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Research Article

Complete $(k, 2)$-Arcs in the Projective Plane Order 5

Elif ALTINTAŞ KAHRİMAN [1], Ayşe BAYAR [2]

Abstract
In this study, the complete $(k,2)$-arcs in the projective plane of order 5 coordinatized by elements of GF(5) are investigated by applying the algorithm (implemented in C#) to determine arcs.

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[1] *Department of Software Engineering, Haliç University, 34060, Istanbul, Türkiye.*
[2] *Department of Mathematics and Computer, Eskişehir Osmangazi University, 26040, Eskişehir, Türkiye.*
[1] ✉elifaltintaskahriman@halic.edu.tr, [2] ✉akorkmaz@ogu.edu.tr
Corresponding author: Elif ALTINTAŞ KAHRİMAN
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1. Introduction

A projective plane π consist of a set $\mathscr{P}$ of points and a set $\mathscr{L}$ of subsets of $\mathscr{P}$, called lines, such that every pair of points is contained in exactly one line, every two distinct lines intersect in exactly one point, and there exist four points in such a position that they pairwise define six distinct lines. A subplane of a projective plane π is a set $\mathscr{B}$ of points and lines which is itself a projective plane, relative to the incidence relation given in π.

Arcs play an important role in projective geometry and have a variety of applications in combinatorics and other fields. In a finite projective plane π (not necessarily Desarguesian) a set K of k $(k \geq 3)$ points such that no three points of K are collinear (on a line) is called a $k-$arc. If the plane π has order p then $k \leq p+2$, however the maximum value of k can only be achieved if p is even. By the Fundamental Theorem of Projective Geometry, in a plane of order p , a $(p+1)-$arc is called an oval and, if p is even, a $(p+2)-$arc is called a hyperoval. A general reference for ovals is Hirschfeld [1]. There are known plenty of examples of arcs in projective planes; all complete $(k,2)$-arcs containing complete quadrangles which generate the Fano planes in the projective plane whose algebraic structure is the left nearfield of order 9 are examined in [2, 3]. The algorithm to determine and classify Fano subplanes of the projective plane of order 9 coordinatized by elements of a left nearfield of order 9 is given in [4].

Fano configurations in $PG(5,2)$ are determined in [5]. The algorithms are given for the classification of $(k,3)$-arcs in the projective plane of order 9 and order 25 over the smallest Cartesian Group in [6, 7]. In [8] by Qassim, $(8,4)-$arcs of the projective plane of order five over GF(5) were examined [8].

The main purpose of this study is to investigate all $(k,2)$-arcs of the projective plane over GF(5) with the help of the numbers $0,1,2,3,4$ and the irreducible polynomial $f(x) = x^3+2x^2+x-1$.

2. PG$(2,5)$ projective plane

In this section, some relevant definitions of projective plane, Left Nearfield and their theorems are reminded.

Definition 1. *An (axiomatic) projective plane P is an incidence structure $(N,D,\circ)$ with N a set of points, D a set of lines and $\circ$ an incidence relations, such that the following axioms are satisfied:*

i. *every pair of distinct points are incident with an unique common line;*

ii. *every pair of distinct lines are incident with an unique common point;*

iii. *P contains a set of four points with the property that no three of them are incident with a common line.*

A closed configuration S of P is a subset of $N\cup D$ that is closed under taking intersection points of any pair of lines in S and lines spanned by any pair of distinct points of S. We denote the line in P spanned by the points p and q by $\langle p,q\rangle$.

Definition 2. *(see [9]) Let $V=V(n+1,K)$ be vector space with $n+1$ dimension on the field K. For an equivalence relation on the vectors of $V-\{0\}$,*

$$X=\{x_1,\ x_2,\ ...,\ x_n\},Y=\{y_1,\ y_2,\ ...,\ y_n\}\in\ V-\{0\},$$

and $\forall t\in K_0$, such that $X\sim Y\Leftrightarrow y_i=tx_i,\ i=1,\ 2,\ ...\ n$, the equivalence classes on $V-\{0\}$ are one-dimensional subspaces formed by subtracting the zero vector from V. The set of these equivalence classes is called the n-dimensional projective space over the field K and it is indicated by $PG(n,K)$. If the qth order Galois field is taken as the field K, the projective space coordinated with the elements of this field is of order q. The obtained n-dimensional projective space is denoted by $PG(n,q)$.

Theorem 1 (see [9]) Let F be any field. A point-line geometry is a triple $(N,D,\circ)$ consisting of the points set N, the lines set D determined algebraically with the elements of the field F and the incidence relation $\circ$. Obviously,

$$\begin{aligned} N= &\quad \{(x_1,\ x_2,\ x_3):x_i\in S,\ (x_1,\ x_2,\ x_3)\neq(0,\ 0,\ 0),\ (x_1,\ x_2,\ x_3)\equiv\lambda\,(x_1,\ x_2,\ x_3),\ \lambda\in F-\{0\}\},\\ D= &\quad \{[a_1,\ a_2,\ a_3]:a_i\in S,\ [a_1,\ a_2,\ a_3]\neq(0,\ 0,\ 0),\ [a_1,\ a_2,\ a_3]\equiv\mu\,[a_1,\ a_2,\ a_3],\ \mu\in F-\{0\}\},\end{aligned}$$

and

$$\circ:(x_1,\ x_2,\ x_3)\circ[a_1,\ a_2,\ a_3]\Leftrightarrow a_1x_1+a_2x_2+a_3\ x_3=0.$$

Definition 3. *(see [9]) Any point in N is represented by a triple (x_1,x_2,x_3) where x_1,x_2,x_3 are not all zero. Nonzero multiples of a triple represent the same point. Similarly, any line in D is represented by a triple $[a_1,a_2,a_3]$ where a_1,a_2,a_3 are not all zero. This point-line geometry $(N,D,\circ)$ defined by F is a projective plane and is denoted by P_2F. Let r and p be a positive integer and a prime number, respectively. The projective plane of order $n=p^r$ over the finite Galois field $F=GF(p^r)$ of p^r elements by $P_2F=PG(2,p^r)$.*

Definition 4. *The set F with the binary operations $+$ and $\cdot$ is called a Left Nearfield if the following conditions hold:*

i. *$(F,+)$ is an abelian group.*

ii. *For $\forall a,b,c\in F$, $(a\cdot b)\cdot c=a\cdot(b\cdot c)$.*

iii. *For $\forall a,b,c\in F$, $a\cdot(b+c)=(a\cdot b)+(a\cdot c)$.*

iv. *For $\forall a\in F$, F contains an element 1 such that $1\cdot a=a\cdot 1=a$.*

v. *For every non-zero element a of F, there exist an element a^{-1} such that $a\cdot a^{-1}=a^{-1}\cdot a=1$.*

The original construction of Hall planes was based on a Hall quasifield (also called a Hall system) [10], [11]. To build a Hall quasifield, start with a Galois field $F=GF(p^n)$, for p a prime and a quadratic irreducible polynomial $f(t)=t^2-rt-s$ over F.

For this study, we consider $PG(2,5)$ which is constructed over $GF(5)$ under irreducible polynomial $f(x)=x^3+2x^2+x-1$ with the elements $0,1,2,3,4$ of $GF(5)$ having 31 points and 31 lines. There are 6 points on every line and 6 lines through every points of $PG(2,5)$ [8].

The point set N of the projective plane of $PG(2,5)$ is $N=\{N_i\,|i=1,2,...,31\}$, where $N_1=(0,\ 0,\ 1)$, $N_2=(1,\ 1,\ 1)$, $N_3=(1,\ 2,\ 2)$, $N_4=(1,\ 4,\ 2)$, $N_5=(1,\ 4,\ 3)$, $N_6=(1,\ 3,\ 4)$, $N_7=(1,\ 0,\ 3)$, $N_8=(1,\ 3,\ 1)$, $N_9=(1,\ 2,\ 4)$, $N_{10}=(1,\ 0,\ 4)$, $N_{11}=(1,\ 0,\ 1)$, $N_{12}=(1,\ 2,\ 1)$, $N_{13}=(1,\ 2,\ 3)$, $N_{14}=(1,\ 3,\ 0)$, $N_{15}=(0,\ 1,\ 3)$, $N_{16}=(1,\ 1,\ 3)$, $N_{17}=(1,\ 3,\ 3)$, $N_{18}=(1,\ 3,\ 2)$, $N_{19}=(1,\ 4,\ 0)$, $N_{20}=(0,\ 1,\ 4)$, $N_{21}=(1,\ 1,\ 0)$, $N_{22}=(0,\ 1,\ 1)$, $N_{23}=(1,\ 1,\ 2)$, $N_{24}=(1,\ 4,\ 4)$, $N_{25}=(1,\ 0,\ 2)$, $N_{26}=(1,\ 4,\ 1)$, $N_{27}=(1,\ 2,\ 0)$, $N_{28}=(0,\ 1,\ 2)$, $N_{29}=(1,\ 1,\ 4)$, $N_{30}=(1,\ 0,\ 0)$ and $N_{31}=(0,\ 1,\ 0)$.

The incident relation table of $PG(2,5)$ is given the following:

L_1	2	3	17	22	24	30
L_2	3	4	18	23	25	31
L_3	4	5	19	24	26	1
L_4	5	6	20	25	27	2
L_5	6	7	21	26	28	3
L_6	7	8	22	27	29	4
L_7	8	9	23	28	30	5
L_8	9	10	24	29	31	6
L_9	10	11	25	30	1	7
L_{10}	11	12	26	31	2	8
L_{11}	12	13	27	1	3	9
L_{12}	13	14	28	2	4	10
L_{13}	14	15	29	3	5	11
L_{14}	15	16	30	4	6	12
L_{15}	16	17	31	5	7	13
L_{16}	17	18	1	6	8	14
L_{17}	18	19	2	7	9	15
L_{18}	19	20	3	8	10	16
L_{19}	20	21	4	9	11	17
L_{20}	21	22	5	10	12	18
L_{21}	22	23	6	11	13	19
L_{22}	23	24	7	12	14	20
L_{23}	24	25	8	13	15	21
L_{24}	25	26	9	14	16	22
L_{25}	26	27	10	15	17	23
L_{26}	27	28	11	16	18	24
L_{27}	28	29	12	17	19	25
L_{28}	29	30	13	18	20	26
L_{29}	30	31	14	19	21	27
L_{30}	31	1	15	20	22	28
L_{31}	1	2	16	21	23	29

Table 1. The incident relation

3. Some properties of arcs

A $k-arc$ in a finite projective or affine plane is a set of k points no three of which are collinear. A $k-arc$ is complete if it is not contained in a $(k+1)-arc$. A line L is secant, tangent or passant to an arc if they have $2,1$ or 0 in common, respectively. In a plane of order q, a $(q+1)-arc$ is called an oval and if q is even, a $(q+2)-arc$ is called a hyperoval.

Now we take the set $A=\{O,I,X,P\}$ such that $I=(1,1,1),X=(1,0,0),O=(0,0,1),P=(1,a,b)$ with $a,b\in GF(5)$. We give an algorithm to find complete $(k,2)-arcs$ of this projective plane by applying the algorithm(implemented C#) as follows:

Steps of algorithm

$A \leftarrow Read(ExcelFile)$
$B \leftarrow Read(TextFile)$
$C \leftarrow A$
$while\ s(C) > 0$
$\quad B_i \leftarrow input(b), \{b | b \in C, b \notin B, i = s(B)+1\}$
$\quad j = 1$
$\quad while\ j \leqslant s(B)$
$\quad\quad for\ k = (j+1)\ to\ s(B)$
$\quad\quad\quad m \leftarrow the\ index\ of\ row\ on\ B_j, B_k$
$\quad\quad\quad D \leftarrow A_{mn}; \{A_{mn} | A_{mn} \neq B_j, A_{mn} \neq B_k, n = 1, ..., 10\}$
$\quad\quad\quad remove\ a\ from\ A; \{a | a \in A, a \in D\}$
$\quad\quad\quad C \leftarrow c; \{c | c \in A, c \notin C\}$
$\quad\quad endfor$
$\quad\quad j = j+1$
$\quad end\ while$
$end\ while$

As a result of application of this algorithm, all complete $(6,2)-$arcs are obtained as follows: $\{1,2,30,4,8,20\}$, $\{1,2,30,4,9,31\}$, $\{1,2,30,4,18,27\}$, $\{1,2,30,5,12,14\}$, $\{1,2,30,5,13,15\}$, $\{1,2,30,5,18,31\}$, $\{1,2,30,6,9,26\}$, $\{1,2,30,6,13,31\}$, $\{1,2,30,6,19,28\}$, $\{1,2,30,8,13,19\}$, $\{1,2,30,8,15,27\}$, $\{1,2,30,9,14,27\}$, $\{1,2,30,12,18,28\}$, $\{1,2,30,12,19,20\}$, $\{1,2,30,14,15,26\}$ and $\{1,2,30,26,27,28\}$.

4. Conclusions

In this study, an algorithm is given to determine complete $(k,2)-$arcs containing the quadrangle $A = O,I,X,P$ such that $I = (1,1,1), X = (1,0,0), O = (0,0,1), P = (1,a,b)$ with $a,b \in GF(5)$. Consequently, only 16 complete $(6,2)-$arcs are found and there is no $(k,2)-$arc where $k \neq 6$. All these determined $(6,2)-$arcs form the basis for projective space and also satisfy the Fano axiom. For the future research, $(k,n)-$arcs $n = 3,4,...$ can be examined in this projective plane.

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