On the boundedness of $B$-Riesz potential and its commutators on generalized weighted $B$-Morrey spaces

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Abstract

In the present paper, we shall investigate a characterization for the boundedness of the $B$-Riesz potential and its commutators on the generalized weighted $B$-Morrey spaces. We also give a characterization for the generalized weighted $B$-Morrey spaces via the boundedness of the Riesz potential and its commutators generated by generalized translate operators associated with Laplace-Bessel differential operator.

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1. Introduction

The classical Morrey spaces were introduced by Morrey [26] to study the local behavior of solutions to second-order elliptic partial differential equations. Moreover, various Morrey spaces are defined in the process of study. In [11, 25, 28], Guliyev, Mizuhara and Nakai introduced generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$. In [22], Komori and Shirai defined weighted Morrey spaces $L^{p,k}(w)$. Guliyev [14] gave a concept of the generalized weighted Morrey spaces $M^{p,\varphi,w}(\mathbb{R}^n)$ which could be viewed as extension of both $M^{p,\varphi}(\mathbb{R}^n)$ and $L^{p,k}(w)$. Authors also studied the boundedness of the classical operators and their commutators in spaces $M^{p,\varphi,w}(\mathbb{R}^n)$.

The boundedness of Riesz potential operator and its commutators on certain function spaces and their characterizations play an important role in various area in harmonic analysis, etc. See for example [1–3, 5–7, 9, 15–18, 22, 25, 26, 28, 29, 31, 32] and the references therein.

Let us now present some of the studies obtained and considered in this study for the maximal operator and Riesz potential. Suppose that $f \in L^{loc}_1(\mathbb{R}^n)$. Let $M$ be a maximal operator and $I^\alpha$ be Riesz potential operator on $\mathbb{R}^n$ defined by

$$Mf(x) = \sup_{t>0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)|dy,$$
If reduces to the above conditions were imposed on singular integral operator \( \omega \) in the general setting of metric measure spaces obtained in \([72x770]\], the following statement was proved, containing the result in \([72x770]\]) obtained sufficient conditions on weights \( \omega_1 \) and \( \omega_2 \) ensuring the boundedness of integral operators \( T \) from \( M_{p,\omega_1}(\mathbb{R}^n) \) to \( M_{p,\omega_2}(\mathbb{R}^n) \). In \([72x770]\], the following statement was proved, containing the result in \([72x770]\] and in the general setting of metric measure spaces obtained in \([72x770]\). In these studies, the authors obtained sufficient conditions on weights \( \omega_1 \) and \( \omega_2 \) for the boundedness of the singular integral operator \( T \) from \( M_{p,\omega_1}(\mathbb{R}^n) \) to \( M_{p,\omega_2}(\mathbb{R}^n) \). In \([72x770]\), the following doubling conditions were imposed on \( \omega(r) \):

\[
c^{-1}\omega(r) \leq \omega(t) \leq c\omega(r), \tag{1.1}
\]
whenever \( r \leq t \leq 2r \), where \( c \geq 1 \) does not depend on \( t \) and \( r \), jointly with the condition:

\[
\int_r^\infty \omega^p(t) \frac{dt}{t} \leq C \omega^p(r) \tag{1.2}
\]

for the maximal or singular integral operator and the condition

\[
\int_r^\infty t^{\alpha p} \omega^p(t) \frac{dt}{t} \leq C t^{\alpha p} \omega^p(r) \tag{1.3}
\]

for potential and fractional maximal operators, where \( C > 0 \) does not depend on \( r \).

**Theorem 1.4** ([28]). Let \( 1 < p < \infty \) and \( \omega(r) \) satisfy conditions (1.1)-(1.2). Then the operators \( M \) and singular integral operator \( T \) are bounded in \( M_{p,\omega}(\mathbb{R}^n) \).

**Theorem 1.5** ([28]). Let \( 1 < p < \infty \), \( 0 < \alpha < \frac{n}{p} \), and \( \omega(t) \) satisfy conditions (1.1) and (1.3). Then the operators \( M^\alpha \) and \( P^\alpha \) are bounded from \( M_{p,\omega}(\mathbb{R}^n) \) to \( M_{q,\omega}(\mathbb{R}^n) \) with \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \).

Note that Theorems 1.6 and 1.7 do not require condition (1.1)

**Theorem 1.6** ([11]). Let \( 1 < p < \infty \) and \( \omega_1(r), \omega_2(r) \) be positive measurable functions satisfying the condition

\[
\int_r^\infty \omega_1(t) \frac{dt}{t} \leq c_1 \omega_2(r) \tag{1.4}
\]

with \( c_1 > 0 \) not depending on \( t > 0 \). Then the operators \( M \) and singular integral operator \( T \) are bounded from \( M_{p,\omega_1}(\mathbb{R}^n) \) to \( M_{p,\omega_2}(\mathbb{R}^n) \).

**Theorem 1.7** ([11]). Let \( 0 < \alpha < n \), \( 1 < p < \infty \), \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \) and \( \omega_1(r), \omega_2(r) \) be positive measurable functions satisfying the condition

\[
\int_r^\infty t^{\alpha} \omega_1(t) \frac{dt}{t} \leq c_1 \omega_2(r). \tag{1.5}
\]

Then the operators \( M^\alpha \) and \( P^\alpha \) are bounded from \( M_{p,\omega_1}(\mathbb{R}^n) \) to \( M_{q,\omega_2}(\mathbb{R}^n) \).

The results given so far are obtained in Morrey space and generalized Morrey spaces of the maximal operator \( M^\alpha \) and Riesz potential operator \( P^\alpha \). In this paper, we shall investigate the maximal operator \( M_{\alpha,\gamma} \) (\( B \)-Maximal operator) and the Riesz potential operator \( I_{\alpha,\gamma} \) (\( B \)-Riesz potential operator) related to the generalized translate operator associated with the Laplace-Bessel differential operator and its commutators on generalized weighted Morrey spaces \( M^{p,\varphi,w}(\mathbb{R}^n) \). We also give a characterization for the \( B \)-BMO space via the boundedness of the commutator of the \( B \)-Riesz potential \( I_{\alpha,\gamma} \). Our aim is to present these two different characterizations of generalized weighted \( B \)-Morrey spaces for \( I^\alpha \) and \( I_{\alpha,\gamma} \) and its commutators.

The maximal operator and potential operator related topics associated with the Laplace-Bessel differential operator have been investigated by many researchers, see B. Muckenhoupt and E. Stein [27], I. Kipriyanov [21], K. Trimeche [38], L. Lyakhov [24], K. Stempak [37], A.D. Gadjiev and I.A. Aliiev [10], V.S. Guliyev [12,13], V.S. Guliyev and J.J. Hasanov [16], J.J. Hasanov [18], A. Serbetci, I. Ekincioglu [2,15,33] and others [19].

Now, let us introduce the Riesz potential operator \( I_{\alpha,\gamma} \) (\( B \)-Riesz potential operator) related to the generalized translate operator associated with the Laplace-Bessel differential operator

\[
\Delta_B = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2} + \sum_{i=1}^{k} \frac{\gamma_i}{x_i} \frac{\partial}{\partial x_i}, \quad \gamma_1 > 0, \ldots, \gamma_k > 0.
\]

At first, we prove that the \( B \)-Riesz potential operator \( I_{\alpha,\gamma} \) and their commutators for \( 0 < \alpha < n + |\gamma| \) is bounded from the generalized weighted \( B \)-Morrey space \( M_{p,\omega_1,\varphi_1,\gamma}^{n}(\mathbb{R}^n_{k,+}) \).
to \( M_{q, \varphi_2, \gamma}(\mathbb{R}^n_{k+}) \), where \( \alpha/(n + |\gamma|) = 1/p - 1/q \), \( 1 \leq p < (n + |\gamma|)/\alpha \), \((\varphi_1, \varphi_2) \in \tilde{A}_{1+q/|\gamma|}(\mathbb{R}^n_{k+}), \frac{1}{p} + \frac{1}{p} = 1 \).

We now consider the generalized translation operator generated by the Laplace-Bessel differential operator \( \Delta_B \). Then, the \( B \)-maximal operator and \( B \)-Riesz potential associated with this operator are investigated in generalized weighted \( B \)-Morrey spaces. We obtain for the operator \( I_{\alpha, \gamma} \) to be bounded from generalized weighted \( B \)-Morrey space \( M_{p, \varphi_1, \varphi_2, \gamma}(\mathbb{R}^n_{k+}) \) to \( M_{q, \varphi_2, \gamma}(\mathbb{R}^n_{k+}) \) and from generalized weighted \( B \)-Morrey space \( M_{1, \varphi_1, \varphi_2, \gamma}(\mathbb{R}^n_{k+}) \) to weak generalized weighted \( B \)-Morrey space \( \text{WM}_{q, \varphi_2, \gamma}(\mathbb{R}^n_{k+}) \).

The structure of the paper is as follows. In first section, we present some definitions and auxiliary results. In second section, we introduce generalized \( B \)-Morrey spaces. In Section 3, the main results of the paper, the boundedness of the \( B \)-potential operator from \( B \)-Morrey space \( M_{p, \varphi_1, \varphi_2, \gamma}(\mathbb{R}^n_{k+}) \) to \( M_{q, \varphi_2, \gamma}(\mathbb{R}^n_{k+}) \), is proved. In the last section, the boundedness of the commutators of the \( B \)-potential operator from generalized weighted \( B \)-Morrey space \( M_{p, \varphi_1, \varphi_2, \gamma}(\mathbb{R}^n_{k+}) \) to \( M_{p, \varphi_2, \gamma}(\mathbb{R}^n_{k+}) \) is obtained.

2. Preliminaries

Let \( \mathbb{R}^n \) be \( n \)-dimensional Euclidean space. For \( 1 \leq k \leq n \), let \( x = (x_1, \ldots, x_n) \in \mathbb{R}^n \), \( x' = (x_1, \ldots, x_k) \in \mathbb{R}^k \) and \( x'' = (x_{k+1}, \ldots, x_n) \in \mathbb{R}^{n-k} \) such that \( x = (x', x'') \in \mathbb{R}^n \). Then, it is defined as \( \mathbb{R}^n_{k+} = \{ x = (x', x'') \in \mathbb{R}^n ; x_1 > 0, \ldots, x_k > 0 \} \).

for \( n \geq 2 \). Recall that \( E(x, r) = \{ y \in \mathbb{R}^n_{k+} ; |x-y| < r \} \) for a measurable subset \( E \subset \mathbb{R}^n_{k+} \). Let \( E_r = E(0, r) \). If \( \gamma = (\gamma_1, \ldots, \gamma_k) \) and \( \gamma_1 > 0, \ldots, \gamma_k > 0 \) then \( |\gamma| = \gamma_1 + \cdots + \gamma_k \) and \( (x')^\gamma = x_1^{\gamma_1} \cdots x_k^{\gamma_k} \). For \( x' \in \mathbb{R}^k \), we define the measures on \( E \) by equality

\[
|E|_\gamma = \int_E (x')^\gamma dx,
\]

then \( |E_r|_\gamma = \omega(n, k, \gamma) r^Q \), where \( Q = n + |\gamma| \) and \( \omega(n, k, \gamma) = \int_{E_1} (x')^\gamma dx = \frac{\pi \frac{n}{2}}{\Gamma \left( \frac{\gamma+1}{2} \right)} = \prod_{i=1}^{k} \frac{\Gamma \left( \frac{\gamma_i+1}{2} \right)}{\Gamma \left( \frac{\gamma_i}{2} \right)} \).

First, we define the generalized translate operator \( (B\text{-translate operator}) \) \( T^x, x \in \mathbb{R}^n \), on \( L^p(\mathbb{R}^n, d\nu) \) by equality

\[
T^x f(y) = C_{\gamma, k} \prod_{i=1}^{n} \int_0^{|y|} f((x', y')_\beta, x'' - y'') d\nu(\beta),
\]

where \( (x_i, y_i)_\beta = (x_i^2 - 2x_iy_i \cos \beta_i + y_i^2)^{\frac{1}{2}}, 1 \leq i \leq k, (x', y')_\beta = ((x_1, y_1)_\beta, \ldots, (x_k, y_k)_\beta), d\nu(\beta) = \prod_{i=1}^{k} \sin^{\gamma_i-1} \beta_i d\beta_1 \cdots d\beta_k, 1 \leq k \leq n \) and

\[
C_{\gamma, k} = \pi^{-\frac{k}{2}} \prod_{i=1}^{k} \frac{\Gamma \left( \frac{\gamma_i+1}{2} \right)}{\Gamma \left( \frac{\gamma_i}{2} \right)} = \frac{2^k}{\pi^k} \omega(2k, k, \gamma).
\]

It acts from \( L^p(\mathbb{R}^n, d\nu) \) to \( L^p(\mathbb{R}^n, d\nu) \) and \( ||T^x f||_p < ||f||_p \) and \( T^x 1 = 1 \) and \( L_p \)-boundedness.

We remark that the generalized translate operator \( T^x \) is closely connected with the Bessel differential operator \( B \) (for example, \( n = k = 1 \) see [23], \( n > 1, k = 1 \) see [21] and \( n, k > 1 \) see [24, 33, 34] for details).

Let \( L_{p, \varphi, \gamma}(\mathbb{R}^n_{k+}) \) be the space of Lebesgue measurable functions \( f \) such that

\[
\|f\|_{L_{p, \varphi, \gamma}(\mathbb{R}^n_{k+})} = \left( \int_{\mathbb{R}^n_{k+}} |f(x)|^p \varphi^p(x)(x')^\gamma dx \right)^{1/p} < \infty
\]
where \( 1 \leq p < \infty \). For \( p = \infty \) the space \( L_{\infty, \gamma}(\mathbb{R}^n_{k,+}) \) is defined by means of the usual modification
\[
\|f\|_{L_{\infty, \gamma}} = \|f\|_{L_{\infty, \infty}} = \text{ess sup}_{x \in \mathbb{R}^n_{k,+}} \varphi(x)|f(x)|.
\]

**Definition 2.1.** The weight function \( \varphi \) belongs to the class \( A_{p, \gamma}(\mathbb{R}^n_{k,+}) \) for \( 1 \leq p < \infty \), if
\[
\sup_{x \in \mathbb{R}^n_{k,+}, r > 0} \left( \frac{1}{|E(x,r)|^\gamma} \int_{E(x,r)} \varphi^p(y)(y')^\gamma dy \right)^{\frac{1}{p}} \left( \frac{1}{|E(x,r)|^\gamma} \int_{E(x,r)} \varphi^{-p}(y)(y')^\gamma dy \right)^{\frac{1}{p}} < \infty
\]
and \( \varphi \) belongs to \( A_{1, \gamma}(\mathbb{R}^n_{k,+}) \), if there exists a positive constant \( C \) such that for any \( x \in \mathbb{R}^n_{k,+} \) and \( r > 0 \)
\[
|E(x,r)|^{-\gamma} \int_{E(x,r)} \varphi(y)(y')^\gamma dy \leq C_{y \in E(x,r)} \frac{1}{\varphi(y)}.
\]

**Definition 2.2.** The weight function \( (\varphi_1, \varphi_2) \) belongs to the class \( \tilde{A}_{p, \gamma}(\mathbb{R}^n_{k,+}) \) for \( 1 < p < \infty \), if
\[
\sup_{x \in \mathbb{R}^n_{k,+}, r > 0} \left( \frac{1}{|E(x,r)|^\gamma} \int_{E(x,r)} \varphi_1^p(y)(y')^\gamma dy \right)^{\frac{1}{p}} \left( \frac{1}{|E(x,r)|^\gamma} \int_{E(x,r)} \varphi_1^{-p}(y)(y')^\gamma dy \right)^{\frac{1}{p}} < \infty.
\]

The generalized translate operator \( T^y \) generates the corresponding \( B \)-convolution
\[
(f \otimes g)(x) = \int_{\mathbb{R}^n_{k,+}} f(y)|T^y g(y)|(y')^\gamma dy,
\]
for which the Young inequality
\[
\|f \otimes g\|_{L_{r, \gamma}} \leq \|f\|_{L_{p, \gamma}} \|g\|_{L_{q, \gamma}}, \quad 1 \leq p, q \leq r \leq \infty, \quad \frac{1}{p} + \frac{1}{q} = \frac{1}{r} + 1
\]
holds.

**Lemma 2.3.** For all \( x \in \mathbb{R}^n_{k,+} \) the following equality is holds
\[
\int_{E_t} T^y g(x)(y')^\gamma dy = \int_{E((x,0), t)} g\left( \sqrt{z''^2 + z_1^2 + \cdots + z_k^2}, z'' \right) d\mu(z, \overline{z}),
\]
where \( E((x,0), t) = \{(z, \overline{z}) \in \mathbb{R}^n \times (0, \infty)^k : \|(z - x, \overline{z})\| < t\} \).

**Lemma 2.4.** For all \( x \in \mathbb{R}^n_{k,+} \) the following equality is holds
\[
\int_{\mathbb{R}^n_{k,+}} T^y g(x)\varphi(y)M_\gamma \chi_{E_t}(y)(y')^\gamma dy
\]
\[
= \int_{\mathbb{R}^n \times (0, \infty)^k} g\left( \sqrt{z''^2 + z_1^2 + \cdots + z_k^2}, z'' \right) \varphi(z, \overline{z}) M_\nu \chi_{E((x,0), t)} (z, \overline{z}) d\nu(z, \overline{z}),
\]
where \( E((x,0), t) = \{(z, \overline{z}) \in \mathbb{R}^n \times (0, \infty)^k : \|(z - x, \overline{z})\| < t\} \).

Lemmas 2.3 and 2.4 are straightforward via the following substitutions
\[
z'' = z''', z_i = y_i \cos \alpha_i, \quad \overline{z_i} = y_i \sin \alpha_i, \quad 0 \leq \alpha_i < \pi, \quad i = 1, \ldots, k,
\]
y \in \mathbb{R}^n_{k,+}, \quad \overline{z} = (\overline{z_1}, \ldots, \overline{z_k}), \quad (z, \overline{z}) \in \mathbb{R}^n \times (0, \infty)^k, \quad 1 \leq k \leq n.

**Definition 2.5** ([13]). Let \( 1 \leq p < \infty \) and \( 0 \leq \lambda \leq Q \). We denote by \( M_{p, \lambda, \gamma}(\mathbb{R}^n_{k,+}) \) Morrey space (\( \equiv B \)-Morrey space), associated with the Laplace-Bessel differential operator the set of locally integrable functions \( f(x), \ x \in \mathbb{R}^n_{k,+} \), with the finite norm
\[
\|f\|_{M_{p, \lambda, \gamma}} = \sup_{t > 0, \ x \in \mathbb{R}^n_{k,+}} \left( t^{-\lambda} \int_{E_t} |T^y f(x)(y')^\gamma dy \right)^{1/p}.
\]
Define the $B$-maximal operator of $f$ by
\[ M_\gamma f(x) = \sup_{r > 0} |E_r|^{-1} r \int_{E_r} |f(y)\gamma \cdot dy, \]
and the fractional $B$-maximal operator by
\[ M_{\alpha,\gamma} f(x) = \sup_{r > 0} |E_r|^{-1} \int_{E_r} |f(y)\gamma \cdot dy, \quad 0 \leq \alpha < Q, \]
and the $B$-Riesz potential by
\[ I_{\alpha,\gamma} f(x) = \int |f(y)\gamma \cdot dy, \quad 0 < \alpha < Q. \]
We write $M_{0,\gamma} f(x) = M_\gamma f(x)$ in the case $\alpha = 0$.

Let $\omega$ and $\varphi$ positive measurable weight functions. The norm in the spaces $M_{p,\omega,\gamma}(\mathbb{R}^n_{k,+})$ and $M_{p,\omega,\varphi,\gamma}(\mathbb{R}^n_{k,+})$ defined in two forms,
\[
\|f\|_{M_{p,\omega,\gamma}} = \sup_{x \in \mathbb{R}^n_{k,+}, t > 0} \omega(t) \left( \int_{E_t} T^\gamma f(x)^P(x) (y)\gamma \cdot dy \right)^{1/p},
\]
and
\[
\|f\|_{M_{p,\omega,\varphi,\gamma}} = \sup_{x \in \mathbb{R}^n_{k,+}, t > 0} \frac{1}{\omega(t)\varphi(t)} \left( \int_{E_t} T^\gamma f(x)^P(x) \varphi(y)(y)\gamma \cdot dy \right)^{1/p}.
\]
If $\omega(t) \equiv r^{-\frac{Q}{p}}$ then $M_{p,\omega,\gamma}(\mathbb{R}^n_{k,+}) \equiv L_{p,\gamma}(\mathbb{R}^n_{k,+})$, if $\omega(t) \equiv t^{-\frac{\lambda - Q}{p}}$, $0 \leq \lambda < Q$, then $M_{p,\omega,\gamma}(\mathbb{R}^n_{k,+}) \equiv M_{p,\lambda,\gamma}(\mathbb{R}^n_{k,+})$.

Denote by $M_1^\gamma$, the sharp $B$-maximal function defined by
\[ M_1^\gamma f(x) = \sup_{t > 0} |E(0,t)|^{-1} t \int_{E(0,t)} |T^\gamma f(x) - f_{E(0,t)}(x)| (y)\gamma \cdot dy, \]
where $f_{E(0,t)}(x) = |E(0,t)|^{-1} \int_{E(0,t)} T^\gamma f(x)(y)\gamma \cdot dy$.

$B - BMO$ space, $BMO_\gamma(\mathbb{R}^n_{k,+})$, defined as the space of locally integrable functions $f$ with finite norm
\[ \|f\|_{BMO_\gamma} = \sup_{t > 0, x \in \mathbb{R}^n_{k,+}} |E(0,t)|^{-1} \int_{E(0,t)} |T^\gamma f(x) - f_{E(0,t)}(x)| (y)\gamma \cdot dy < \infty, \]
or
\[ \|f\|_{BMO_\gamma} = \inf_C \sup_{t > 0, x \in \mathbb{R}^n_{k,+}} |E(0,t)|^{-1} \int_{E(0,t)} |T^\gamma f(x) - C(\gamma) (y)\gamma \cdot dy < \infty. \]

The following theorem was proved in [4].

**Theorem 2.6.**  
\[ \text{i) If Let } f \in L^{loc}_{1,\gamma}(\mathbb{R}^n_{k,+}). \]
\[ \sup_{t > 0, x \in \mathbb{R}^n_{k,+}} \left( |E(0,t)|^{-1} \int_{E(0,t)} |T^\gamma f(x) - f_{E(0,t)}(x)|^P(y)^\gamma \cdot dy \right)^{1/p} = \|f\|_{BMO_{p,\gamma}} < \infty, \]
then for any $1 < p < \infty$
\[ \|f\|_{BMO_\gamma} \leq \|f\|_{BMO_{p,\gamma}} \leq A_p \|f\|_{BMO_\gamma}, \]
where the constant $A_p$ depends only on $p$. 

ii) Let \( f \in BMO_\gamma(\mathbb{R}^n_{k,+}) \). Then, there is a constant \( C > 0 \) such that

\[
|f_{E(0,r)} - f_{E(0,t)}| \leq C\|f\|_{BMO_\gamma} \ln \frac{t}{r}, \quad 0 < 2r < t
\]

where \( C \) is independent of \( f, x, r \) and \( t \).

**Lemma 2.7** ([20]). Let \( 1 < p < \infty, \varphi \in A_{p,\gamma}(\mathbb{R}^n_{k,+}), b \in BMO_\gamma(\mathbb{R}^n_{k,+}) \). Then

\[
\|b\|_{BMO_\gamma} \approx \sup_{x \in \mathbb{R}^n_{k,+}, r > 0} \frac{\|T b(x) - b_{E(0,r)}\|_{L_{p,p,\gamma}(E(0,r))}}{\|\varphi\|_{L_{p,\gamma}(E(0,r))}}.
\]

3. \( B \)-Riesz potentials on generalized weighted \( B \)-Morrey spaces

**Theorem 3.1.** Let \( 0 < \alpha < Q, 1 < p < \frac{Q}{\alpha}, \frac{1}{p} - \frac{1}{q} = \frac{\alpha}{Q} \), \( (\varphi_1, \varphi_2) \in \tilde{A}_{1+\frac{n}{\alpha},\gamma}(\mathbb{R}^n_{k,+}) \), \( \frac{1}{p} + \frac{1}{p'} = 1 \) and \( \omega_1(r), \omega_2(r) \) be positive measurable functions satisfying the condition

\[
\int_0^\infty \omega_1(t)\|\varphi_1\|_{L_{p,\gamma}(E(0,r))} \frac{dr}{r} \leq C\omega_2(t). \tag{3.1}
\]

Then \( I_{\alpha,\gamma} \) is bounded from \( M_{p,\omega_1,\varphi_1,\gamma}(\mathbb{R}^n_{k,+}) \) to \( M_{q,\omega_2,\varphi_2,\gamma}(\mathbb{R}^n_{k,+}) \).

**Proof.** Let \( f \in M_{p,\omega_1,\varphi_1,\gamma}(\mathbb{R}^n_{k,+}) \). Then

\[
I_{\alpha,\gamma}f(x) = I_{\alpha,\gamma}f_1(x) + I_{\alpha,\gamma}f_2(x). \tag{3.2}
\]

Firstly, we estimate \( I_{\alpha,\gamma}f_1(x) \). By using the Hölder’s inequality, we have

\[
|I_{\alpha,\gamma}f_1(x)| \leq \int_{E(0,t)} T^y |f(x)| |y|^{\alpha - Q}(y')^\gamma dy
\]

\[
\leq \sum_{j=-\infty}^{-1} \left( 2^j t \right)^{\alpha - Q} \int_{E(0,2^{j+1}t)\setminus E(0,2^jt)} T^y |f(x)|(y')^\gamma dy
\]

\[
\leq \sum_{j=-\infty}^{-1} \left( 2^j t \right)^{\alpha - Q} \left( \int_{E(0,2^{j+1}t)\setminus E(0,2^jt)} T^y |f(x)|^p \varphi_1^p(y)(y')^\gamma dy \right)^{1/p}
\]

\[
\times \left( \int_{E(0,2^{j+1}t)\setminus E(0,2^jt)} \varphi_1^{-p'}(y)(y')^\gamma dy \right)^{1/p'}
\]

\[
\leq C\|f\|_{L_{p,\varphi_1,\gamma}(E(0,t))}\|\varphi_2\|_{L_{q,\gamma}(E(0,t))}^{-1}.
\]

By the inequality (3.1), we obtain

\[
\|I_{\alpha,\gamma}f_1\|_{L_{q,\varphi_2,\gamma}(E(0,t))} \leq C\|f\|_{L_{p,\varphi_1,\gamma}(E(0,t))}
\]

\[
\leq C\|f\|_{L_{p,\varphi_1,\gamma}(E(0,t))}\|\varphi_2\|_{L_{q,\gamma}(E(0,t))} \int_0^\infty \omega_1(r)\|\varphi_1\|_{L_{p,\gamma}(E(0,r))} \frac{dr}{r}
\]

\[
\leq C\|f\|_{M_{p,\omega_1,\varphi_1,\gamma}}\omega_2(t)\|\varphi_2\|_{L_{q,\gamma}(E(0,t))}.
\]

Hence, we have

\[
\|I_{\alpha,\gamma}f_1\|_{L_{q,\varphi_2,\gamma}(E(0,t))} \leq C\|f\|_{M_{p,\omega_1,\varphi_1,\gamma}}\omega_2(t)\|\varphi_2\|_{L_{q,\gamma}(E(0,t))} \tag{3.3}
\]
Now we estimate $I_{α,γ}f_2(x)$. By using the Hölder’s inequality, we get
\[
|I_{α,γ}f_2(x)| \leq \int_{\mathbb{R}^n_{k,+}\setminus E(0,t)} T^y|f(x)||y|^{α−Q}(y')^γ \, dy \\
\leq \sum_{j=0}^∞ \left(2^j t\right)^{α−Q} \int_{E(0,2^{j+1}t)\setminus E(0,2^jt)} T^y|f(x)|(y')^γ \, dy \\
\leq \sum_{j=0}^∞ \left(2^j t\right)^{α−Q} \left( \int_{E(0,2^{j+1}t)\setminus E(0,2^jt)} \varphi_1^{-p'}(y)(y')^γ \, dy \right)^{1/p'} \\
\times \left( \int_{E(0,2^{j+1}t)\setminus E(0,2^jt)} T^y |f(x)|^p \varphi_1^{p'}(y)(y')^γ \, dy \right)^{1/p} \\
\leq C\|f\|_{M_{p,ω_1,φ_1,γ}} \int_{t}^{∞} \frac{ω_1(r)}{r} \|φ_1\|_{L_{p,γ}(E(0,r))} \, dr.
\]
Thus, by the inequality (3.1), we obtain
\[
|I_{α,γ}f_2(x)| \leq C\|f\|_{M_{p,ω_1,φ_1,γ}} \omega_2(t). \tag{3.4}
\]
So, from (3.3) and (3.4), we have
\[
\|I_{α,γ}f\|_{L_{q,φ_2,γ}(E(0,t))} \leq \|I_{α,γ}f_1\|_{L_{q,φ_2,γ}(E(0,t))} + \|I_{α,γ}f_2\|_{L_{q,φ_2,γ}(E(0,t))} \\
\leq C\|f\|_{M_{p,ω_1,φ_1,γ}} \omega_2(t) \|φ_2\|_{L_{q,γ}(E(0,t))}.
\]
Finally $I_{α,γ}f \in M_{q,ω_2,φ_2,γ}(\mathbb{R}^n_{k,+})$ and
\[
\|I_{α,γ}f\|_{M_{q,ω_2,φ_2,γ}} \leq C\|f\|_{M_{p,ω_1,φ_1,γ}}.
\]

\begin{corollary}
Let $0 < α < Q$, $1 < p < \frac{Q}{α}$, $\frac{1}{p} - \frac{1}{q} = \frac{α}{Q}$, $(φ_1, φ_2) \in \tilde{A}_{1+\frac{α}{p},γ}(\mathbb{R}^n_{k,+})$. Then the operator $I_{α,γ}$ is bounded from $L_{p,φ_1,γ}(\mathbb{R}^n)$ to $L_{q,φ_2,γ}(\mathbb{R}^n)$.
\end{corollary}

\begin{corollary}
Let $0 < α < Q$, $1 < p < \frac{Q}{α}$, $\frac{1}{p} - \frac{1}{q} = \frac{α}{Q}$, $(φ_1, φ_2) \in \tilde{A}_{1+\frac{α}{p},γ}(\mathbb{R}^n_{k,+})$. Then the operator $M_{α,γ}$ is bounded from $L_{p,φ_1,γ}(\mathbb{R}^n)$ to $L_{q,φ_2,γ}(\mathbb{R}^n)$.
\end{corollary}

4. Commutators of $B$-Riesz potential on generalized weighted $B$-Morrey spaces

In this section, we consider commutators of the $B$-Riesz potential defined as the following equality
\[
[b, I_{α,γ}]f(x) = \int_{\mathbb{R}^n_{k,+}} (b(x) - b(y))|y|^{α−Q} T^y f(x)(y')^γ \, dy, \quad 0 < α < Q.
\]
Given a measurable function $b$ the operator $|b, I_{α,γ}|$ is defined by
\[
|b, I_{α,γ}]f(x) = \int_{\mathbb{R}^n_{k,+}} |b(x) - b(y)||y|^{α−Q} T^y f(x)(y')^γ \, dy, \quad 0 < α < Q.
\]

\begin{theorem}
Let $0 < α < Q$, $1 < p < \frac{Q}{α}$, $\frac{1}{p} - \frac{1}{q} = \frac{α}{Q}$, $b \in BMO_γ(\mathbb{R}^n_{k,+})$, $(φ_1, φ_2) \in \tilde{A}_{1+\frac{α}{p},γ}(\mathbb{R}^n_{k,+})$, $φ_1 \in A_{p,γ}(\mathbb{R}^n_{k,+})$ and $ω_1(r)$, $ω_2(r)$ be positive measurable functions satisfying the condition (3.1). Then $|b, I_{α,γ}|$ is bounded from $M_{p,ω_1,φ_1,γ}(\mathbb{R}^n_{k,+})$ to $M_{q,ω_2,φ_2,γ}(\mathbb{R}^n_{k,+})$.
\end{theorem}
Proof. Let $f \in \mathcal{M}_{p,\omega_1,\varphi_1,\gamma} \left( \mathbb{R}^n_{k,+} \right)$. Then
\[
|b, I_{\alpha,\gamma} f(x)| = \left( \int_{E(0,t)} + \int_{E(0,t)} \right) T^y \| |b - b(x)||f(x)||y|^{-Q}(y')^\gamma \, dy
\]
\[
= F_1(x, t) + F_2(x, t).
\]
Firstly, we estimate $F_1(x, t)$. By using the Hölder’s inequality, we have
\[
F_1(x, t) = \int_{E(0,t)} T^y \| |b - b(x)||f(x)||y|^{-Q}(y')^\gamma \, dy
\]
\[
\leq \sum_{j=-\infty}^{-1} (2^j t)^{-Q} \int_{E(0,2^{j+1}t) \setminus E(0,2jt)} T^y \| |b - b(x)||f(x)||y|^{-Q}(y')^\gamma \, dy
\]
\[
\leq \sum_{j=-\infty}^{-1} (2^j t)^{-Q} \left( \int_{E(0,2^{j+1}t) \setminus E(0,2jt)} |T^y b(x) - b|^{p'} \varphi_1^{-1}(y)(y')^\gamma \, dy \right)^{1/p'}
\]
\[
\times \left( \int_{E(0,2^{j+1}t) \setminus E(0,2jt)} |f(x)|^p \varphi_1(y)(y')^\gamma \, dy \right)^{1/p}
\]
\[
\leq C \|b\|_{BMO, \gamma} \|f\|_{L_{p,\varphi_1,\omega_1}(E(0,t))} \|\varphi_2\|_{L_{q,\gamma}(E(0,t))}^{-1}.
\]
By the inequality (3.1), we obtain
\[
\|F_1\|_{L_{q,\varphi_2,\omega_2,\gamma}(E(0,t))} \leq C \|b\|_{BMO, \gamma} \|f\|_{L_{p,\varphi_1,\omega_1,\gamma}(E(0,t))} \int_0^\infty \omega_1(r) \|\varphi_1\|_{L_{p,\gamma}(E(0,r))} \, dr
\]
\[
\leq C \|b\|_{BMO, \gamma} \|f\|_{L_{p,\varphi_1,\omega_1,\gamma}} \|\varphi_2\|_{L_{q,\gamma}}^{-1}.
\]
Hence we have
\[
\|F_1\|_{L_{q,\varphi_2,\omega_2,\gamma}(E(0,t))} \leq C \|b\|_{BMO, \gamma} \|f\|_{L_{p,\varphi_1,\omega_1,\gamma}} \|\varphi_2\|_{L_{q,\gamma}}^{-1}.
\] (4.1)
Now we estimate $F_2(x, t)$. By using the Hölder’s inequality, we get
\[
F_2(x, t) \leq \int_{\mathbb{R}^n_{k,+} \setminus E(0,t)} T^y \| |b - b(x)||f(x)||y|^{-Q}(y')^\gamma \, dy
\]
\[
\leq \sum_{j=0}^\infty (2^j t)^{-Q} \int_{E(0,2^{j+1}t) \setminus E(0,2jt)} T^y \| |b - b(x)||f(x)||y|^{-Q}(y')^\gamma \, dy
\]
\[
\leq \sum_{j=0}^\infty (2^j t)^{-Q} \left( \int_{E(0,2^{j+1}t) \setminus E(0,2jt)} |T^y b(x) - b|^{p'} \varphi_1^{-1}(y)(y')^\gamma \, dy \right)^{1/p'}
\]
\[
\times \left( \int_{E(0,2^{j+1}t) \setminus E(0,2jt)} |f(x)|^p \varphi_1(y)(y')^\gamma \, dy \right)^{1/p}
\]
\[
\leq C \|b\|_{BMO, \gamma} \|f\|_{M_{p,\varphi_1,\omega_1,\gamma}} \int_0^\infty \omega_1(r) \|\varphi_1\|_{L_{p,\gamma}(E(0,r))} \, dr \frac{1}{r}.
\]
Thus by the inequality (3.1), we have
\[
F_2(x, t) \leq C \|b\|_{BMO, \gamma} \|f\|_{M_{p,\varphi_1,\omega_1,\gamma}} \varphi_2(t).
\] (4.2)
Therefore, from (4.1) and (4.2), we obtain
\[ \| b, I_{\alpha,\gamma} |f| \|_{L^{q,\varphi,\gamma}(E(0,t))} \leq \| F_1 \|_{L^{q,\varphi,\gamma}(E(0,t))} + \| F_2 \|_{L^{q,\varphi,\gamma}(E(0,t))} \]
\[ \leq C \| b \|_{\text{BMO}_\gamma} \| f \|_{M_{p,\varphi_1,\varphi_2}(t)} \| \varphi_2 \|_{L^{q,\gamma}(E(0,t))}. \]
Finally, we get \( |b, I_{\alpha,\gamma}|f| \in M_{q,\omega_2,\varphi_2,\gamma}(\mathbb{R}^n_k) \) and
\[ \| b, I_{\alpha,\gamma} |f| \|_{M_{q,\omega_2,\varphi_2,\gamma}} \leq C \| b \|_{\text{BMO}_\gamma} \| f \|_{M_{p,\varphi_1,\varphi_2,\gamma}}. \]

\[ \square \]

**Corollary 4.2.** Let \( 0 < \alpha < Q, \ 1 < p < \frac{Q}{\alpha}, \ \frac{1}{p} - \frac{1}{Q} = \frac{\gamma}{\gamma}, \ b \in \text{BMO}_\gamma(\mathbb{R}^n_k), \ (\varphi_1, \varphi_2) \in \tilde{A}_{1+\frac{\gamma}{\alpha}}(\mathbb{R}^n_k) \) and \( \varphi_1 \in A_{p,\gamma}(\mathbb{R}^n_k) \). Then the operator \( |b, I_{\alpha,\gamma}| \) is bounded from \( L_{p,\varphi_1,\gamma}(\mathbb{R}^n) \) to \( L_{q,\varphi_2,\gamma}(\mathbb{R}^n) \).

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**References**

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