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THE NAKAGAMI FRECHET DISTRIBUTION IN MODELING REAL-LIFE DATA

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ABSTRACT

This work focused on the generalised Nakagami Frechet (NF) distribution. In this study, the probability density function (PDF), cumulative distribution function (CDF), survival, and hazard functions of the NF distribution were derived. In addition, some fundamental mathematical and statistical properties were derived. Moreover, the parameters of NF-distribution were estimated using maximum likelihood estimators (MLE). Furthermore, we presented the simulation studies to corroborate the reliability of MLE in estimating NF-distribution parameters. Finally, the application and utilization of the proposed distribution were demonstrated using real-life data sets.

1. INTRODUCTION

Statistics experts have formulated new flexible distributions by adding at least one shape parameter to the baseline distribution in order to address the continuous probability distribution's fundamental issues and limitations in modelling real-world data sets. Experts in mathematical statistics have developed methods for developing new families of distributions. The concept of a generalised family of distribution was first proposed by [1], the Kumaraswamy generator by [2], the Weibull generated family by [3-4], the Generalised Odd Gamma generated family of distribution by [5] and many more. Over the past few years, numerous researchers have proposed and investigated various classes of the Nakagami generated family of distributions. The generalised odd Nakagam [6], new family of odd generalised Nakagami [7] and a new family of odd nakagami exponential (NE-G) distributions [8]. The Nakagami Frechet distribution is a new generalised distribution introduced in this study. Also, some of its statistical properties will be thoroughly examined and MLE method will be explored to estimate the NF-distribution parameters. Finally, numerical results using simulation studies and application to real life data sets will be investigated.

2. EXPERIMENTAL

One of the probability distributions associated with the gamma distribution is called the Nakagami distribution. The Nakagami distribution family has two parameters: one shape parameter $\beta \geq 1/2$ and the other scale parameter $\lambda > 0$.Its probability density function (PDF) is given by:

$$f(x;\lambda,\beta)=\frac{2\lambda^{\lambda}}{\Gamma(\lambda)\beta^{\lambda}}x^{2\lambda-1}e^{-\frac{\lambda}{\beta}x^{2}} \tag{1}$$

The cumulative distribution function (CDF) corresponding to (1) is given by:

$$F(x;\beta,\lambda)=P\left(\lambda,\frac{\lambda}{\beta}x^{2}\right) \tag{2}$$

where P is the regularized lower incomplete gamma function.

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2.1. Theoretical Framework of Nakagami Frechet Distribution

Nakagami generalised family of distribution (Nakagami-G) due to [6] is given by:

$$F(x;\lambda,\beta,\tau,\rho,\nu)=\gamma_*\left[\lambda,\frac{\lambda}{\beta}\left(\frac{G(x;\rho,\nu)^\tau}{1-G(x;\rho,\nu)^\tau}\right)^2\right] \quad (3)$$

And

$$f(x;\lambda,\beta,\tau,\rho,\nu)=\frac{2\lambda^\lambda\tau g(x;\rho,\nu)G(x;\rho,\nu)^{2\lambda\tau-1}}{\Gamma(\lambda)\beta^\lambda(1-G(x;\rho,\nu)^\tau)^{2\lambda+1}}e^{-\frac{\lambda}{\beta}\left(\frac{G(x;\rho,\nu)^\tau}{1-G(x;\rho,\nu)^\tau}\right)^2} \quad (4)$$

Using the [6] framework, we will now introduce the Nakagami Frechet (NF) distribution, whose CDF and PDF are defined as

$$F(x;\lambda,\beta,\tau,\rho,\nu)=\gamma_*\left[\lambda,\frac{\lambda}{\beta}\left(\frac{e^{-\tau\left(\frac{\rho}{x}\right)^\nu}}{1-e^{-\tau\left(\frac{\rho}{x}\right)^\nu}}\right)^2\right] \quad (5)$$

$$f(x;\lambda,\beta,\tau,\rho,\nu)=\frac{2\lambda^\lambda\tau\rho^\nu\nu e^{-2\lambda\tau\left(\frac{\rho}{x}\right)^\nu}}{x^{\nu+1}\Gamma(\lambda)\beta^\lambda\left(1-e^{-\tau\left(\frac{\rho}{x}\right)^\nu}\right)^{2\lambda+1}}e^{-\frac{\lambda}{\beta}\left(e^{\tau\left(\frac{\rho}{x}\right)^\nu}-1\right)^{-2}} \quad (6)$$

Where $G(x;\rho,\nu)=e^{-\left(\frac{\rho}{x}\right)^\nu}$ and $g(x;\rho,\nu)=\frac{\rho^\nu\nu}{x^{\nu+1}}e^{-\left(\frac{\rho}{x}\right)^\nu}$

2.2. Investigation of the Proposed NF Distribution

To show that the proposed NF distribution is a pdf, we proceed as follows:

$$\int_0^\infty\frac{2\lambda^\lambda\tau\rho^\nu\nu e^{-2\lambda\tau\left(\frac{\rho}{x}\right)^\nu}}{x^{\nu+1}\Gamma(\lambda)\beta^\lambda\left(1-e^{-\tau\left(\frac{\rho}{x}\right)^\nu}\right)^{2\lambda+1}}e^{-\frac{\lambda}{\beta}\left(e^{\tau\left(\frac{\rho}{x}\right)^\nu}-1\right)^{-2}}\partial x=1 \quad (7)$$

Let

$$u=\frac{\lambda}{\beta}\left(e^{\tau\left(\frac{\rho}{x}\right)^\nu}-1\right)^{-2} \quad (8)$$

$$\partial x=\frac{\beta\left(e^{\tau\left(\frac{\rho}{x}\right)^\nu}-1\right)^3x}{2\lambda\tau\nu\left(\frac{\rho}{x}\right)^\nu e^{\tau\left(\frac{\rho}{x}\right)^\nu}}\partial u \quad (9)$$

substitute Equations (8) and (9) into (7) becomes.

$$\frac{\lambda^{\lambda-1}}{\Gamma(\lambda)\beta^{\lambda-1}}\int_0^\infty\left(\frac{\beta u}{\lambda}\right)^{\lambda-1}e^{-u}\partial u \quad (10)$$

the integral in Equation (10) becomes.

$$\frac{1}{\Gamma(\lambda)}\int_0^\infty u^{\lambda-1}e^{-u}\partial u=\frac{1}{\Gamma(\lambda)}\Gamma(\lambda) \quad (11)$$

$$\int_0^\infty f(x)\partial x=1$$

Hence Nakagami-Frechet distribution is a PDF.

2.3. Expansion of NF Distribution

The probability density function of the NF distribution is given in a simple form in this section. We can obtain the Taylor expansion and generalised binomial from Equation (6).

$$e^{-\frac{\lambda}{\beta}\left(e^{\tau\left(\frac{\rho}{x}\right)^{\nu}}-1\right)^{-2}} = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{\lambda^i}{\beta^i} \left(\frac{\mathrm{e}^{-\tau\left(\frac{\rho}{x}\right)^{\nu}}}{1-\mathrm{e}^{\tau\left(\frac{\rho}{x}\right)^{\nu}}} \right)^{2i} \tag{12}$$

insert Equation (12) into (6)

$$f(x;\lambda,\beta,\tau,\rho,\nu) = \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} \frac{2\lambda^{\lambda+i}\tau\rho^{\nu}\nu \mathrm{e}^{-2\tau(\lambda+i)\left(\frac{\rho}{x}\right)^{\nu}}}{x^{\nu+1}\Gamma(\lambda)\beta^{\lambda+i}\left(1-\mathrm{e}^{-\tau\left(\frac{\rho}{x}\right)^{\nu}}\right)^{2(\lambda+i)+1}} \tag{13}$$

$$\left(1-\mathrm{e}^{-\tau\left(\frac{\rho}{x}\right)^{\nu}}\right)^{-[2(\lambda+i)+1]} = \sum_{j=0}^{\infty} \binom{2(\lambda+i)+j}{j} \mathrm{e}^{-j\tau\left(\frac{\rho}{x}\right)^{\nu}} \tag{14}$$

insert Equation (14) into (13)

$$f(x;\lambda,\beta,\tau,\rho,\nu) = \sum_{i,j=0}^{\infty} \frac{2\tau\rho^{\nu}\nu}{x^{\nu+1}\Gamma(\lambda)} \frac{(-1)^i}{i!} \left(\frac{\lambda}{\beta}\right)^{\lambda+i} \binom{2(\lambda+i)+j}{j} e^{-\tau(2i+2\lambda+j)\left(\frac{\rho}{x}\right)^{\nu}} \tag{15}$$

therefore, (15) reduced to

$$f(x;\lambda,\beta,\tau,\rho,\nu) = \frac{2\tau\rho^{\nu}\nu}{\Gamma(\lambda)} \sum_{i,j=0}^{\infty} \varpi_{i,j} \left(\frac{\lambda}{\beta}\right)^{\lambda+i} x^{-(\nu+1)} e^{-\tau(2i+2\lambda+j)\left(\frac{\rho}{x}\right)^{\nu}} \tag{16}$$

where

$$\varpi_{i,j} = \frac{(-1)^i}{i!} \binom{2(\lambda+i)+j}{j}.$$

2.4. Reliability Analysis for NF Distribution

The respective survival and hazard functions of the proposed NF distribution follow:
R(x)=1-F(x)

$$R(x) = 1 - \frac{1}{\Gamma(\lambda)} \gamma \left[\lambda, \frac{\lambda}{\beta} \left(\frac{e^{-\tau\left(\frac{\rho}{x}\right)^{\nu}}}{1-e^{-\tau\left(\frac{\rho}{x}\right)^{\nu}}} \right)^2 \right] \tag{17}$$

$$h(x) = \frac{f(x)}{R(x)}$$

$$h(x) = \frac{\dfrac{2\lambda^{\lambda}\tau\rho^{\nu}\nu e^{-2\lambda\tau\left(\frac{\rho}{x}\right)^{\nu}}}{x^{\nu+1}\Gamma(\lambda)\beta^{\lambda}\left(1-e^{-\tau\left(\frac{\rho}{x}\right)^{\nu}}\right)^{2\lambda+1}} e^{-\frac{\lambda}{\beta}\left(e^{\tau\left(\frac{\rho}{x}\right)^{\nu}}-1\right)^{-2}}}{1-\dfrac{1}{\Gamma(\lambda)}\gamma\left[\lambda, \dfrac{\lambda}{\beta}\left(\dfrac{e^{-\tau\left(\frac{\rho}{x}\right)^{\nu}}}{1-e^{-\tau\left(\frac{\rho}{x}\right)^{\nu}}}\right)^2\right]} \tag{18}$$

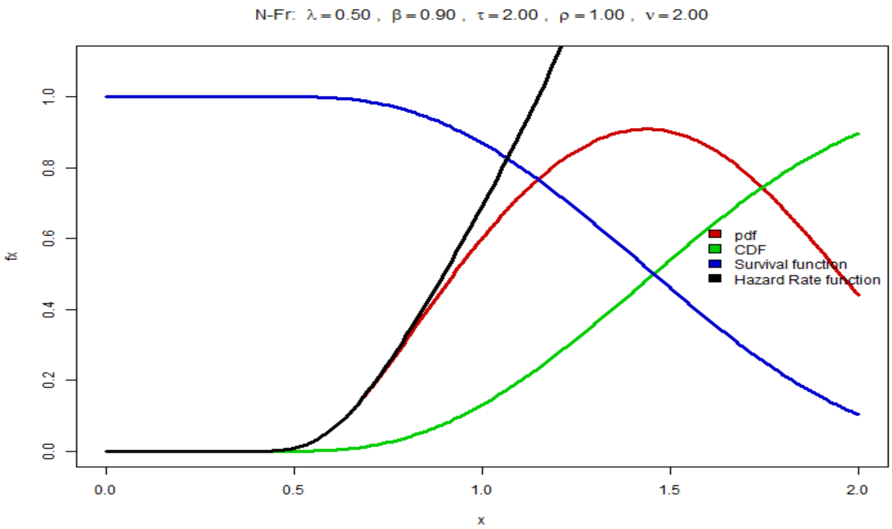

Fig 1. Plots of PDF, CDF, Survival and Hazard of NF distribution with parameters λ=0.5, β=0.9, τ=2.0, ρ=1.0, v=2.0

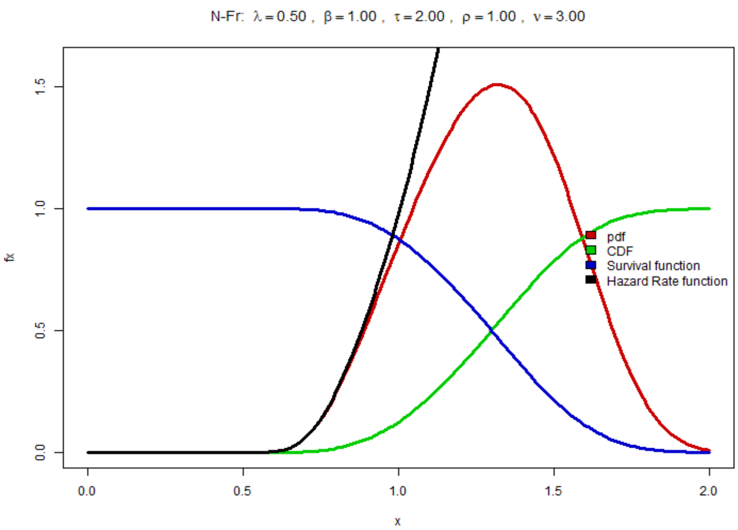

Fig 2. Plots of PDF, CDF, Survival and Hazard of NF distribution with parameters λ=0.5, β=1.0, τ=2.0, ρ=1.0, v=3.0

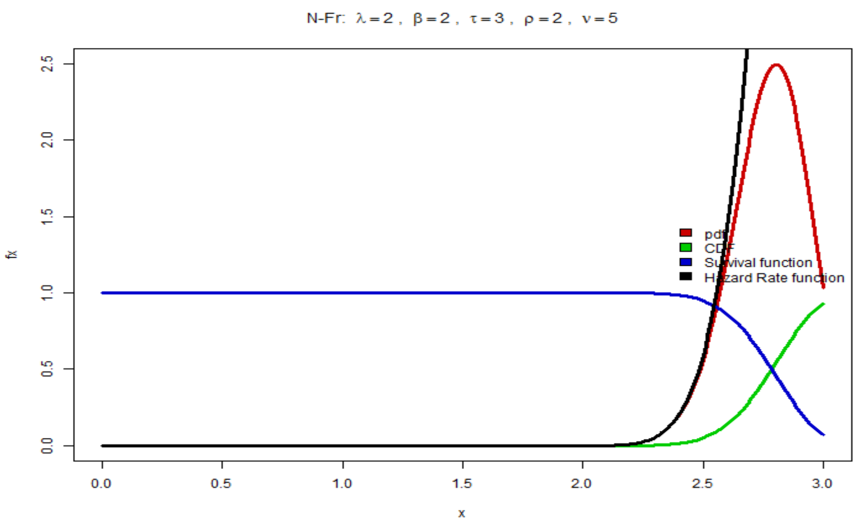

Fig 3. Plots of PDF, CDF, Survival and Hazard of NF- distribution with parameters λ=2, β=2, τ=3, ρ=2, v=5

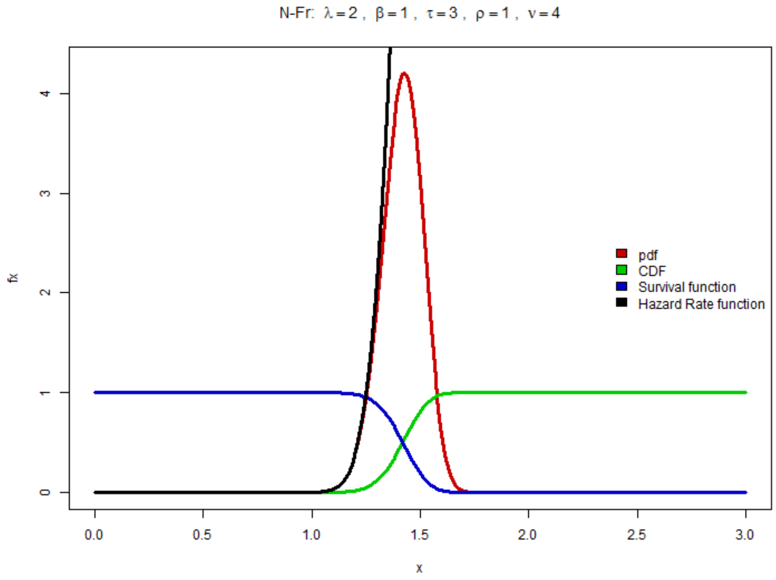

Fig 4. Plots of PDF, CDF, Survival and Hazard of NF distribution with parameters λ=2.0, β=1.0, τ=3.0, ρ=1.0, v=4.0

From the graphs of the Nakagami Frechet (NF) distribution, PDF, CDF, survival, and hazard functions for varying selected values of parameter are shown in Figs. 1–4.

From Fig.1. The pdf is platykurtic, CDF and Survival function are inverse of each other intersected at 1.4, while the hazard rate is an increasing function.

From Fig.2. The pdf is mesokurtic or approximately normal, CDF and Survival function are inverse of each other intersected at 1.25, while the hazard rate is an increasing function but less increasing than in Fig.1.

From Fig.3. The pdf is kurtic but less kurtic than Fig.1 and Fig.2, CDF and Survival function are inverse of each other intersected at 2.75, while the hazard rate is fairly a constant.

From Fig.4. The pdf is leptokurtic, CDF and Survival function are inverse of each other intersected at 1.35, while the hazard rate is fairly constant at 1.2.

2.5. Mathematical and Statistical Properties for NF Distribution

In this section, we provide some mathematical properties of the NF distribution.

2.5.1. The Raw Moments of the NF Distribution

The r^{th} moment of random variable X can be obtained from the PDF in Equation (16) as follows:

$$E(X^r) = \frac{2\tau\rho^\nu \nu}{\Gamma(\lambda)} \sum_{i,j=0}^{\infty} \varpi_{i,j} \left(\frac{\lambda}{\beta}\right)^{\lambda+i} \int_0^\infty x^{-(\nu+1)+r} e^{-\tau(2i+2\lambda+j)\left(\frac{\rho}{x}\right)^\nu} \partial x \tag{19}$$

let

$$u = \tau(2i + 2\lambda + j)\left(\frac{\rho}{x}\right)^\nu \tag{20}$$

$$x = \left[\frac{u}{\tau(2i + 2\lambda + j)\rho^\nu}\right]^{\frac{-1}{\nu}} \tag{21}$$

$$\partial x = -\frac{x^{(\nu+1)}\partial u}{\nu\tau(2\lambda + 2i + j)\rho^\nu} \tag{22}$$

by inserting Equation (20), (21) and (22) into (19) yield

$$E(X^r) = \frac{2\rho^r \tau^{\frac{r}{\nu}}}{\Gamma(\lambda)} \sum_{i,j=0}^{\infty} \varpi_{i,j} \left(\frac{\lambda}{\beta}\right)^{\lambda+i} [(2\lambda + 2i + j)]^{\frac{r}{\nu}-1} \int_0^{\infty} u^{-\frac{r}{\nu}} e^{-u} \partial u \quad (23)$$

for $r < \nu$ we obtain,

$$\mu'_r = \frac{2\rho^r \tau^{\frac{r}{\nu}}}{\Gamma(\lambda)} \sum_{i,j=0}^{\infty} \varpi_{i,j} \left(\frac{\lambda}{\beta}\right)^{\lambda+i} [(2\lambda + 2i + j)]^{\frac{r}{\nu}-1} \Gamma\left(1 - \frac{r}{\nu}\right), r = 1,2,3 \ldots \quad (24)$$

setting $r = 1$ in Equation (24) we obtain the mean of NF distribution.

$$\mu = \frac{2\rho \tau^{\frac{1}{\nu}}}{\Gamma(\lambda)} \sum_{i,j=0}^{\infty} \varpi_{i,j} \left(\frac{\lambda}{\beta}\right)^{\lambda+i} [(2\lambda + 2i + j)]^{\frac{1}{\nu}-1} \Gamma\left(1 - \frac{1}{\nu}\right), \quad (25)$$

setting $r = 2$ in Equation (24) we obtain the second moment of NF distribution.

$$E(X^2) = \frac{2\rho^2 \tau^{\frac{2}{\nu}}}{\Gamma(\lambda)} \sum_{i,j=0}^{\infty} \varpi_{i,j} \left(\frac{\lambda}{\beta}\right)^{\lambda+i} [(2\lambda + 2i + j)]^{\frac{2}{\nu}-1} \Gamma\left(1 - \frac{2}{\nu}\right) \quad (26)$$

the variance of NF distribution

$$Var(X) = E(X^2) - [E(X)]^2 \quad (27)$$

inserting Equation (25) and (26) into (27) yield

$$Var(X) = \frac{2\rho^2 \tau^{\frac{2}{\nu}}}{\Gamma(\lambda)} \sum_{i,j=0}^{\infty} \varpi_{i,j} \left(\frac{\lambda}{\beta}\right)^{\lambda+i} [(2\lambda + 2i + j)]^{\frac{2}{\nu}-1} \Gamma\left(1 - \frac{2}{\nu}\right) - \left[\frac{2\rho \tau^{\frac{1}{\nu}}}{\Gamma(\lambda)} \sum_{i,j=0}^{\infty} \varpi_{i,j} \left(\frac{\lambda}{\beta}\right)^{\lambda+i} [(2\lambda + 2i + j)]^{\frac{1}{\nu}-1} \Gamma\left(1 - \frac{1}{\nu}\right)\right]^2 \quad (28)$$

Moment generating function of NF will take this form:

$$M_X(t) = E(e^{tX}) = \sum_{r=0}^{\infty} \frac{t^r \mu'_r}{r!} \quad (29)$$

Insert Equation (24) into, (29), which yield the moment generating function of Nakagami Frechet distribution.

$$M_X(t) = E(e^{tX}) = \frac{2t^r \rho^r \tau^{\frac{r}{\nu}}}{r!\,\Gamma(\lambda)} \sum_{i,j,r=0}^{\infty} \varpi_{i,j} \left(\frac{\lambda}{\beta}\right)^{\lambda+i} [(2\lambda + 2i + j)]^{\frac{r}{\nu}-1} \Gamma\left(1 - \frac{r}{\nu}\right) \quad (30)$$

Characteristics function of NF will take this form:

$$\varphi_X(t) = E(e^{itX}) = \sum_{r=0}^{\infty} \frac{(it)^r \mu'_r}{r!} \quad (31)$$

Insert Equation (24) into, (31), which yield the characteristics function of Nakagami Frechet distribution.

$$\varphi_X(t) = \frac{2(it)^r \rho^r \tau^{\frac{r}{\nu}}}{r!\,\Gamma(\lambda)} \sum_{i,j,r=0}^{\infty} \varpi_{i,j} \left(\frac{\lambda}{\beta}\right)^{\lambda+i} [(2\lambda + 2i + j)]^{\frac{r}{\nu}-1} \Gamma\left(1 - \frac{r}{\nu}\right) \quad (32)$$

2.5.2. Incomplete Moments of NF Distribution

Considering the PDF of NF distribution obtained in Equation (16) the r^{th} incomplete moment for NF is derived as follows:

$$M_r^q = \int_0^t x^r f(x) \partial x \quad (33)$$

$$M_r^q = \frac{2\rho^r \tau^{\frac{r}{\nu}}}{\Gamma(\lambda)} \sum_{i,j=0}^{\infty} \varpi_{i,j} \left(\frac{\lambda}{\beta}\right)^{\lambda+i} [(2\lambda + 2i + j)]^{\frac{r}{\nu}-1} \int_0^t u^{-\frac{r}{\nu}} e^{-u} \partial u \quad (34)$$

$$M_r^q = \frac{2\rho^r \tau^{\frac{r}{\nu}}}{\Gamma(\lambda)} \sum_{i,j=0}^{\infty} \varpi_{i,j} \left(\frac{\lambda}{\beta}\right)^{\lambda+i} [(2\lambda + 2i + j)]^{\frac{r}{\nu}-1} \gamma\left(1 - \frac{r}{\nu}, \tau(2i + 2\lambda + j)\left(\frac{\rho}{t}\right)^{\nu}\right) \quad (35)$$

2.5.3. Quantile Function of NF Distribution

The quantile function of NF distribution is given by:

$$u = \frac{1}{\Gamma(\lambda)}\gamma\left[\lambda, \frac{\lambda}{\beta}\left(\frac{e^{-\tau\left(\frac{\rho}{x}\right)^{\nu}}}{1-e^{-\tau\left(\frac{\rho}{x}\right)^{\nu}}}\right)^{2}\right] \tag{36}$$

we obtained the quantile function of NF distribution that follows:

$$x = \left\{\frac{-\tau\rho^{\nu}}{\log\left\{\frac{\left[\frac{\beta}{\lambda}\gamma^{-1}(\lambda, u\Gamma(\lambda))\right]^{1/2}}{1+\left[\frac{\beta}{\lambda}\gamma^{-1}(\lambda, u\Gamma(\lambda))\right]^{1/2}}\right\}}\right\}^{1/\nu} \tag{37}$$

from Equation (37) we obtained the median of NF distribution

$$Med(x) = \left\{\frac{-\tau\rho^{\nu}}{\log\left\{\frac{\left[\frac{\beta}{\lambda}\gamma^{-1}(\lambda, 0.5\Gamma(\lambda))\right]^{1/2}}{1+\left[\frac{\beta}{\lambda}\gamma^{-1}(\lambda, 0.5\Gamma(\lambda))\right]^{1/2}}\right\}}\right\}^{1/\nu} \tag{38}$$

2.6. Maximum Likelihood Estimates of the Parameters of NF Distribution

In this section, the parameter of the NF distribution will be estimated using maximum likelihood estimators. The likelihood function of the NF distribution is given by:

$$L(x) = \prod_{i=1}^{n}\frac{2\lambda^{\lambda}\tau\rho^{\nu}\nu e^{-2\lambda\tau\left(\frac{\rho}{x}\right)^{\nu}}}{x^{\nu+1}\Gamma(\lambda)\beta^{\lambda}\left(1-e^{-\tau\left(\frac{\rho}{x}\right)^{\nu}}\right)^{2\lambda+1}}e^{-\frac{\lambda}{\beta}\left(e^{\tau\left(\frac{\rho}{x}\right)^{\nu}}-1\right)^{-2}} \tag{39}$$

The log- likelihood function of the NF distribution is given by:

$$\ell(x) = n\log(2) + n\lambda\log(\lambda) + n\log(\tau) + n\nu\log(\rho) + n\log(\nu) - (\rho+1)\sum_{i=1}^{n}\log(x) - n\log(\Gamma(\lambda$$

$$- n\lambda\log(\beta) - (2\lambda+1)\sum_{i=1}^{n}\log\left(1-\mathrm{e}^{-\tau\left(\frac{\rho}{x}\right)^{\nu}}\right) - \frac{\lambda}{\beta}\sum_{i=1}^{n}\left(\mathrm{e}^{\tau\left(\frac{\rho}{x}\right)^{\nu}}-1\right)^{-2} \tag{40}$$

Taken the partial derivative of Equation (40) with respect to $\lambda, \beta, \tau, \rho, \upsilon$ and solve for $\lambda, \beta, \tau, \rho, \upsilon$ to obtained the maximum likelihood estimators.

$$\frac{\partial\ell(x)}{\partial\lambda} = n + n\log(\lambda) - n\Psi(\lambda) - n\log(\beta) - 2\sum_{i=1}^{n}\log\left(1-\mathrm{e}^{-\tau\left(\frac{\rho}{x}\right)^{\nu}}\right) - \frac{\sum_{i=1}^{n}\left(\mathrm{e}^{\tau\left(\frac{\rho}{x}\right)^{\nu}}-1\right)^{-2}}{\beta}=0 \tag{41}$$

$$\frac{\partial\ell(x)}{\partial\beta} = -\frac{n\lambda}{\beta} + \frac{\lambda}{\beta^{2}}\sum_{i=1}^{n}\left(\mathrm{e}^{\tau\left(\frac{\rho}{x}\right)^{\nu}}-1\right)^{-2} = 0 \tag{42}$$

$$\frac{\partial\ell(x)}{\partial\tau} = \frac{n}{\tau} - (2\lambda+1)\sum_{i=1}^{n}\frac{\left(\frac{\rho}{x}\right)^{\nu}\mathrm{e}^{-\tau\left(\frac{\rho}{x}\right)^{\nu}}}{1-\mathrm{e}^{-\tau\left(\frac{\rho}{x}\right)^{\nu}}} + \frac{2\lambda}{\beta}\sum_{i=1}^{n}\frac{\left(\frac{\rho}{x}\right)^{\nu}\mathrm{e}^{\tau\left(\frac{\rho}{x}\right)^{\nu}}}{\left(\mathrm{e}^{\tau\left(\frac{\rho}{x}\right)^{\nu}}-1\right)^{3}}=0 \tag{43}$$

$$\frac{\partial \ell(x)}{\partial \nu} = n\log(\rho) + \frac{n}{\nu} - \frac{n}{\tau} - (2\lambda+1)\sum_{i=1}^{n} \frac{\tau(\frac{\rho}{x})^{\nu}\log(\frac{\rho}{x})\mathrm{e}^{-\tau(\frac{\rho}{x})^{\nu}}}{1-\mathrm{e}^{-\tau(\frac{\rho}{x})^{\nu}}} + \frac{2\lambda}{\beta}\sum_{i=1}^{n} \frac{\tau\left(\frac{\rho}{x}\right)^{\nu}\log\left(\frac{\rho}{x}\right)\mathrm{e}^{\tau\left(\frac{\rho}{x}\right)^{\nu}}}{\left(\mathrm{e}^{\tau\left(\frac{\rho}{x}\right)^{\nu}}-1\right)^{3}}=0 \tag{44}$$

$$\frac{\partial \ell(x)}{\partial \rho} = \frac{n\nu}{\rho} - \sum_{i=1}^{n}\log(x) - (2\lambda+1)\sum_{i=1}^{n} \frac{\tau\nu(\frac{\rho}{x})^{\nu}\mathrm{e}^{-\tau(\frac{\rho}{x})^{\nu}}}{\rho(1-\mathrm{e}^{-\tau(\frac{\rho}{x})^{\nu}})} + \frac{2\lambda}{\beta}\sum_{i=1}^{n} \frac{\tau\nu\left(\frac{\rho}{x}\right)^{\nu}\mathrm{e}^{\tau\left(\frac{\rho}{x}\right)^{\nu}}}{\rho\left(\mathrm{e}^{\tau\left(\frac{\rho}{x}\right)^{\nu}}-1\right)^{3}}=0 \tag{45}$$

Since these equations (41-45) are nonlinear, they cannot be solved analytically but can be solved through iterative methods. This article employed Newton-Raphson's function in the R programming language to estimate the model parameters.

3. SIMULATION STUDIES

The Nakagami Frechet (NF) distribution random number generator procedure was generated by using the uniform distribution method from the quantile function, and the simulations were repeated 1000 times for the NF distribution at varying sample sizes, n = 30, 50, 80, and parameter values.

Table 1. Monte Carlo simulation results: The mean, bias, MSE, and RMSE of the MLE of NF distribution parameters

N	Par	Init.	Mean	Bias	MSE	RMSE
30	λ	1.9	1.921912	0.02191181	0.2934972	0.54175
	β	1	0.7579875	-0.2420125	0.3211022	0.56666
	τ	0.05	0.07510589	0.02510589	0.005165396	0.07187
	ρ	0.1	0.4260937	0.3260937	0.233132	0.48284
	ϑ	0.1	0.100441	0.0004410424	0.0006184553	0.02487
50	λ	1.9	2.047306	0.1473057	0.2804175	0.52954
	β	1	0.8074895	-0.1925105	0.2136727	0.46225
	τ	0.05	0.07016362	0.02016362	0.00300192	0.05479
	ρ	0.1	0.3736567	0.2736567	0.1805187	0.42487
	ϑ	0.1	0.09640465	-0.003595352	0.0003887303	0.01972
80	λ	1.9	2.074211	0.1742114	0.1718437	0.41454
	β	1	0.8633808	-0.1366192	0.1097647	0.33131
	τ	0.05	0.06506434	0.01506434	0.001630674	0.04038
	ρ	0.1	0.2509945	0.1509945	0.07381414	0.27169
	ϑ	0.1	0.09516572	-0.004834276	0.0002119342	0.01456

Table 2. Monte Carlo simulation results: The mean, bias, MSE, and RMSE of the MLE of NF distribution parameters

N	Par	Init.	Mean	Bias	MSE	RMSE
30	λ	1.5	1.2139480	-0.2860515	2.314285	1.52128
	β	0.9	2.516777	1.616777	7.410622	2.7222
	τ	0.5	1.250645	0.7506453	2.089474	1.44550
	ρ	0.7	0.7051115	0.005111494	0.5164915	0.71867
	ϑ	1	2.073479	1.073479	2.395852	1.54785
50	λ	1.5	1.295309	-0.2046905	1.851503	1.36070
	β	0.9	2.026286	1.126286	4.208077	2.05136
	τ	0.5	1.20487	0.7048697	1.974426	1.40514
	ρ	0.7	0.6864917	-0.01350828	0.4387352	0.66237
	ϑ	1	1.784429	0.784429	1.547356	1.24393
80	λ	1.5	1.366522	-0.1334781	1.52735	1.23586
	β	0.9	1.719037	0.8190373	2.952958	1.71842
	τ	0.5	1.096512	0.5965116	1.506792	1.22751
	ρ	0.7	0.7168961	0.01689611	0.398427	0.63121
	ϑ	1	1.546149	0.5461493	0.9271532	0.96289

4. NUMERICAL EXAMPLES USING REAL-LIFE DATA SETS

In this section, we present an application to a real-life data set to illustrate the potentiality of the proposed NF distribution. We consider the Nakagami Frechet distribution to compare its performance to other generated models. For model comparison, the Anderson-Darling (A), Kolmogorov-Smirnov (K.S.), and P-value of the K.S. tests were used.
[9] provided the data, which refers to the period between failures for a repairable component. The following is the data set: 1.43, 0.11, 0.71, 0.77, 2.63, 1.49, 3.46,2.46, 0.59, 0.74, 1.23, 0.94, 4.36, 0.40, 1.74, 4.73, 2.23, 0.45, 0.70, 1.06, 1.46, 0.30,1.82, 2.37, 0.63, 1.23, 1.24, 1.97, 1.86, 1.17.

Table 3. MLEs and Goodness-of-fit measures for Data Set

Models	MLE	$-\ell$	W	A	K.S	P value
NF	$\lambda = 1.0283164$	39.68584	0.02016843	0.1553593	0.062814	0.9998
	$\beta = 1.0162492$					
	$\tau = 1.0082819$					
	$\rho = 0.7554704$					
	$\nu = 0.4975756$					
ExpWei-wei	$\lambda = 1.5066570$	40.73648	0.03281623	0.2466412	0.11028	0.8589
	$\beta = 1.5867188$					
	$\tau = 1.0290464$					
	$\rho = 0.4147790$					
	$\nu = 0.7418513$					
OGG-Fr	$\lambda = 1.6977351$	39.89123	0.02322876	0.1803743	0.075102	0.9958
	$\beta = 0.8171655$					
	$\tau = 1.0543589$					
	$\rho = 0.6438243$					
GOG-Fr	$\lambda = 0.9051306$	40.25291	0.02883905	0.216739	0.10206	0.9135
	$\beta = 1.8603620$					
	$\tau = 0.5355734$					
	$\rho = 0.7624326$					

Clearly, the NF distribution with five parameter values provides a better fit than their competitors' models. It has the smallest W, A and K-S values among those considered in tables (3).

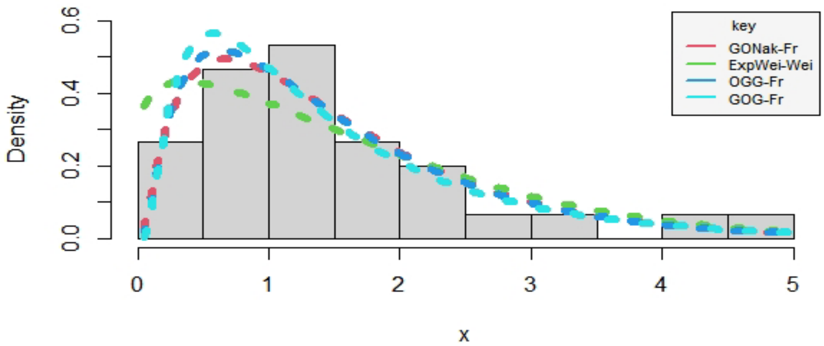

Fig 5. Histogram and Estimated pdfs of NF

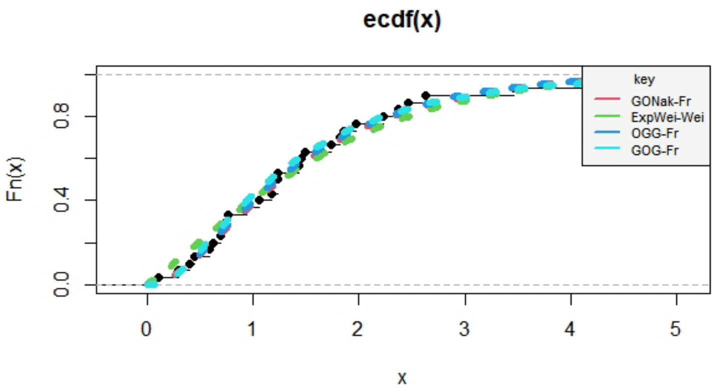

Fig 6. Estimated cdfs NF

The fitted densities of the models were compared to the empirical histogram of the [9] data set in figure (6). The fitted probability density of the NF distribution is clearly closer to the empirical histogram than the Exp.Wei-Wei, OGG-Fr, and GOG-Fr models in figure (5) the fitted CDF of the NF distribution is closer to the empirical CDF of the [9] data set than the Exp.Wei-Wei, OGG-Fr, and GOG-Fr models.

5. CONCLUSION

This study presents the NF distributions. Quantile functions, moments, and moments-generating functions as some of the statistical and mathematical properties of the NF distribution being studied. The maximum likelihood estimates of model parameters were obtained. The simulation study was carried out to assess the behaviour of the maximum likelihood estimators for the NF-distribution parameters. For future work, the NF distribution parameters estimation could be extended with a viewing to using other methods as well as examination of other properties of the NF distribution.

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