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On Relaxing the Identity Operator in Korovkin Theorem via Statistical Convergence

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Abstract. An operator version of the Korovkin theorem has recently been obtained by D. Popa. With the motivation of this result, we have extended it by using a more powerful convergence which also includes ordinary convergence. We have also presented an example to illustrate the strength of our theorem.

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1. Introduction and Background

The natural density of G is given by

$$\delta(G) = \lim_{n\to\infty} \frac{1}{n} |\{k \leq n : k \in G\}|,$$

provided that the limit exists where $G \subseteq \mathbb{N}$. With the use of this notion, ordinary convergence has been generalized by introducing statistical convergence: $\delta(G_\varepsilon) = \delta(\{k \in \mathbb{N} : |s_k - l| \geq \varepsilon\}) = 0$ holds for every $\varepsilon > 0$ then it is said that $s = (s_k)$ is statistically convergent to l and this is denoted by $st - \lim s = l$ [11, 12, 20]. Note that ordinary convergence implies statistical convergence to same limit, but the converse implication is not always valid. Not all properties of ordinary convergence are valid for statistical convergence, for example, a statistically convergent sequence is not necessarily bounded while a convergent sequence is. As it can be seen, statistical convergence replaces finite sets of indices in ordinary convergence with sets of density zero. Since it generalizes ordinary convergence, it is more effective. With these properties statistical convergence has many uses in different areas of mathematics and it has also been extended by using infinite matrices, power series [5, 6, 13, 18, 21]. Until 2002, this concept has not been used in approximation theory and from this direction Gadjiev and Orhan have been the first to combine statistical convergence and approximation theory [16]. Later than, this convergence and various extensions have all been used in approximation theory, especially Korovkin theory [2, 3, 7–10, 15, 16]. By using only three functions 1, x and x^2, Korovkin-type theorems state the uniform convergence of positive linear operators to the identity operator in $C[0, 1]$, the space of all real valued continuous functions on $[0, 1]$. Since these theorems are simple and efficient, Korovkin-type approximation theory is popular and well studied [1, 4, 14, 17, 22]. Also a very recent paper by D. Popa has dealt with the operator version of this theorem. It is noteworthy to recall this lemma and main theorem from [19].

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Lemma 1.1. *[19] (i) For every $\epsilon > 0$ there exists $\nu_\epsilon > 0$ such that*

$$|L(f)(t)T(1)(t) - L(1)(t)T(f)(t)| \leq \epsilon L(1)(t)T(1)(t) + \nu_\epsilon(L(e_2)(t)T(1)(t) - 2L(e_1)(t)T(e_1)(t) + L(1)(t)T(e_2)(t))$$

holds for every L, T positive linear operators from $C[a,b]$ to $C(H)$ and for every $t \in H$.

(ii) For every $\epsilon > 0$ there exists ν_ϵ such that

$$\|L(f)T(1) - L(1)T(f)\| \leq \epsilon\|L(1)\|\|T(1)\| + \nu_\epsilon(\|L(e_2) - T(e_2)\|\|T(1)\| + 2\|L(e_1) - T(e_1)\|\|T(e_1)\| + \|L(1) - T(1)\|\|T(e_2)\| + 2\Delta)$$

holds for every L, T positive linear operators from $C[a,b]$ to $C(H)$. Here, $\Delta = \|T(1)T(e_2) - [T(e_1)]^2\|$ and $1, e_j$ are defined from $[a,b]$ to $\mathbb{R}$ by $e_j(x) = x^j$, $j \in \mathbb{N}; a, b \in \mathbb{R}, a < b$, H is a compact Hausdorff space and $f \in C[a,b]$.

Theorem 1.2. *[19] Assume that, H be a compact Hausdorff space, L_n, T from $C[a,b]$ to $C(H)$ be positive linear operators such that $T(1)T(e_2) = [T(e_1)]^2$ and $T(1)(t) > 0$, for every $t \in H$. If $\lim L_n(1) = T(1)$, $\lim L_n(e_1) = T(e_1)$, $\lim L_n(e_2) = T(e_2)$ all uniformly on H, then $\lim L_n(f) = T(f)$ uniformly on H holds for any $f \in C[a,b]$.*

Furthermore, many and different kinds of examples which satisfy Theorem 1.2 can also be found in [19].

Here, our main goal is to study the operator version of Korovkin theorem by using statistical convergence which is more effective than ordinary convergence. We also present an example which is an interesting contribution to this new literature. We believe that this direction includes many open problems.

2. Main Results

In this section, we are ready to present our operator version of Korovkin theorem by using statistical convergence.

Theorem 2.1. *Assume that, H be a compact Hausdorff space, L_n, T from $C[a,b]$ to $C(H)$ be positive linear operators such that $T(1)T(e_2) = [T(e_1)]^2$ and $T(1)(t) > 0$, for every $t \in H$. If $st-\lim \|L_n(1) - T(1)\| = 0$, $st-\lim \|L_n(e_1) - T(e_1)\| = 0$, $st - \lim \|L_n(e_2) - T(e_2)\| = 0$, then we have $st - \lim \|L_n(f) - T(f)\| = 0$ for any $f \in C[a,b]$.*

Proof. Consider $f \in C[a,b]$ and arbitrary $\epsilon > 0$. From Lemma 1.1 and by using $\Delta = 0$, we have $\nu_\epsilon > 0$ such that

$$\|L_n(f)T(1) - L_n(1)T(f)\| \leq \epsilon\|L_n(1)\|\|T(1)\| + \nu_\epsilon(\|L_n(e_2) - T(e_2)\|\|T(1)\| + 2\|L_n(e_1) - T(e_1)\|\|T(e_1)\| + \|L_n(1) - T(1)\|\|T(e_2)\|)$$

holds for every $n \in \mathbb{N}$. From the hypotheses, we have subsets $G_i \subseteq \mathbb{N}$, $i = 0, 1, 2$ such that

$$\lim_{n \in G_i} \|L_n(e_i) - T(e_i)\| = 0.$$

Define $G = G_0 \cap G_1 \cap G_2$. One can easily observe that $\delta(G) = 1$. Therefore,

$$\lim_{n \in G} \|L_n(f)T(1) - L_n(1)T(f)\| = 0.$$

Since

$$\begin{aligned}\|L_n(f)T(1) - T(1)T(f)\| &\leq \|L_n(f)T(1) - L_n(1)T(f)\| + \|L_n(1)T(f) - T(1)T(f)\| \\ &\leq \|L_n(f)T(1) - L_n(1)T(f)\| + \|T(f)\|\|L_n(1) - T(1)\|,\end{aligned}$$

the limit $\lim_{n \in G} \|[L_n(f) - T(f)]T(1)\| = 0$ is obtained. Also, it is well-known

$$\inf_{t \in H} \phi(t)\|h\| \leq \|\phi h\|$$

holds for any continuous function $\phi : H \to [0, \infty), h : H \to \mathbb{R}$. This implies that

$$[\inf_{t \in H} T(1)(t)]\|L_n(f) - T(f)\| \leq \|[L_n(f) - T(f)]T(1)\|.$$

From the hypotheses $\inf_{t \in H} T(1)(t) > 0$ and we obtain that $\lim_{n \in G} \|L_n(f) - T(f)\| = 0$ which implies $st - \lim \|L_n(f) - T(f)\| = 0$.

□

Our next result will be a general example for Kantorovich type operators. Assume that $(\Omega_n, \mu_n)_{n \in \mathbb{N}}$ be a sequence of probability measure spaces and $p_n : \Omega_n \to \mathbb{R}$ be a sequence of μ_n-integrable functions satisfying that for each $n \in \mathbb{N}$ there exists $K_n \geq 0$ such that $0 \leq p_n(\omega_n) \leq K_n$, for every $\omega_n \in \Omega_n$.

Example 2.2. Assume that, r_n and r defined on a compact Hausdorff space H to $[0,1]$ be continuous functions and also define L_n from $C[0,1]$ to $C(H)$ by

$$L_n(f)(t) = \sum_{k=0}^{n} \binom{n}{k} (r_n(t))^k (1 - r_n(t))^{n-k} \int_{\Omega_n} f(\frac{k + p_n(w_n)}{n + K_n}) d_{\mu_n}(\omega_n), n \in \mathbb{N}.$$

Then, construct $\tilde{L_n}$ as follows:

$$\tilde{L_n}(f)(t) = (1 + s_n) L_n(f)(t),$$

where (s_n) is divergent in the ordinary sense but statistically convergent to 0. If $\alpha = st-\lim \frac{K_n}{n}, \beta = st-\lim \frac{\int_{\Omega_n} p_n d_{\mu_n}}{n}, \gamma = st - \lim \frac{\int_{\Omega_n} p_n^2 d_{\mu_n}}{n^2}$ and $\gamma = \beta^2$, then the followings are equivalent:
$(i) For\ any\ f \in C[0,1], st - \lim \|\tilde{L_n}(f)(.) - f(\frac{r(.)+\beta}{\alpha+1})\| = 0.$
$(ii) st - \lim \|r_n(.) - r(.)\| = 0.$

It has been obtained that for every $n \in \mathbb{N}$ and $t \in H$,

$$L_n(e_1)(t) = \frac{r_n(t)}{1 + \frac{K_n}{n}} + \frac{1}{1 + \frac{K_n}{n}} \frac{\int_{\Omega_n} p_n d_{\mu_n}}{n}$$

holds. Therefore, we can write that

$$\tilde{L_n}(e_1)(t) = (1 + s_n)[\frac{r_n(t)}{1 + \frac{K_n}{n}} + \frac{1}{1 + \frac{K_n}{n}} \frac{\int_{\Omega_n} p_n d_{\mu_n}}{n}].$$

Since $\alpha = st-\lim \frac{K_n}{n}, \beta = st-\lim \frac{\int_{\Omega_n} p_n d_{\mu_n}}{n}$ we deduce that $st-\lim \tilde{L_n}(e_1)(t) = \frac{r(t)+\beta}{\alpha+1}$ is equivalent to $st-\lim r_n(t) = r(t)$. This proves the necessity.

For the sufficiency, it has been proved that

$$L_n(e_2(t)) = \frac{B_n(e_2)(r_n(t))}{(1 + \frac{K_n}{n})^2} + \frac{2r_n(t)}{(1 + \frac{K_n}{n})^2} \frac{\int_{\Omega_n} p_n d_{\mu_n}}{n} + \frac{1}{(1 + \frac{K_n}{n})^2} \frac{\int_{\Omega_n} p_n^2 d_{\mu_n}}{n^2},$$

where $B_n(f)$ is the well-known Bernstein operators. Again for our operators $\tilde{L_n}$ we can write that

$$\tilde{L_n}(e_2(t)) = (1 + s_n)[\frac{B_n(e_2)(r_n(t))}{(1 + \frac{K_n}{n})^2} + \frac{2r_n(t)}{(1 + \frac{K_n}{n})^2} \frac{\int_{\Omega_n} p_n d_{\mu_n}}{n} + \frac{1}{(1 + \frac{K_n}{n})^2} \frac{\int_{\Omega_n} p_n^2 d_{\mu_n}}{n^2}].$$

With the use of our assumptions, we find that

$$st - \lim \tilde{L_n}(e_2(t)) = \frac{r^2(t) + 2\beta r(t) + \gamma}{(\alpha + 1)^2} = (\frac{r(t) + \beta}{\alpha + 1})^2.$$

Now, let $T : C[0,1] \to C(H)$ be an operator such that $T(f)(t) = f(\frac{r(t)+\beta}{\alpha+1})$ and observe that $T(1) = 1, T(e_2) = [T(e_1)]^2$. As a consequence of Theorem 2.1, we obtain $st - \lim \tilde{L_n}(f) = T(f)$ for every $f \in C[0,1]$.

3. Conclusion

Under the light of operator version of Korovkin theorem which has recently been given by D. Popa, we have obtained its statistical analog in this study. Note that our results can also be extended by using A-statistical convergence which reduces to ordinary convergence if $A = I$, identity matrix and reduces to statistical convergence if $A = C_1$, Cesàro matrix. We believe that this direction of study includes many open problems.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

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