where* F *is a forest with no tree equal to* $\bullet$.

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